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CENTERLESS GROUPS—AN ALGEBRAIC FORMULATION OF GOTTLIEB'S THEOREM

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INTRODUCTION

D. GOTTLIEB [1] has proved that the center of the fundamental group of a finite, aspherical polyhedron, whose Euler characteristic is non-zero, is trivial. His proof uses an idea of Nielsen [2] and Wecken [3], who classify the fixed points of a mapping $K \to K$ according to certain homotopy properties.

This proof of Gottlieb's can be transformed into algebra, and that is what will be done here. We discuss the Lefschetz fixed point theorem in an algebraic context; we define "trace" in such a way that homotopic endomorphisms of a chain complex have the same "Lefschetz number". We are able to define "rank" of a finitely generated projective module and the "Euler characteristic" of a complex; and then we prove Gottlieb's theorem.

An algebraic interpretation of the Nielsen–Wecken fixed-point theory is described in [3], part II. Gottlieb's theorem involves only the simple case of maps $K \to K$ homotopic to the identity; the technique seems to be previously unused in group theory.

§1. THE TRACE

Let Λ be a ring. A *trace function* is a function T, defined on square matrices over Λ, with values in an abelian group, such that:

(a) (Additivity) If M and N are square matrices of the same size, then $T(M + N) = T(M) + T(N)$.

(b) (Commutativity) If M is a $p \times q$ matrix and N a $q \times p$ matrix, then $T(MN) = T(NM)$.

If $\lambda \in \Lambda$, then $T(\lambda)$ will mean $T(M)$ where M is the 1×1 matrix (λ). 0 will mean any matrix of zeros; I will mean a square matrix whose diagonal entries are 1 and whose non-diagonal entries are 0. The symbol M_n will imply that M is an $n \times n$ matrix.

1.1. *If* $M = \begin{pmatrix} 0_p & X \\ 0 & 0_q \end{pmatrix}$, *then* $T(M) = 0$.

For, let $J = \begin{pmatrix} I_p & 0 \\ 0 & 0_q \end{pmatrix}$. Then $M = JM - MJ$, and hence, using (a) and (b), $T(M) = T(JM) - T(MJ) = 0$.

Similarly:

1.2. *If* $M = \begin{pmatrix} 0_p & 0 \\ X & 0_q \end{pmatrix}$, *then* $T(M) = 0$.

1.3. *If* $M = \begin{pmatrix} A_p & X \\ Y & B_q \end{pmatrix}$, *then* $T(M) = T(A) + T(B)$.

For, by 1.2 and additivity, $T(M) = T\begin{pmatrix} A_p & 0 \\ 0 & 0_q \end{pmatrix} + T\begin{pmatrix} 0_p & 0 \\ 0 & B_q \end{pmatrix}$.

Now, $\begin{pmatrix} A_p & 0 \\ 0 & 0_q \end{pmatrix} = \begin{pmatrix} I_p \\ 0 \end{pmatrix}(A_p 0)$ and $A_p = (A_p 0)\begin{pmatrix} I_p \\ 0 \end{pmatrix}$. Hence, by commutativity, $T\begin{pmatrix} A_p & 0 \\ 0 & 0_q \end{pmatrix} = T(A)$. Similarly, $T\begin{pmatrix} 0_p & 0 \\ 0 & B_q \end{pmatrix} = T(B)$.

1.4. *If* $M_n = (m_{ij})$, *then* $T(M) = \sum_{i=1}^{n} T(m_{ii})$.

This clearly follows inductively from 1.3.

1.5. Suppose T is a function defined only on elements of Λ, such that $T(\lambda_1 + \lambda_2) = T(\lambda_1) + T(\lambda_2)$ and $T(\lambda_1\lambda_2) = T(\lambda_2\lambda_1)$. Then we can define $T(M)$ to be $\Sigma T(m_{ii})$; this function is a trace function.

Clearly it is additive. To show commutativity, let $M = (m_{ij})$ be $p \times q$ and $N = (n_{jk})$ be $q \times p$. Then:

$$\begin{aligned} T(MN) &= \sum_i T\left(\sum_j m_{ij}n_{ji}\right) \\ &= \sum_i \sum_j T(m_{ij}n_{ji}) \\ &= \sum_j \sum_i T(n_{ji}m_{ij}) \\ &= \sum_j T\left(\sum_i n_{ji}m_{ij}\right) \\ &= T(NM) \end{aligned}$$

Therefore, we can define a *universal trace function*, through which every trace function can be factored. We define $\mathscr{T}$ to be the additive group of Λ modulo the subgroup generated by elements $(\lambda_1\lambda_2 - \lambda_2\lambda_1)$. The function $T\colon \Lambda \to \mathscr{T}$ is the obvious one, and is extended to square matrices by the above method.

1.6. *The trace of an endomorphism.* Let P, Q, R be free left Λ-modules with bases $\{x_1, \ldots, x_p\}$, $\{y_1, \ldots, y_q\}$, $\{z_1, \ldots, zr\}$ and let $\phi\colon P \to Q$, $\psi\colon Q \to R$ be homomorphisms. To ϕ we assign a $p \times q$ matrix $A = (a_{ij})$ such that $\phi(x_i) = \sum_k a_{ij}y_j$. To ψ the $q \times r$ matrix $B = (b_{jk})$, such that $\psi(y_j) = \sum_k b_{jk}z_k$. Then to $\psi\phi\colon P \to R$ we assign the matrix $C = \{c_{ik}\}$ where $\psi\phi(x_i) = \sum_k c_{ik}z_k$, and the obvious computation shows that $C = A \cdot B$.

Now consider an endomorphism $\phi\colon P \to P$. With one basis for P, to ϕ corresponds one square matrix M; with respect to another basis, to ϕ will correspond a matrix N; and there will be matrices A, B such that AB and BA are both I and $N = AMB$. Now $T(N) = T(AM \cdot B) = T(B \cdot AM) = T(M)$, if T is a trace function. Hence $T(\phi)$ can be defined unambiguously as $T(M)$, where M is any matrix corresponding to ϕ and a basis of P.

And if $\phi: P \to Q$, $\psi: Q \to P$, it follows that $T(\phi\psi) = T(\psi\phi)$.

1.7. *Trace for projective modules.* Let P be a finitely generated projective module over Λ. Then there is a module Q such that the direct sum $P \oplus Q$ is a finitely generated free module. Let $\phi: P \to P$ and define $\phi + 0: P \oplus Q \to P \oplus Q$ to be ϕ on P and zero on Q; $\phi + 0$, being an endomorphism of a free module, has a well-defined trace, by 1.6, and we define $T(\phi) = T(\phi + 0)$.

We must show this is independent of choice of Q. Suppose $P \oplus Q = F$ is free and $P \oplus Q' = F'$ is also free. Then $Q \oplus F'$ and $Q' \oplus F$ are isomorphic, both being isomorphic to $Q \oplus P \oplus Q'$. Let $\alpha: P \oplus Q \oplus F' \to P \oplus Q' \oplus F$ be the isomorphism which is the identity on P and some isomorphism $Q \oplus F' \to Q' \oplus F$. Then, by 1.3, $T(\phi + 0_Q) = T(\phi + 0_Q + 0_{F'})$ and $T(\phi + 0_{Q'}) = T(\phi + 0_{Q'} + 0_F)$; and since $\phi + 0_Q + 0_{F'} = \alpha^{-1}(\phi + 0_{Q'} + 0_F)\alpha$, by commutativity, $T(\phi + 0_Q + 0_{F'}) = T(\phi + 0_{Q'} + 0_F)$.

Thus the trace of an endomorphism of a finitely generated projective module is well-defined.

1.8. *Rank and Euler characteristic.* With respect to a given trace function T, the rank of a finitely generated projective P is defined to be $\mathrm{rank}_T(P) = T$ (identity map $P \to P$). In other words, if $\phi: F \to F$ is an idempotent endomorphism of a free module, whose image $\phi(F)$ is isomorphic to P, then $\mathrm{rank}_T(P) = T(\phi)$. By 1.7, this is well-defined.

Let $\mathscr{C} = \{C_n \to \dots \to C_0\}$ be a finite projective Λ-complex; that is, a finite-dimensional chain complex of finitely generated projective left Λ-modules. Then the *Euler characteristic* of $\mathscr{C}$ is defined to be

$$\chi_T(\mathscr{C}) = \Sigma(-1)^i \,\mathrm{rank}_T(C_i).$$

§2. THE LEFSCHETZ THEOREM

2.1. Let $\phi: \mathscr{C} \to \mathscr{C}$ be an endomorphism of a finite projective Λ-complex; that is, a set of maps $\phi_i: \mathscr{C}_i \to \mathscr{C}_i$, of left Λ-modules, which commute with the boundary homomorphisms.

The *Lefschetz number* of ϕ, relative to a trace function T, is defined to be:

$$L_T(\phi) = \Sigma\,(-1)_i T(\phi_i).$$

In particular, the Euler characteristic of $\mathscr{C}$ is the Lefschetz number of the identity map.

2.2. THEOREM. *Let f and g be chain-homotopic endomorphisms of $\mathscr{C}$. Then $L_T(f) = L_T(g)$.*

Proof. Denote the boundary maps of $\mathscr{C}$ by $\partial_i: C_i \to C_{i-1}$, and a chain homotopy between f and g by

$$d_i: C_i \to C_{i+1}.$$

Then $f_i - g_i = \partial_{i+1} d_i + d_{i-1}\partial_i$.

So, by additivity of T,

$$T(f_i) - T(g_i) = T(\partial_{i+1}d_i) + T(d_{i-1}\partial_i).$$

Hence

$$\begin{aligned} L_T(f) - L_T(g) &= \Sigma\,(-1)^i[T(\partial_{i+1}d_i) + T(d_{i-1}\partial_i)] \\ &= \Sigma\,(-1)^i[T(\partial_{i+1}d_i) - T(d_i\partial_{i+1})] \\ &= 0, \text{ by commutativity of } T. \end{aligned}$$

2.3. COROLLARY. *If the finite projective complexes $\mathscr{C}$ and $\mathscr{D}$ have the same homotopy type, then* $\chi_T(\mathscr{C}) = \chi_T(\mathscr{D})$.

For, let $f: \mathscr{C} \to \mathscr{D}$ be a homotopy equivalence, and let $\mathscr{M}$ be the mapping cone of f, so that $M_i = C_{i-1} \oplus D_i$ and $\partial(c, d) = (-\partial c, f(c) + \partial d)$. Clearly $\chi_T(\mathscr{M}) = \chi_T(\mathscr{D}) - \chi_T(\mathscr{C})$ and since f is a homotopy equivalence, the identity map on $\mathscr{M}$ is homotopic to the zero map. By the theorem, $\chi_T(\mathscr{M}) = L_T$ (identity map) $= L_T$ (zero) $= 0$.

§3. GROUP RINGS

Let R be a commutative ring and Π a multiplicative group. The ring $\Lambda = R(\Pi)$ is defined, as usual, to be the free R-module with basis Π, and multiplication so that $(r_1\pi_1)\cdot(r_2\pi_2) = (r_1r_2)\cdot(\pi_1\pi_2)$.

Consider the universal trace function for Λ, as discussed in 1.5. We construct $\mathscr{T}$ as the additive group of Λ modulo the subgroup generated by $(\lambda_1\lambda_2 - \lambda_2\lambda_1)$ This is $R(\Pi)$ modulo the subgroup generated by elements of the form $r(\pi_1\pi_2 - \pi_2\pi_1)$.

Now, the relation on Π, that $x \sim y$ if and only if there are π_1, π_2 such that $x = \pi_1\pi_2$ and $y = \pi_2\pi_1$, is an equivalence relation, known as *conjugacy*. Therefore, $\mathscr{T}$ is just the free R-module with basis the set of conjugacy classes of Π. The trace function T is the extension to square matrices by the formula of 1.4 of the function

$$T\left(\sum_\pi r(\pi)\cdot\pi\right) = \sum_C \left(\sum_{\pi\in C} r(\pi)\right)C,$$

where C denotes a conjugacy class.

Certain conjugacy classes consist of only one element; one such is the class [1] consisting of 1 alone.

3.1. Suppose F is a free $R(\Pi)$-module with a basis of n elements. Then rank $(F) = n\cdot[1]$. For, the identity map $F \to F$ defines the matrix I_n, whose trace according to T is n times the conjugacy class [1]. But if P is merely projective, it may happen that its rank will involve other conjugacy classes.

3.2. It may happen that there is a finite projective $R(\Pi)$-resolution of R: An exact sequence

$$0 \to C_n \to \ldots \to C_0 \to R \to 0,$$

where the C_i are finitely generated projective modules.

It is well-known from homological algebra that any two such resolutions have the same homotopy type, so that the homotopy type of the complex $\mathscr{C} = (C_n \to \ldots \to C_0)$ is well determined. And so by 2.3, its Euler characterisitc $\chi(\mathscr{C})$ is unique; we denote this element of $\mathscr{T}$ by $\chi(\Pi; R)$.

If there is a finite resolution by free modules, this Euler characteristic is some integral multiple of [1]; otherwise it is just some crazy element of $\mathscr{T}$.

3.3. *$\mathscr{T}$ is a module over the center of Π.* For if x and y belong to the same conjugacy class, so that $x = aya^{-1}$, and γ belongs to the center of Π, then $\gamma x = \gamma aya^{-1} = a\gamma ya^{-1}$; so that γx and γy belong to the same conjugacy class. Thus γC, where C is a conjugacy class, is a conjugacy class.

Let A be a square matrix whose diagonal entries are γ and whose non-diagonal entries are 0. Then $T(AB) = \gamma \cdot T(B)$.

3.4. THEOREM. *If there is a finite projective $R(\Pi)$-resolution of R, and γ is an element of the center of Π, then $\gamma \cdot \chi(\Pi; R) = \chi(\Pi; R)$.*

Proof. Let $\mathscr{C}$ be such a resolution. Define $\gamma_\# : \mathscr{C} \to \mathscr{C}$ by $\gamma_\#(c) = \gamma c$. Then $\gamma_\#$ is a map of left $R(\Pi)$-modules since $\gamma_\#(\lambda c) = \gamma\lambda c = \lambda\gamma c = \lambda\gamma_\#(c)$; and $\gamma_\#$ is a chain mapping. It extends the identity map on R, since elements of Π act trivially on R; it is therefore, by a well-known theorem of homological algebra, homotopic to the identity map, $1_\#$. And so, by 2.2, $L(\gamma_\#) = L(1_\#) = \chi(\Pi; R)$. The trace $T(\gamma_\# : C_i \to C_i)$ is the trace of the matrix AB, where A is the matrix of γ's on the diagonal and 0's elsewhere, and B is the matrix of a retraction of a free module onto C_i. Thus $T(AB) = \gamma \cdot T(B)$, by 3.3; and $T(B) = \text{rank } C_i$.

Hence $L(\gamma_\#) = \gamma \cdot L(1_\#)$. Q.E.D.

3.5. COROLLARY. *If the coefficient of* [1] *in $\chi(\Pi; R)$ is non-zero, then the center of Π is finite.*

For, each element γ of the center of Π makes up its own conjugacy class $[\gamma]$ and $\gamma \cdot [1] = [\gamma]$. Thus, by 3.4, the coefficient of $[\gamma]$ in $\chi(\Pi; R)$ is the same as that of [1]. Since each element of $\mathscr{T}$ is a finite linear combination of conjugacy classes, either all these equal coefficients are zero, or else the center is finite.

3.6. COROLLARY. *If $\chi(\Pi; R)$ is a non-zero multiple of* [1], *then the center of Π consists of the single element* 1.

For, by 3.4, if $\chi(\Pi; R) = r \cdot [1]$, and if γ belongs to the center of Π, then $\gamma \cdot \chi(\Pi; R) = r \cdot [\gamma] = \chi(\Pi; R) = r \cdot [1]$. So, if $r \neq 0$, then $\gamma = 1$.

This corollary contains Gottlieb's theorem, for if Π is the fundamental group of an aspherical polyhedron K, then by means of K we obtain a finite *free* resolution of $Z(\Pi)$ over Z, where Z is the ring of integers; and $\chi(\Pi; Z) = \chi(K) \cdot [1]$, where $\chi(K)$ is the ordinary Euler characteristic of K. And so, if $\chi(K) \neq 0$, the center of Π is trivial.

§4. REMARKS

4.1. In the first place, there seeems to be no known example of a group Π for which there is a finite projective resolution of Z over $Z(\Pi)$ such that $\chi(\Pi; Z)$ is not a multiple of [1]. Such a group would be very interesting.

4.2. If Π is a finite group of order n and R is a ring in which n is invertible, then R itself is a projective $R(\Pi)$ module, since R is the image of an idempotent endomorphism of $R(\Pi)$—namely multiplication by $\frac{1}{n}(\sum)$, where $\sum$ is the sum of the elements of Π. It follows that the rank of $R = \chi(\Pi; R)$ is the image of $\mathscr{T}$ of $\frac{1}{n}\sum$; in particular the coefficient of [1] in $\chi(\Pi; R)$ is $\frac{1}{n}$.

We can therefore conclude from 3.5 that the center of a finite group is finite!

4.3. The idea which seems most obvious for proving Gottlieb's theorem does not work. Out of the assumption that the center of the fundamental group G of a finite aspherical polyhedron is non-trivial, we find an infinite cyclic central subgroup Z of G. From a spectral sequence argument, using the Euler characteristic defined as the alternating sum of the ranks of the homology groups, we have

$$\chi(G) = \chi(Z) \cdot \chi(G/Z).$$

Since $\chi(Z) = 0$, therefore $\chi(G) = 0$.

The flaw in this argument is that perhaps $\chi(G/Z) = \infty$. The spectral sequence argument cannot rule out the possibility that G has the homology of a point, so that $\chi(G) = 1$; all that it would prove is that then G/Z has the homology of infinite-dimensional complex projective space.

It would be interesting to know whether such an argument can be somehow corrected. Then Gottlieb's theorem might be proved along more classical homological lines.

REFERENCES

1. D. H. Gottlieb, A certain subgroup of the fundamental group, *Amer. J. Math.*, to appear.
2. J. Nielsen, Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen I, *Acta Math.* **50** (1927), 189–358.
3. F. Wecken, Fixpunktklassen I, II, III, *Math. Ann.* **117–118** (1940–42).

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