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Contracting planar graphs to contractions of triangulations ☆

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A B S T R A C T

For every graph H, there exists a polynomial-time algorithm deciding if a planar input graph G can be contracted to H. However, the degree of the polynomial depends on the size of H. We identify a class of graphs $\mathcal{C}$ such that for every fixed $H \in \mathcal{C}$, there exists a linear-time algorithm deciding whether a given planar graph G can be contracted to H. The class $\mathcal{C}$ is the closure of planar triangulated graphs under taking of contractions. In fact, we prove that a graph $H \in \mathcal{C}$ if and only if there exists a constant c_H such that if the treewidth of a graph is at least c_H, it contains H as a contraction. We also provide a characterization of $\mathcal{C}$ in terms of minimal forbidden contractions.

1. Introduction

We consider simple graphs without loops and multiple edges. For a graph G, let $V(G)$ be its vertex set and $E(G)$ its edge set. For notions not defined here, we refer the reader to the monograph on graph theory by Diestel [6].

1.1. Planar graphs

All graphs in this paper are planar. Plane graphs are always assumed to be drawn on the unit sphere and their edges are arbitrary polygonal arcs (not necessarily straight line segments).

Embeddings. In this work, we only need to distinguish between essentially different embeddings of a planar graph. This motivates the following definition. Two plane graphs G and H are *combinatorially equivalent* ($G \simeq H$) if there exists a homeomorphism of the unit sphere (in which they are embedded) that transforms one into the other. The relation of being combinatorially equivalent is reflexive, symmetric and transitive, and thus an equivalence relation. Let $\mathcal{G}$ be the class of all plane graphs isomorphic to a planar graph G, and let us consider the quotient set $\mathcal{G}/\simeq$. The equivalence classes (i.e., the elements of the quotient set) can be thought of as *embeddings*. In fact, we will work with embeddings but for simplicity, we will pick a plane graph representative for each embedding.

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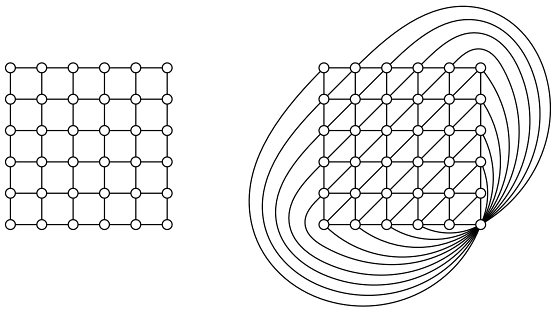

Fig. 1. Graphs M_6 and Γ_6, respectively.

Dual. The dual of a plane graph G will be denoted by G^*. Note that there is a one-to-one correspondence between the edges of G and the edges of G^*. We keep the convention that e^* is the edge of G^* corresponding to edge e of G.

Triangulation. A planar graph is called *triangulated* if it has an embedding in which every face is incident with exactly three vertices. Let us recall two useful facts related to planar 3-connected graphs that we will need later.

Lemma 1.1. *Triangulated planar graphs are* 3*-connected.*

Lemma 1.2. *A* 3*-connected planar graph has a unique embedding.*

For proofs of these lemmas, see for instance the monograph of Mohar and Thomassen [16], Lemma 2.3.3, p. 31 and Lemma 2.5.1, p. 39, respectively. From these two lemmas, every triangulated graph has a unique embedding.

Grids and walls. The $k \times k$ *grid* M_k has as its vertex set all pairs (i, j) for $i, j = 0, 1, \dots, k-1$, and two vertices (i, j) and (i', j') are joined by an edge if and only if $|i - i'| + |j - j'| = 1$.

For $k \geqslant 2$, let Γ_k denote the graph obtained from M_k by triangulating its faces as follows: add an edge between vertices (i, j) and (i', j') if $i - i' = 1$ and $j' - j = 1$, and add an edge between corner vertex $(k-1, k-1)$ and every external vertex that is not already adjacent to $(k-1, k-1)$, i.e., every vertex (i, j) with $i \in \{0, k-1\}$ or $j \in \{0, k-1\}$, apart from the vertices $(k-2, k-1)$ and $(k-1, k-2)$. The graph Γ_k is called a *triangulated grid*. See Fig. 1 for graphs M_6 and Γ_6. The dual Γ_k^* of a triangulated grid is called a *wall*.

Treewidth and MSOL. The fragment of second-order logic where quantified relation symbols must have arity 1 is called *monadic second-order logic* (MSOL). A seminal result of Courcelle [4] is that on any class of graphs of bounded treewidth, every problem definable in monadic second-order logic can be solved in time linear in the number of vertices of the graph. Moreover, Courcelle's result holds not just when graphs are given in terms of their edge relation, but also when the domain of a structure encoding a graph G consists of the disjoint union of the set of vertices and the set of edges, as well as unary relations V and E to distinguish the vertices and the edges, respectively, and also a binary incidence relation I which denotes when a particular vertex is incident with a particular edge (thus, $I \subseteq V \times E$). The reader is referred to Courcelle [4] for more details and also for the definition of treewidth.

Lemma 1.3. *(See [4].) For every fixed k and every problem $\mathcal{P}$ expressible in MSOL, there exists a linear-time algorithm for $\mathcal{P}$ in the class of graphs of treewidth at most k.*

Below we recall some other results related to treewidth that we will need later in the paper.

Lemma 1.4. *(See (6.2) in [18].) Let $m \geqslant 1$ be an integer. Every planar graph with no $m \times m$ grid minor has treewidth $\leqslant 6m - 5$.*

Lemma 1.5. *(See (1.5) in [18].) If H is a planar graph with $|V(H)| + 2|E(H)| \leqslant n$, then H is isomorphic to a minor of the $2n \times 2n$ grid.*

Lemma 1.6. *(See Theorem 6 in [2].) For any plane graph G and its dual G^*, $\mathbf{tw}(G)^* \leqslant \mathbf{tw}(G) + 1$.*

Lemma 1.7. *(See Theorem 1.1 in [1].) For every fixed k, there exists a linear-time algorithm deciding whether the input graph has treewidth at most k.*

Pasting along vertices and edges. Let G be a graph with induced subgraphs G_1 and G_2 such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Let $G_1 \cap G_2$ denote the subgraph of G induced by $V(G_1) \cap V(G_2)$. Then we say that G arises from G_1 and G_2 by *pasting along* $G_1 \cap G_2$. In this paper, we only paste along vertices and edges (that is, if $G_1 \cap G_2$ is a vertex or an edge). We note that pasting planar graphs along vertices and edges creates planar graphs.

1.2. Containment relations

An *edge contraction* of an edge e in a graph is the graph obtained by removing e, identifying its two endpoints, and eliminating parallel edges that may appear. Some basic properties of contractions are collected by Wolle and Bodlaender [20]. Formally, for an edge e with endpoints u and w, the contraction of e, denoted by G/e, is the graph with vertex set $V(G/e) = V(G) \setminus \{u, w\} \cup \{v_{uw}\}$ and edge set

$$E(G/e) = E \setminus \big\{\{x, y\} \in E\colon\ x \in \{u, w\},\ y \in V\big\} \cup \big\{\{v_{uw}, x\}\colon\ \{x, u\} \in E \vee \{x, w\} \in E\big\}.$$

A graph H is a *contraction* of a graph G (or G is *contractible* to H) if H can be obtained from G by a sequence of edge contractions. We denote it by $H \leqslant_c G$.

A *dissolution* of a vertex v of degree 2 in a plane graph G is a contraction of one of the two edges to which v is incident in G. A graph H that can be obtained from G be a sequence of dissolutions and edge/vertex deletions is called a *topological minor* of G ($H <_{tm} G$). Finally, a graph H that is a contraction of a subgraph of G is called its *minor* ($H <_m G$).

In the paper, when we speak about different containment relations like contraction or topological minor, the graph H will be called a *pattern*.

1.3. Parameterized complexity

Parameterized complexity is a paradigm in computational complexity and analysis of algorithms that has received a lot of attention in the last 20 years. The idea is to evaluate the efficiency of an algorithm not only in terms of the size n of an input I, but also by some parameter k that is assumed to be natural for the problem under consideration. A decision problem is called *fixed parameter tractable* if an instance (I, k) can be solved by an algorithm that runs in time $f(k) \cdot n^{\mathcal{O}(1)}$ where f is a computable function that only depends on k, thus independent of n. Such an algorithm is considered efficient from the parameterized point of view. The class FPT is the class of all fixed-parameter tractable decision problems. For more information on parameterized complexity we refer to monographs of Downey and Fellows [7], Flum and Grohe [8], and Niedermeier [17].

2. Previous work

The problem of checking whether a graph is a contraction of another has already attracted some attention. In this section we briefly survey known results. Let P_n, C_n and K_n denote the path, cycle and complete graph on n vertices, respectively. Let $K_{p,q}$ denote the complete bipartite graph with partition classes of size p and q, respectively. The graph $K_{1,m}$ for $m \geqslant 1$ is also called a *star*.

Perhaps the first systematic study of contractions was undertaken by Brouwer and Veldman [3]. The two main results from their paper are as follows.

Theorem 2.1. *(See Theorem 3 in [3].) A graph G is contractible to $K_{1,m}$ if and only if G is connected and contains an independent set S of m vertices such that $G - S$ is connected.*

In particular, a graph is contractible to P_3 if and only if it is connected and is neither a cycle nor a complete graph. Theorem 2.1 also allows us to detect, in polynomial time, if a graph is contractible to $K_{1,m}$ for a fixed integer m. It suffices to enumerate over all sets S with m independent vertices and check if the graph $G - S$ is connected. This gives a $|V(G)|^{\mathcal{O}(m)}$ algorithm, which is polynomial for every fixed integer m.

Theorem 2.2. *(See Theorem 9 in [3].) If H is a connected triangle-free graph other than a star, then contractibility to H is* NP*-complete.*

Hence, checking if a graph is contractible to P_4 or C_4 is NP-complete. More generally, it is NP-complete for every bipartite graph with at least one connected component that is not a star.

The research direction initiated by Brouwer and Veldman was continued by Levin, Paulusma, and Woeginger [13,14]. Here is the main result established in these two papers.

Theorem 2.3. *(See Theorem 3 in [13].) Let H be a connected graph on at most* 5 *vertices. If H has a dominating vertex, then contractibility to H can be decided in polynomial time. If H does not have a dominating vertex, then contractibility to H is* NP*-complete.*

However, the existence of a dominating vertex in the pattern H is not enough to ensure that contractibility to H can be decided in polynomial time. A pattern on 69 vertices for which contractibility to H is NP-complete was exhibited by van 't Hof et al. [19].

Looking at contractions to fixed pattern graphs is justified by the following theorems proved by Matoušek and Thomas [15].

Theorem 2.4. *(See Theorem 4.1 in [15].) The problem of deciding, given two input graphs G and H, whether G is contractible to H is* NP*-complete even if we impose one of the following restrictions on G and H:*

(i) *H and G are trees of bounded diameter,*
(ii) *H and G are trees with at most one vertex of degree more than* 5.

Theorem 2.5. *(See Theorem 4.3 in [15].) For every fixed k, the problem of deciding, given two input graphs G and H, whether G is contractible to H is* NP*-complete even if we restrict G to partial k-trees and H to k-connected graphs.*

The authors also proved a positive result.

Theorem 2.6. *(See Theorem 5.14 in [15].) For every fixed Δ, k, there exists an $\mathcal{O}(|V(H)|^{k+1} \cdot |V(G)|)$ algorithm to decide, given two input graphs G and H, whether G is contractible to H, when the maximum degree of H is at most Δ and G is a partial k-tree.*

In a previous paper [11], we studied the problem of deciding whether a given planar graph can be contracted to a fixed pattern. Here, we will need some of the definitions and results of that paper.

An *embedded contraction* of an edge e of a plane graph G is a plane graph G' that is obtained by homeomorphically mapping the endpoints of e in G to a single vertex without any edge crossings and recursively removing one of the two parallel edges bounding a 2-face, if a graph has such a pair. Note that there are many embedded contractions of an edge of a plane graph G but they are all combinatorially equivalent.

An *embedded dissolution* of a vertex v of degree 2 in a plane graph G is an embedded contraction of one of the two edges to which v is incident in G.

Let G and H be two plane graphs. We say that H is an embedded contraction of G ($H \leqslant_{ec} G$), if H is combinatorially equivalent to a graph that can be obtained from G by a series of embedded contractions. We say that H is an *embedded topological minor* of G ($H \leqslant_{etm} G$), if H is combinatorially equivalent to a graph that can be obtained from G by a series of vertex and edge deletions, and embedded dissolutions of vertices of degree 2.

The main technical result of [11] is an equivalence between embedded contractions in a planar graph and embedded topological minors in its dual. A multigraph is called *thin* if it has no two parallel edges bounding a 2-face. Simple graphs are in particular thin.

Lemma 2.7. *(See Lemma 2 in [11].) Let H and G be two thin planar graphs and H^*, G^* their respective duals. Then*

$$H \leqslant_{ec} G \quad \Leftrightarrow \quad H^* \leqslant_{etm} G^*.$$

This equivalence is used to reduce the problem of finding a contraction in a planar graph G to finding an embedded topological minor in its dual graph G^*. This consequently leads to the main result of that paper.

Theorem 2.8. *(See Theorem 12 in [11].) For every graph H, there exists a polynomial-time algorithm that given a planar graph G decides whether H is a contraction of G, and if so finds a series of contractions transforming G into H.*

3. Our motivation and results

Theorem 2.8 tells us that for every graph H, one can decide in polynomial time if a planar graph can be contracted to H. However, the running time of the algorithm, as mentioned in [11], is bounded by a polynomial whose degree depends on the size of H. Therefore, this is not an FPT algorithm, when parameterized by $|H|$.

Recently, Grohe et al. [10] announced that topological minor testing is FPT. We emphasize that due to the difference between embedded topological minors and topological minors their result does not imply that contraction testing is FPT in planar graphs.

In this paper, given the duality of Theorem 2.8, we focus on contractions in planar graphs and identify a class of patterns for which an FPT algorithm exists. We show that the problem of testing whether a planar graph G can be contracted to a triangulated planar graph H is equivalent to testing whether the dual of G contains the dual of H as a topological minor; so for such cases embedded topological minors and topological minors coincide. This means that we could use the result of Grohe et al. to immediately find that contraction testing is FPT in planar graphs as long as the pattern graphs are triangulated. However, we aim for a stronger result. Let $\mathcal{C}$ be the closure of the class of triangulated planar graphs with respect to taking of contractions. We present an FPT algorithm running in time linear in n, that tests if a given n-vertex planar graph can be contracted to a pattern graph $H \in \mathcal{C}$.

Our approach is as follows. We prove that for every graph $H \in \mathcal{C}$, there is a constant c_H such that if $\mathbf{tw}(G) > c_H$, then G contains H as a contraction. Our algorithm first checks if the treewidth of the input graph is large enough. If so, then the input graph contains H as a contraction. Otherwise, we use the celebrated result by Courcelle [4] to solve the problem on the class of graphs with bounded treewidth.

We also study properties of $\mathcal{C}$ and provide a characterization of the class in terms of forbidden contractions.

4. Algorithm

Definition 4.1. Let $\mathcal{T}$ be the class of triangulated planar graphs and $\mathcal{C}$ be the minimal contraction-closed class containing $\mathcal{T}$.

We note that $\mathcal{C}$ is well-defined. It is the set of all graphs that can be obtained as a contraction of a graph in $\mathcal{T}$. Because contraction is transitive, $\mathcal{C}$ is closed under contractions.

In order to illustrate that $\mathcal{C}$ is a proper superclass of $\mathcal{T}$ we consider the diamond, which is the graph D given by vertices u, v, w, x and edges uv, uw, vw, vx, wx. We observe that D is a planar graph that is not triangulated. Hence, D does not belong to $\mathcal{T}$. However, D does belong to $\mathcal{C}$, because it is a contraction of the triangulated planar graph in $\mathcal{T}$ that is obtained from D by adding a new vertex y adjacent to u, v, w, x. We generalize this observation in Lemma 4.2.

Lemma 4.2. *$\mathcal{C}$ is the closure of $\mathcal{T} \cup \{K_2\}$ with respect to pasting along vertices and edges.*

Proof. For the forward implication, let us note that triangulated planar graphs are 3-connected by Lemma 1.1. All cut-sets of a triangulated planar graph are isomorphic to the clique on 3 vertices. Let us observe that contractions of a graph whose cut-sets are cliques are graphs with clique cut-sets. Also, the size of a cut-set will not increase after contraction.

Let $G \in \mathcal{C}$. By definition, G is a contraction of a triangulated planar graph. The maximal 3-connected components of G are triangulated, and the minimal cut-sets of size less than 3 are isomorphic to complete graphs. Hence, G can be obtained from triangulated graphs by pasting along vertices and edges.

For the backward implication, let G be a minimal graph that belongs to the closure of $\mathcal{T}$ with respect to pasting along vertices and edges but that is not a contraction of a triangulated planar graph.

First, suppose that G has a cut-vertex v. Let $C_1, \ldots, C_k$ denote the connected components of $G - v$ after restoring v in each of them. Let $C'_1, \ldots, C'_k$ be triangulated planar graphs such that $C_1, \ldots, C_k$ are their contractions respectively. Let us consider drawings of $C'_1, \ldots, C'_k$ such that v belongs to the outerface in each of these drawings (clearly, such drawings exist). Let x_i, y_i be the other two vertices of C_i, for $i = 1, \ldots, k$, besides v, incident with the outerface. Now let us identify v from different drawings of $C'_1, \ldots, C'_k$ in such a way that the cyclic order of vertices around v is $x_1, y_1, \ldots, x_k, y_k$. Let us then add new vertices $w_1, \ldots, w_k$ to the drawing and make them all adjacent to v and each w_i adjacent to y_i and x_{i+1}, for $k = 1, \ldots, k-1$, and w_k adjacent to y_k and x_1. At the last step, we add a vertex u adjacent to all x_i, y_i, w_i, for $i = 1, \ldots, k$. It is easy to verify that the graph we have created is planar and triangulated, and that G is a contraction of that graph; a contradiction.

Second, we suppose that G has a cut-edge vw. Let us consider a planar drawing of G and put a new vertex in every 4-face of G. Next, we make the new vertices adjacent to all 4 vertices incident with that face. It is easy to verify that the graph we have created is planar and triangulated, and that G is a contraction of that graph; a contradiction. This completes the proof of Lemma 4.2. $\square$

Lemma 4.3. *Let H be a triangulated planar graph and H^* be its dual. For every planar graph G and its dual G^*,*

$$H \leqslant_c G \quad \Leftrightarrow \quad H^* \leqslant_{tm} G^*.$$

Proof. H is a triangulated planar graph and has a unique embedding by Lemmas 1.1 and 1.2. Hence, the contraction and embedded contraction relations coincide. As the dual of a planar triangulated graph also has a unique embedding, the embedded topological minor and topological relations minor coincide. Thus, $H \leqslant_{ec} G \Leftrightarrow H \leqslant_c G$ and $H^* \leqslant_{etm} G^* \Leftrightarrow H^* \leqslant_{tm} G^*$, and the lemma follows from Lemma 2.7. $\square$

Theorem 4.4. *Let H be a graph. Then H belongs to $\mathcal{C}$ if and only if there exists a constant c_H such that for every graph G, if $\mathbf{tw}(G) > c_H$ then $H \leqslant_c G$.*

Proof. Let H be a graph in $\mathcal{C}$. For the forward implication we note that by definition H is a contraction of a triangulated planar graph. Hence, there exists a triangulated planar graph H_T such that $H \leqslant_c H_T$. For a graph G, we have that $H_T \leqslant_c G \Leftrightarrow H_T^* \leqslant_{tm} G^*$ by Lemma 4.3.

The graph H_T^* is cubic, and as the dual of a triangulation it has a unique embedding into the sphere. The minor and topological minor relations are equivalent for graphs of maximum degree 3 (see Proposition 1.7.2, p. 20, [6]), therefore $H_T \leqslant_c G \Leftrightarrow H_T^* \leqslant_m G^*$.

However, every planar graph is a minor of some grid by Lemma 1.5. (As explained in [6], "To see this take a drawing of the graph, fatten its vertices and edges, and superimpose a sufficiently fine plane grid.") Let $m_{H_T^*}$ be the minimum size of a grid that contains H_T^* as a minor and $c_H > 6m_{H_T^*} - 4$. Then, for every graph G of treewidth at least c_H, the treewidth of its dual is at least $c_H - 1 > 6m_{H_T^*} - 5$ by Lemmas 1.4 and 1.6. Therefore, G^* contains H_T^* as a minor and, as explained before, G contains H as a contraction.

For the backward implication, we need to prove that if there exists a constant c_H such that every planar graph G with $\mathbf{tw}(G) > c_H$ contains H as a contraction, for some graph H, then $H \in \mathcal{C}$. To see this, choose G to be a triangulated grid with $\mathbf{tw}(G) > c_H$. Then, G contains H as a contraction, so $H \in \mathcal{C}$. This completes the proof of Theorem 4.4. □

Theorem 4.5. *For every fixed graph $H \in \mathcal{C}$, there exists a linear-time algorithm deciding whether a given planar graph can be contracted to H.*

Proof. Let $H \in \mathcal{C}$. This means that H is a contraction of a graph in $\mathcal{T}$. The latter graph is planar by definition. Then H is planar as well, because planar graphs are closed under contractions. Let c_H be the constant from Theorem 4.4. Let G be our planar input graph. First, we test if $\mathbf{tw}(G) > c_H$ using the linear-time algorithm from Lemma 1.7. If so then G contains H as a contraction by Theorem 4.4. Otherwise, we use Lemma 1.3. We point out that testing whether a graph can be contracted to a fixed pattern is expressible in MSOL. We explain this below.

Let G and H be two graphs. Then G contains H as a contraction if and only if there is a partition of $V(G)$ into $|V(H)|$ sets $W(h)$, called witness sets, such that each $W(h)$ induces a connected subgraph of G and for every two $h_i, h_j \in V(H)$, witness sets $W(h_i)$ and $W(h_j)$ are adjacent in G if and only if h_i and h_j are adjacent in H. Here, two subsets A, B of $V(G)$ are called adjacent if there is an edge $ab \in E(G)$ with $a \in A$ and $b \in B$. By contracting all the edges in each of the witness sets, we obtain the graph H. Due to this alternative definition, we only need to express in MSOL the notions of adjacency between sets, nonadjacency between sets and connectivity. This can be done by standard techniques as explained in the survey paper of Grohe [9]. This completes the proof of Theorem 4.5. □

5. Characterizations of $\mathcal{C}$

In this section, we give two equivalent characterizations of $\mathcal{C}$. First, we show that a graph belongs to $\mathcal{C}$ if and only if it is a contraction of a triangulated grid. Then, we provide a characterization of $\mathcal{C}$ in terms of minimal forbidden contractions.

5.1. Triangulated grids

Lemma 5.1. *$G \in \mathcal{C}$ if and only if G is a contraction of a triangulated grid.*

Proof. For the forward implication, let us recall that by Lemma 1.5, for every planar graph H, there exists a constant m_H such that H is a minor of the $m_H \times m_H$ grid. One can easily see that a grid is a minor of some wall. Hence, for every planar graph H, there exists a constant m'_H such that H is a minor of the $\Gamma^*_{m'_H}$ wall. In particular, every cubic planar graph is a minor of a big enough wall. Since the minor and topological minor relations are equivalent for graphs of maximum degree 3 (see Proposition 1.7.2, p. 20, [6]), every cubic planar graph is a topological minor of a big enough wall.

If $G \in \mathcal{C}$, then there exists a triangulated planar graph H_T such that $H \leqslant_c H_T$. Let G^* be a wall big enough that it contains H_T^* as a topological minor (H_T^* is cubic). From Lemma 4.3, $H_T \leqslant_c G \Leftrightarrow H_T^* \leqslant_{tm} G^*$. Therefore, H_T is a contraction of a triangulated grid.

For the backward implication, if G is a contraction of a triangulated graph, then $G \in \mathcal{C}$ by the definition of $\mathcal{C}$. This completes the proof of Lemma 5.1. □

5.2. Minimal forbidden contraction

In this subsection, we characterize our class $\mathcal{C}$ in terms of minimal forbidden contractions. Such a characterization already exists for the class of planar graphs as shown by Demaine, Hajiaghayi, and Kawarabayashi [5]. Before we present their characterization (we need it for ours), we first define some graph terminology. Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ denote the partition classes of $K_{3,3}$. Adding edge $\{a_1, a_2\}$ leads to the graph $K^1_{3,3}$, adding edges $\{a_1, a_2\}, \{a_2, a_3\}$ to the graph $K^2_{3,3}$ and adding edges $\{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}$ to the graph $K^3_{3,3}$. The 5-vertex *wheel* W_4 is obtained from adding an extra vertex adjacent to every vertex of a C_4.

We are now ready to present the two characterizations; also see Figs. 2 and 3.

Theorem 5.2. *(See Corollary 29 in [5].) A connected graph is planar if and only if it does not contain any graph from $\{K_{3,3}, K^1_{3,3}, K^2_{3,3}, K^3_{3,3}, K_5\}$ as a contraction.*

Theorem 5.3. *A graph belongs to $\mathcal{C}$ if and only if it does not contain any graph from $\{K_{2,r} \mid r \geqslant 2\} \cup \{K^3_{3,3}, K_5, W_4\}$ as a contraction.*

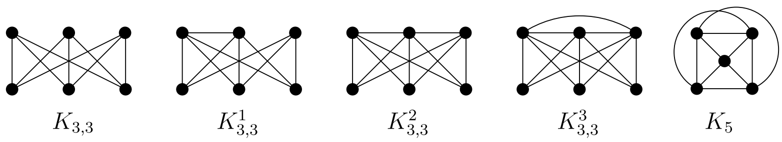

Fig. 2. The minimal forbidden contractions in Theorem 5.2.

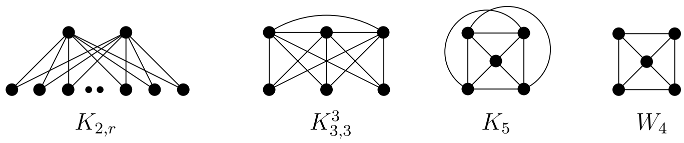

Fig. 3. The minimal forbidden contractions in Theorem 5.3.

Proof. We first prove the forward implication. Suppose H is a graph in $\mathcal{C}$. Because $\mathcal{C}$ is contraction-closed and every graph in $\{K_{2,r} \mid r \geqslant 2\} \cup \{K_{3,3}^3, K_5, W_4\}$ does not belong to $\mathcal{C}$, we find that H cannot be contracted to such a graph.

We now prove the backward implication. Let H be a graph. Suppose H does not contain any graph from $\{K_{2,r} \mid r \geqslant 2\} \cup \{K_{3,3}^3, K_5, W_4\}$ as a contraction. We must show that $H \in \mathcal{C}$. In order to derive a contradiction, suppose $H \notin \mathcal{C}$. We may without loss of generality assume that H is *minimal*, i.e., contracting an arbitrary edge in H results in a graph $H' \in \mathcal{C}$.

Claim 1. *H is planar.*

We prove Claim 1 as follows. Suppose H is not planar. Then H can be contracted to a graph $F \in \{K_{3,3}, K_{3,3}^1, K_{3,3}^2, K_{3,3}^3, K_5\}$ due to Theorem 5.2. By our assumptions, $F \notin \{K_{3,3}^3, K_5\}$. Hence, $F \in \{K_{3,3}, K_{3,3}^1, K_{3,3}^2\}$. However, in all these three cases, F contains W_4 as a contraction by contracting the edge $\{a_2, b_2\}$. This is not possible. Hence, we have proven Claim 1.

Claim 2. *H is 3-connected.*

We prove Claim 2 as follows. Suppose H is not 3-connected. Then H either contains a cut-vertex x or a cut-set $\{x, y\}$ of size two.

Consider the first case. Because $H \notin \mathcal{C}$, there exists a component D of $H - x$ such that $D + x \notin \mathcal{C}$; otherwise $H \in \mathcal{C}$ because of Lemma 4.2. We observe that $D + x$ contains no graph from $\{K_{2,r} \mid r \geqslant 2\} \cup \{K_{3,3}^3, K_5, W_4\}$ as a contraction, because otherwise H would contain such a graph as contraction as well, and this is not possible. Hence, we could take $D + x$ instead of H. This means that H is not minimal, a contradiction.

Now consider the second case. We find that H is not minimal either by the same kind of arguments as in the first case, unless x and y are not adjacent. However, in that case contracting the connected components of $H - \{x, y\}$ to a single vertex yields the graph $K_{2,r}$, where r denotes the number of such components. This is not possible, and Claim 2 has been proven.

Claim 3. *H allows an embedding with an outer face of exactly 4 vertices.*

We prove Claim 3 as follows. Because H is 3-connected (by Claim 2) and $H \notin \mathcal{C}$, we find that H is not triangulated. This means that H contains a face of at least 4 vertices. Without loss of generality we may assume that this face is the outer face. If it contains more than 4 vertices, we can contract one of its edges and obtain a graph H' that contradicts the minimality of H. This proves Claim 3.

Now let C be the outer face of H. By Claim 3, we may assume that C consists of exactly 4 vertices $z_1, \ldots, z_4$. Because H is planar by Claim 1, $H - C$ contains at most one connected component adjacent to all four vertices of C. If this is the case we can contract this component to one single vertex c. We contract the other connected components of $H - C$ to single vertices as well. This leads to a graph in which the vertices $c, z_1, \ldots, z_4$ induce a W_4. We get rid of any remaining vertex c' as follows. If c' is adjacent to c then we contract the edge $\{c, c'\}$. Otherwise, c' is adjacent to a vertex z_i or two vertices z_i, z_j of C, and in the latter case z_i and z_j are adjacent. We contract the edge $\{c', z_i\}$. In this way we find that H can be contracted to W_4. This is not possible.

If $H - C$ contains no component adjacent to all four vertices of C, then H contains C_4 as a contraction by similar arguments as used above. This completes the proof of Theorem 5.3. □

We observe that none of the graphs $\{K_{2,r} \mid r \geqslant 2\} \cup \{K_{3,3}^3, K_5, W_4\}$ is a contraction of another. Thus, Theorem 5.3 characterizes $\mathcal{C}$ in terms of minimal forbidden contractions.

6. Conclusions

This paper can be read as an introductory study of the class $\mathcal{C}$. We define the class, discover some of its properties and provide an algorithmic application. The graphs that belong to $\mathcal{C}$ are exactly those that are contractions of triangulated graphs, or equivalently, contractions of triangulated grids. Note that membership in $\mathcal{C}$ can be tested in polynomial time. The input graph should be decomposed along K_1- and K_2-cuts and each component should be tested for being a planar triangulated graph.

We believe that $\mathcal{C}$ is an interesting class of graphs in its own right for the following reasons. First, Lemma 5.1 tells us that a graph belongs to $\mathcal{C}$ if and only if it is a contraction of a triangulated grid. This could be viewed as an analog to the well-known statement that a graph is planar if and only if it is a minor of a grid. Second, Theorem 4.4 tells us that a graph $H \in \mathcal{C}$ if and only if there exists a constant c_H such that if the treewidth of a graph is at least c_H, it contains H as a contraction. Third, the well-known theorem by Kruskal states that the set of trees over a well-quasi-ordered set of labels is itself well-quasi-ordered [12]. The set of triangulated planar graphs is known to be well-quasi-ordered [5]. Every graph in $\mathcal{C}$ can be seen as a tree (a decomposition tree with respect to K_1- and K_2-cuts) whose vertices are labeled by triangulated graphs. Hence, the graphs in $\mathcal{C}$ are well-quasi-ordered with respect to contractions.

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