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# Schröder quasigroups with a specified number of idempotents

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ABSTRACT

Schröder quasigroups have been studied quite extensively over the years. Most of the attention has been given to idempotent models, which exist for all the feasible orders $v$, where $v \equiv 0, 1 \pmod 4$ except for $v = 5, 9$. There is no Schröder quasigroup of order 5 and the known Schröder quasigroup of order 9 contains 6 non-idempotent elements. It is known that the number of non-idempotent elements in a Schröder quasigroup must be even and at least four. In this paper, we investigate the existence of Schröder quasigroups of order $v$ with a specified number $k$ of idempotent elements, briefly denoted by $SQ(v, k)$. The necessary conditions for the existence of $SQ(v, k)$ are $v \equiv 0, 1 \pmod 4$, $0 \le k \le v$, $k \ne v - 2$, and $v - k$ is even. We show that these conditions are also sufficient for all the feasible values of $v$ and $k$ with few definite exceptions and a handful of possible exceptions. Our investigation relies on the construction of holey Schröder designs (HSDs) of certain types. Specifically, we have established that there exists an HSD of type $4^n u^1$ for $u = 1, 9$, and 12 and $n \ge max\{(u + 2)/2, 4\}$. In the process, we are able to provide constructions for a very large variety of non-idempotent Schröder quasigroups of order $v$, all of which correspond to $v^2 \times 4$ orthogonal arrays that have the Klein 4-group as conjugate invariant subgroup.



## 1. Introduction

A *quasigroup* is an ordered pair $(Q, *)$, where $Q$ is a set and $(*)$ is a binary operation on $Q$ such that the equations

$$a * x = b \quad \text{and} \quad y * a = b \tag{1}$$

are uniquely solvable for every pair of elements $a, b \in Q$. A quasigroup is called *idempotent* if the identity

$$x * x = x \tag{2}$$

is satisfied for all $x \in Q$. An element $e \in Q$ is called *idempotent* if $e * e = e$; otherwise *non-idempotent*. If the identity

$$(x * y) * (y * x) = x \tag{3}$$

holds for all $x, y \in Q$, then it is called a *Schröder quasigroup*. The *order* of the quasigroup is $|Q|$.

Idempotent Schröder quasigroups, or ISQs, are associated with other combinatorial configurations such as a class of edge-colored block designs with block size 4, triple tournaments and self-orthogonal Latin squares with Weisner property (see [7,2,10,11]). A pair of Latin squares, say $(Q, *)$ and $(Q, \cdot)$, are said to have the *Weisner property* if $x * y = z$ and $x \cdot y = w$ whenever $z * w = x$ and $z \cdot w = y$ for all $x, y, z, w \in Q$. If $(Q, \cdot)$ is the transpose of $(Q, *)$, then $z \cdot w = w * z$. If $(Q, *)$ is an ISQ, then from $z * w = x$ and $z \cdot w = y$, we have $x * y = (z * w) * (w * z) = z$. Similarly, we also have $x \cdot y = w$ [11]. The following theorem states the known results relating to the existence of Schröder quasigroups and in particular ISQs.

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**Theorem 1.1** (*[10,5,7]*).

(a) *A Schröder quasigroup of order* $v$ *exists if and only if* $v \equiv 0, 1 \pmod 4$ *and* $v \neq 5$.
(b) *An idempotent Schröder quasigroup of order* $v$ *exists if and only if* $v \equiv 0, 1 \pmod 4$ *and* $v \neq 5, 9$.

Let $Q$ be a set and $\mathcal{H} = \{S_1, S_2, \ldots, S_k\}$ be a set of subsets of $Q$. A *holey idempotent Schröder quasigroup* having hole set $\mathcal{H}$ is a triple $(Q, \mathcal{H}, *)$, which satisfies the following properties:

1. $(*)$ is a binary operation defined on $Q$, however, when both points $a$ and $b$ belong to the same set $S_i$, there is no definition for $a * b$,
2. the Eqs. (1) hold when $a, b$ are not contained in the same set $S_i$, $1 \le i \le k$,
3. the identity (2) holds for any $x \notin \cup_{1 \le i \le k} S_i$,
4. the identity (3) holds when $x$ and $y$ are not contained in the same set $S_i$, $1 \le i \le k$.

We denote the holey ISQ by HISQ$(v; s_1, s_2, \ldots, s_k)$, where $v = |Q|$ is the *order* and $s_i = |S_i|$, $1 \le i \le k$. Each $S_i$ is called a *hole*. When $\mathcal{H} = \emptyset$, we obtain an ISQ, and denote it by ISQ$(v)$. When $\mathcal{H} = \{S_1\}$, we obtain an *incomplete ISQ*, and denote it by IISQ$(v, |S_1|)$. The definition of holey ISQ can be extended to non-idempotent Schröder quasigroups (NISQ). In particular, when an NISQ has only $b$ disjoint holes of the same size $a$, we denote it by HNISQ$(v, a^b)$.

From the definition of HISQ, we can obtain the definition of frame ISQ as follows. If $\mathcal{H} = \{S_1, S_2, \ldots, S_k\}$ is a partition of $Q$, then a holey ISQ is called *frame ISQ*. The *type* of the frame ISQ is defined to be the multiset $\{|S_i| : 1 \le i \le k\}$. We shall use an "exponential" notation $s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$ to describe the type of $n_i$ occurrences of $s_i$, $1 \le i \le t$ in the multiset. We briefly denote a frame ISQ of type $s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$ by FISQ$(s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t})$.

An ISQ$(v)$ is equivalent to an edge-colored design CBD$[G_6; v]$ which is investigated in [7]. An *edge-colored design* CBD$[G_6; v]$ on a $v$-set $Q$ is a partition of the colored edges of a triplicate complete graph $3K_v$, each $K_v$ receives one color for its edges from three different colors, into blocks $\{a, b, c, d\}$ each containing edges $\{a, b\}, \{c, d\}$ colored with color 1, edges $\{a, c\}, \{b, d\}$ with color 2, and edges $\{a, d\}, \{b, c\}$ with color 3. If we define a binary operation $(\cdot)$ as $a \cdot b = c$, $b \cdot a = d$, $c \cdot d = a$ and $d \cdot c = b$ from the block $\{a, b, c, d\}$ and define $x \cdot x = x$ for every $x \in Q$, an ISQ$(v)$ is obtained on set $Q$. On the other hand, suppose $Q$ is an ISQ. If $a \cdot b = c$, $b \cdot a = d$, then we must have $c \cdot d = (a \cdot b) \cdot (b \cdot a) = a$ and $d \cdot c = (b \cdot a) \cdot (a \cdot b) = b$. So the block $\{a, b, c, d\}$ is determined and a CBD$[G_6; v]$ can be obtained in this way.

Now from an FISQ$(s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t})$, we can use the same method to obtain an edge-colored design which is called a holey Schröder design and denoted by HSD$(s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t})$. A *holey Schröder design* is a triple $(X, \mathcal{H}, \mathcal{B})$ which satisfies the following properties:

1. $\mathcal{H}$ is a partition of $X$ into subsets called *holes*,
2. $\mathcal{B}$ is a family of 4-subsets of $X$ (called *blocks*) such that a hole and a block contain at most one common point,
3. the pairs of points in a block $\{a, b, c, d\}$ are colored as $\{a, b\}$ and $\{c, d\}$ with color 1, $\{a, c\}$ and $\{b, d\}$ with color 2, and $\{a, d\}$ and $\{b, c\}$ with color 3,
4. every pair of points from distinct holes occurs in 3 blocks with different colors.

The *type* of the HSD is the multiset $\{|H| : H \in \mathcal{H}\}$ and it is also described by an exponential notation.

An HSD can be viewed as a generalization of CBD$[G_6; v]$. An HSD of type $\{s_1, s_2, \ldots, s_k\}$ is a partition of the colored edges of a triplicate graph $3K_{s_1, s_2, \ldots, s_k}$ into blocks $\{a, b, c, d\}$ each containing edges $\{a, b\}, \{c, d\}$ with color 1, edges $\{a, c\}, \{b, d\}$ with color 2, and edges $\{a, d\}, \{b, c\}$ with color 3, where each $K_{s_1, s_2, \ldots, s_k}$ receives one color for its edges from three different colors.

On the other hand, it is well known that the multiplication table of a quasigroup defines a Latin square. For a Schröder quasigroup, the corresponding Latin square is self-orthogonal with the Weisner property. For the equivalence of the Schröder quasigroup to SOLS with Weisner property, the reader is referred [11]. In this way, an HSD is equivalent to a frame self-orthogonal Latin square (FSOLS) with Weisner property.

For the existence of FSOLS of type $4^n u^1$, [13] gives the following theorem.

**Theorem 1.2.** *There exists an* FSOLS$(4^n u^1)$ *if and only if* $n \ge 4$ *and* $0 \le u \le 2n - 2$.

By a simple calculation and in light of Theorem 1.2, we have the following lemma.

**Lemma 1.3.** *If an* HSD$(4^n u^1)$ *exists, then* $n \ge 4$ *and* $u \le 2n - 2$.

Another class of designs related to HSDs is *group divisible design* (GDD). A GDD is a 4-tuple $(X, \mathcal{G}, \mathcal{B}, \lambda)$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of $X$ into subsets called *groups*,
2. $\mathcal{B}$ is a family of subsets of $X$ (called *blocks*) such that a group and a block contain at most one common point,
3. every pair of points from distinct groups occurs in exactly $\lambda$ blocks.

The *type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We also use the notation GD$(K, M; \lambda)$ to denote the GDD when its block sizes belong to $K$ and group sizes belong to $M$.

If $M = \{1\}$, then the GDD becomes a *PBD*. If $K = \{k\}$, $M = \{n\}$ and with the type $n^k$, then the GDD becomes TD$(k, n)$. It is well known that the existence of a TD$(k, n)$ is equivalent to the existence of $k - 2$ MOLS$(n)$. For more information on GDDs and PBDs, the reader is referred to [9,12]. It is easy to see that if we erase the colors in the blocks, the HSD becomes a GDD with block size 4 and $\lambda = 3$. But the converse may be not true. It is proved in [6] that a $\{4\}$-GDD with $\lambda = 3$ and of type $h^u$ exists if and only if $h^2u(u-1) \equiv 0 \pmod 4$, while in [3,16], the following theorem is proved.

**Theorem 1.4.** *An* HSD$(h^u)$ *exists if and only if* $h^2u(u-1) \equiv 0 \pmod 4$ *with exceptions of* $(h, u) \in \{(1, 5), (1, 9), (2, 4)\}$.

Theorem 1.1 gives the necessary and sufficient conditions for the existence of Schröder quasigroups in general and for the existence of an ISQ. Non-idempotent Schröder quasigroups, or NISQs, are the main subject of our current investigation. There has been no concerted effort made in the past to construct NISQs. An additional basic necessary condition for the existence of NISQs is that the number of non-idempotent elements must be even and at least four. For any $v$ where we have a NISQ of order $v$, briefly NISQ$(v)$, we are interested in the construction of the NISQ$(v)$ with all the feasible number $k$ of idempotent elements. The necessary conditions for $k$ are that $0 \le k \le v$, $k \ne v - 2$, and $v - k$ is even (see, for example, [10]). Given any $v$, whenever $k$ satisfies these conditions, we say $k$ is *feasible*. We show that these conditions for $k$ are also sufficient with few definite exceptions and only a handful of possible exceptions.

From now on, let SQ$(v, n)$ denote a Schröder quasigroup of order $v$ with $n$ idempotent elements. Note that SQ$(v, v)$ is an ISQ$(v)$. The remainder of the paper will be devoted to proving the following theorem:

**Theorem 1.5.** *An* SQ$(v, n)$ *exists if and only if* $v \equiv 0, 1 \pmod 4$, $0 \le n \le v$, $n \ne v - 2$, *and* $v - n$ *is even, except for* $(v, n) \in \{(5, 1), (5, 5), (8, 2), (8, 4), (9, 1), (9, 5)\}$, *and except possibly for* $(v, n)$, *where* $v \in \{20, 21, 24, 25, 28, 29, 32, 36\}$ *and* $n \equiv 2 \pmod 4$, *when* $v$ *is even, and* $n \equiv 3 \pmod 4$, *when* $v$ *is odd.*

## 2. Direct constructions

The results in this section were obtained using computer search. We often list the SQ's multiplication table for its presentation. In some of what follows, we shall tacitly make use of the following construction:

**Construction 2.1.** (a) *If there exist an* IISQ$(v, n)$ *and an* SQ$(n, k)$, *then there exists a* SQ$(v, v - n + k)$.
(b) *If there exist an* HNISQ$(v, a^b)$ *with k idempotent elements and an* SQ$(a, d)$, *then there exists an* SQ$(v, k + bd)$.

**Lemma 2.2.** *There exists an* SQ$(v, n)$ *for* $(v, n) \in \{(4, 0), (4, 4), (8, 0), (8, 8), (9, 3)\}$ *and there are no* SQ$(v, n)$ *for* $(v, n) \in \{(8, 2), (8, 4), (9, 1), (9, 5)\}$.

**Proof.** SQ(4, 0) is given below:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 3 | 1 | 0 | 2 |
| 1 | 0 | 2 | 3 | 1 |
| 2 | 2 | 0 | 1 | 3 |
| 3 | 1 | 3 | 2 | 0 |

For SQ$(v, v)$, $v = 4, 8$, they come from idempotent models.
Here are SQ(8, 0) and SQ(9, 3):

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 7 | 2 | 6 | 1 | 4 | 0 | 5 | 3 |
| 1 | 6 | 5 | 7 | 3 | 0 | 4 | 2 | 1 |
| 2 | 1 | 4 | 3 | 7 | 2 | 5 | 0 | 6 |
| 3 | 4 | 1 | 0 | 2 | 7 | 6 | 3 | 5 |
| 4 | 0 | 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 5 | 5 | 6 | 2 | 0 | 3 | 1 | 7 | 4 |
| 6 | 3 | 0 | 1 | 6 | 5 | 2 | 4 | 7 |
| 7 | 2 | 7 | 5 | 4 | 1 | 3 | 6 | 0 |

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 5 | 8 | 7 | 0 | 2 | 4 | 6 | 1 | 3 |
| 1 | 6 | 1 | 8 | 7 | 0 | 3 | 5 | 2 | 4 |
| 2 | 1 | 5 | 6 | 8 | 4 | 2 | 7 | 3 | 0 |
| 3 | 3 | 2 | 0 | 4 | 8 | 5 | 1 | 6 | 7 |
| 4 | 8 | 7 | 2 | 1 | 3 | 6 | 4 | 0 | 5 |
| 5 | 7 | 6 | 5 | 3 | 1 | 0 | 8 | 4 | 2 |
| 6 | 0 | 4 | 3 | 5 | 6 | 7 | 2 | 8 | 1 |
| 7 | 4 | 0 | 1 | 2 | 5 | 8 | 3 | 7 | 6 |
| 8 | 2 | 3 | 4 | 6 | 7 | 1 | 0 | 5 | 8 |

An exhaustive computer search has confirmed the non-existence in each of the four cases stated. □

**Lemma 2.3.** *There exists an* SQ$(12, k)$ *for any feasible k.*

**Proof.** The case $k = 12$ comes from idempotent models. For the other cases, please see Appendix A.1. □

**Example 2.4.** An IISQ(13, 4):

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | x | y | z | w |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 7 | z | 8 | 3 | w | x | 5 | y | 1 | 2 | 4 | 6 |
| 1 | y | 1 | 8 | z | 0 | 4 | w | x | 6 | 2 | 3 | 5 | 7 |
| 2 | 7 | y | 2 | 0 | z | 1 | 5 | w | x | 3 | 4 | 6 | 8 |
| 3 | x | 8 | y | 3 | 1 | z | 2 | 6 | w | 4 | 5 | 7 | 0 |
| 4 | w | x | 0 | y | 4 | 2 | z | 3 | 7 | 5 | 6 | 8 | 1 |
| 5 | 8 | w | x | 1 | y | 5 | 3 | z | 4 | 6 | 7 | 0 | 2 |
| 6 | 5 | 0 | w | x | 2 | y | 6 | 4 | z | 7 | 8 | 1 | 3 |
| 7 | z | 6 | 1 | w | x | 3 | y | 7 | 5 | 8 | 0 | 2 | 4 |
| 8 | 6 | z | 7 | 2 | w | x | 4 | y | 8 | 0 | 1 | 3 | 5 |
| x | 4 | 5 | 6 | 7 | 8 | 0 | 1 | 2 | 3 | | | | |
| y | 3 | 4 | 5 | 6 | 7 | 8 | 0 | 1 | 2 | | | | |
| z | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 0 | 1 | | | | |
| w | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 0 | | | | |

**Example 2.5.** An SQ(13, 9) can be obtained from an IISQ(13, 4) by filling the hole of size 4 with an SQ(4, 0).

**Lemma 2.6.** *There exists an* SQ(13, $k$) *for any feasible* $k$.

**Proof.** The case $k = 13$ comes from idempotent models. The case $k = 9$ is covered in Example 2.5. For the other cases, please see Appendix A.2. □

**Example 2.7.** An HNISQ(16, $4^2$) with two idempotent elements, where the two holes are $S_1 = \{0, 1, 2, 3\}$ and $S_2 = \{12, 13, 14, 15\}$, and an HNISQ(17, $4^2$) with three idempotent elements, where the two holes are $S_1 = \{0, 2, 4, 6\}$ and $S_2 = \{1, 3, 5, 7\}$, are given below and on the top of the next page.

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | 13 | 11 | 12 | 14 | 15 | 7 | 6 | 4 | 8 | 9 | 10 | 5 |
| 1 | | | | | 15 | 14 | 4 | 13 | 7 | 8 | 9 | 12 | 10 | 5 | 6 | 11 |
| 2 | | | | | 14 | 4 | 15 | 10 | 13 | 12 | 5 | 6 | 7 | 11 | 8 | 9 |
| 3 | | | | | 8 | 12 | 7 | 15 | 5 | 14 | 13 | 9 | 4 | 6 | 11 | 10 |
| 4 | 6 | 9 | 8 | 15 | 4 | 13 | 5 | 12 | 10 | 3 | 11 | 14 | 2 | 0 | 7 | 1 |
| 5 | 15 | 11 | 12 | 10 | 1 | 9 | 14 | 5 | 8 | 0 | 4 | 13 | 6 | 2 | 3 | 7 |
| 6 | 5 | 15 | 7 | 13 | 10 | 1 | 8 | 3 | 14 | 6 | 12 | 11 | 0 | 4 | 9 | 2 |
| 7 | 4 | 10 | 14 | 5 | 2 | 7 | 13 | 11 | 12 | 9 | 8 | 15 | 1 | 3 | 0 | 6 |
| 8 | 10 | 12 | 9 | 14 | 7 | 5 | 2 | 0 | 6 | 13 | 15 | 8 | 11 | 1 | 4 | 3 |
| 9 | 14 | 13 | 11 | 6 | 12 | 15 | 9 | 7 | 0 | 5 | 2 | 10 | 3 | 8 | 1 | 4 |
| 10 | 12 | 14 | 13 | 11 | 9 | 6 | 1 | 4 | 3 | 15 | 10 | 0 | 5 | 7 | 2 | 8 |
| 11 | 13 | 8 | 15 | 12 | 3 | 2 | 6 | 1 | 11 | 4 | 14 | 7 | 9 | 10 | 5 | 0 |
| 12 | 7 | 6 | 4 | 9 | 5 | 0 | 10 | 8 | 1 | 11 | 3 | 2 | | | | |
| 13 | 8 | 4 | 5 | 7 | 11 | 10 | 0 | 6 | 9 | 2 | 1 | 3 | | | | |
| 14 | 11 | 5 | 6 | 4 | 0 | 8 | 3 | 9 | 2 | 10 | 7 | 1 | | | | |
| 15 | 9 | 7 | 10 | 8 | 6 | 3 | 11 | 2 | 4 | 1 | 0 | 5 | | | | |

**Lemma 2.8.** *There exists an* SQ(16, $k$) *for* $k \in \{2, 6, 10\}$.

**Proof.** For the two holes of HNISQ(16, $4^2$) in the previous example, we fill in either an SQ(4, 0) or an SQ(4, 4) to get the desired result. □

**Lemma 2.9.** *There exists an* SQ(17, $k$) *for* $k \in \{3, 7, 11\}$.

**Proof.** For the two holes of HNISQ(17, $4^2$) in the previous example, we fill in either an SQ(4, 0) or an SQ(4, 4) to get the desired result. □

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | 14 | | 10 | | 12 | | 11 | 7 | 16 | 13 | 9 | 3 | 8 | 5 | 1 | 15 |
| 1 | 12 | | 15 | | 9 | | 8 | | 13 | 6 | 2 | 16 | 14 | 4 | 11 | 10 | 0 |
| 2 | | 13 | | 14 | | 11 | | 16 | 3 | 8 | 15 | 12 | 10 | 9 | 7 | 5 | 1 |
| 3 | 15 | | 13 | | 10 | | 12 | | 9 | 14 | 6 | 8 | 4 | 11 | 0 | 16 | 2 |
| 4 | | 11 | | 13 | | 10 | | 15 | 12 | 1 | 16 | 14 | 7 | 5 | 9 | 8 | 3 |
| 5 | 9 | | 14 | | 11 | | 16 | | 15 | 13 | 0 | 6 | 2 | 10 | 8 | 12 | 4 |
| 6 | | 10 | | 11 | | 8 | | 9 | 1 | 3 | 7 | 13 | 15 | 16 | 12 | 14 | 5 |
| 7 | 8 | | 9 | | 14 | | 13 | | 4 | 11 | 12 | 2 | 16 | 0 | 15 | 6 | 10 |
| 8 | 13 | 0 | 16 | 2 | 15 | 4 | 9 | 12 | 8 | 5 | 1 | 7 | 11 | 14 | 10 | 3 | 6 |
| 9 | 11 | 8 | 3 | 4 | 13 | 2 | 10 | 0 | 14 | 12 | 9 | 1 | 5 | 6 | 16 | 15 | 7 |
| 10 | 5 | 16 | 1 | 9 | 7 | 14 | 15 | 2 | 6 | 10 | 11 | 4 | 12 | 3 | 13 | 0 | 8 |
| 11 | 7 | 12 | 8 | 15 | 1 | 16 | 3 | 14 | 0 | 4 | 5 | 10 | 6 | 13 | 2 | 11 | 9 |
| 12 | 14 | 6 | 7 | 16 | 8 | 15 | 5 | 13 | 2 | 0 | 10 | 3 | 9 | 12 | 1 | 4 | 11 |
| 13 | 1 | 9 | 5 | 6 | 16 | 0 | 14 | 8 | 10 | 7 | 4 | 11 | 13 | 15 | 3 | 2 | 12 |
| 14 | 10 | 4 | 11 | 12 | 3 | 9 | 1 | 6 | 16 | 15 | 8 | 5 | 0 | 2 | 14 | 7 | 13 |
| 15 | 16 | 2 | 12 | 0 | 5 | 6 | 7 | 10 | 11 | 9 | 3 | 15 | 8 | 1 | 4 | 13 | 14 |
| 16 | 3 | 15 | 10 | 8 | 12 | 13 | 11 | 4 | 5 | 2 | 14 | 0 | 1 | 7 | 6 | 9 | 16 |

To construct HSDs directly, sometimes we can use *starter blocks*. Suppose the block set $\mathcal{B}$ of an HSD is closed under the action of some Abelian group $G$, then we are able to list only part of the blocks (starter or base blocks) which determines the structure of the HSD. We can also attach some infinite points to an Abelian group $G$. When the group acts on the blocks, the infinite points remain fixed. Formally, let $\mathcal{B}$ be the block set of an HSD over the point set $S = G \cup X$, where $(G, +)$ is a group, $X$ is a set of *infinite points*, $G \cap X = \emptyset$. The addition $(+)$ is extended over $X$ as follows: $g + x = x + g = x$ for any $g \in G$ and $x \in X$. A set $\mathcal{A} \subset \mathcal{B}$ is called *starter blocks* of $\mathcal{B}$ if $\mathcal{A}$ is a minimum subset of $\mathcal{B}$ satisfying the property that for any $\mathbf{a} \in \mathcal{A}$ and any $g \in G$, $\mathbf{a} + g \in \mathcal{B}$, and for any $\mathbf{b} \in \mathcal{B}$, there exist $\mathbf{a} \in \mathcal{A}$ and $g \in G$ such that $\mathbf{b} = \mathbf{a} + g$, where $\mathbf{a} + g = \{a_1 + g, a_2 + g, a_3 + g, a_4 + g\}$ when $\mathbf{a} = \{a_1, a_2, a_3, a_4\}$.

**Example 2.10.** Example 2.4 corresponds to an HSD$(1^9 4^1)$ with the following:
points: $Z_9 \cup \{x, y, z, w\}$, where $x, y, z, w$ are infinite points for the hole.
starter blocks: $\{x, 0, 4, 1\}$, $\{y, 0, 3, 2\}$, $\{z, 0, 2, 4\}$, $\{w, 0, 1, 6\}$.

In this example, the entire set of blocks are developed from the starter blocks by adding 1 (mod 9) to each point of the starter blocks. For instance, the block $\{x, 0, 4, 1\}$ will generate eight more blocks such as $\{x, 1, 5, 2\}$, $\{x, 2, 6, 3\}$, and so on.

To check the starter blocks, we need only calculate whether the differences $\pm(x - y)$ from all pairs $\{x, y\}$ with color $i$ in the starter blocks are precisely $G \setminus S$ for $1 \le i \le 3$, where $S$ is the set of the differences of the holes. For the above example, for color 1, the set of differences from the four blocks is $\{\pm(4-1), \pm(3-2), \pm(4-2), \pm(6-1)\}$, which is exactly $Z_9 - \{0\}$. This is also true for colors 2 and 3.

We have pointed out in the previous section that there is a correspondence between an HISQ and an HSD. That is, for all distinct $a, b, c, d \in Q$, $a * b = c$, $b * a = d$, $c * d = a$, $d * c = b$ in the HISQ if and only if $\{a, b, c, d\}$ is a block of the HSD. So we are free to use either form. In fact, all the designs found by computer in this paper are in the form of Schröder quasigroups. To allow the existence of starter blocks with a group $G$, for quasigroup $(Q, *)$, we require that $Q = G \cup X$ and for all $x, y, z \in Q$, $x * y = z$ if and only if $(x + g) * (y + g) = (z + g)$ for any $g \in G$ [14,15]. Since HSDs have a more compact form than quasigroups, we will present them as HSDs in this paper.

**Example 2.11.** An HSD$(4^4 2^1)$
points: $Z_{16} \cup \{x, y\}$, where $x, y$ are infinite points.
holes: $\{\{i, i+4, i+8, i+12\} : 0 \le i \le 3\} \cup \{x, y\}$
starter blocks: $\{0, 1, 6, 15\}$, $\{0, 3, 9, 14\}$, $\{0, 6, 13, x\}$, $\{0, 14, 1, y\}$.

In this example, the entire set of blocks are developed from the starter blocks by adding 1 (mod 16) to each point of the starter blocks; the infinite points are unchanged for addition. The above idea of starter blocks can be also generalized: Instead of adding 1 to each point of the starter blocks, we may add $k$, where $k > 1$, to develop the block set; we refer to this as the $+k$ method. In this case, for a set $\mathcal{A}$ to be starter blocks, we require that for any $\mathbf{a} \in \mathcal{A}$ and any $g \in G$, $\mathbf{a} + kg \in \mathcal{B}$. For quasigroups, we require that for all $x, y, z \in Q$, $x * y = z$ if and only if $(x + kg) * (y + kg) = (z + kg)$ for any $g \in G$ [14,15].

**Example 2.12.** An HSD$(4^4 1^1)$
points: $Z_{16} \cup \{x\}$
holes: $\{\{i, i+4, i+8, i+12\} : 0 \le i \le 3\} \cup \{x\}$
starter blocks (+2 mod 16): $\{0, 2, 13, 7\}$, $\{0, 3, 2, 9\}$, $\{0, 5, 3, 6\}$, $\{0, 6, 11, 13\}$, $\{0, 9, 10, x\}$, $\{0, 11, 1, 2\}$, $\{1, 0, 3, x\}$.

By adding 2 (mod 16) to the 7 starter blocks, we obtain a set of 56 blocks.

**Lemma 2.13.** *There exists an* HSD$(4^n 1^1)$ *for* $4 \le n \le 13$ *and* $n \not\equiv 0 \pmod 3$.

**Proof.** The case $n = 4$ is given in Example 2.12. The starter blocks of the other cases are given in Appendix A.3. □

**Lemma 2.14.** *There exist HSDs of type $4^n u^1$ for $n = 4, 5$ and $0 \le u \le 6$.*

**Proof.** For $u = 0, 4$ the HSDs come from Theorem 1.4. For $u = 1$, the HSDs come from Lemma 2.13. For all other cases, the HSDs can be found in Appendix A.4. □

Construction 2.1 shows how holey Schröder quasigroups are useful in the study of non-idempotent Schröder quasigroups. In the next section, we present some results related to holey or frame Schröder quasigroups.

## 3. Recursive constructions

In this section we present several recursive constructions of HSDs, which are commonly used in other block designs. The following construction comes from the weighting construction of GDDs [12].

**Construction 3.1** (*Weighting*)**.** *Suppose $(X, \mathcal{H}, \mathcal{B})$ is a GDD with $\lambda = 1$ and let $w : X \mapsto Z^+ \cup \{0\}$. Suppose there exist HSDs of type $\{w(x) : x \in B\}$ for every $B \in \mathcal{B}$. Then there exists an HSD of type $\{\sum_{x\in H} w(x) : H \in \mathcal{H}\}$.*

The following result about TD$(4, m)$ is well known (see [1,4], for example).

**Lemma 3.2.** *There exists a* TD$(4, m)$ *for any positive integer $m$, $m \ne 2, 6$.*

Using Lemma 3.2, if we give every point of an HSD weight $m$ and input TD$(4, m)$ to each block of the HSD, we can obtain the following construction.

**Construction 3.3.** *Suppose there exists an* HSD$(h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k})$*, then there exists an* HSD$((mh_1)^{n_1} (mh_2)^{n_2} \cdots (mh_k)^{n_k})$*, where $m \ne 2, 6$.*

The next construction may be called "filling in holes". It is used commonly in constructing designs.

**Construction 3.4.** *Suppose there exist an HSD of type $\{s_i : 1 \le i \le k\}$ and HSDs of type $\{h_{i_j} : 1 \le j \le n_i\} \cup \{a\}$, where $\sum_{j=1}^{n_i} h_{i_j} = s_i$ and $1 \le i \le k - 1$, then there exists an HSD of type $\{h_{i_j} : 1 \le j \le n_i, 1 \le i \le k - 1\} \cup \{s_k + a\}$.*

**Construction 3.5.** *If there exist an* HSD$(4^m t^1)$ *and an* HSD$(4^s u^1)$*, where $4s + u = t$, then there exists an* HSD$(4^{m+s} u^1)$*.*

The next construction comes from [7].

**Construction 3.6.** *Suppose there exists an* FSOLS$(h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k})$*, then there exists an* HSD$((4h_1)^{n_1} (4h_2)^{n_2} \cdots (4h_k)^{n_k})$*.*

For some of our recursive constructions, we shall utilize the following lemma.

**Lemma 3.7.** *For any $m \ge 4$, there exists an* HSD$(4^{4m}(4u + k)^1)$ *whenever $0 \le u \le 2m - 2$ and $0 \le k \le 6$.*

**Proof.** For all integers $m$ and $u$ satisfying the conditions above, we have an FSOLS$(4^m u^1)$ ([13], Theorem 4.4). From this, we first get an HSD$(16^m (4u)^1)$ by Construction 3.6. To this HSD we can adjoin $k$ infinite points using an HSD$(4^4 k^1)$ from Lemma 2.14 to fill in the holes of size 16 and creating one hole of size $4u + k$ to get the desired HSD. □

We also need to make use of the following result relating to a TD$(6, m)$ (see [1,4], for example).

**Lemma 3.8.** *For $m \ge 5$ and $m \notin \{6, 10, 14, 18, 22\}$, there exists a* TD$(6, m)$*.*

**Lemma 3.9.** *If there exists a* TD$(6, m)$*, then there exist HSDs of type* HSD$(4^{4m+k} u^1)$*, where $k = 0, 1, 4, 5, \dots, m$ and $0 \le u \le 6m$.*

**Proof.** Give weight 4 to each point of first four groups of a TD$(6, m)$. Give weight 4 to $k$ points of the fifth group and weight 0 to the remaining points of this group, and give weight 0, 1, 2, 3, 4, 5 or 6 to the points of the sixth group such that the sum of these weights of the points is equal to $u$. As there exist HSDs of type $4^m$ and $4^k$ and also HSD$(4^n t^1)$ for $n = 4, 5$ and $0 \le t \le 6$ by Lemma 2.14, we obtain the desired HSD by Construction 3.1. □

In most of what follows, we shall rely quite heavily on the following useful construction in going from HSDs of various types to SQ$(v, k)$s:

**Construction 3.10.** (a) *If there exists an* HSD$(4^n)$*, then there exists an* SQ$(4n, k)$ *for any feasible $k \equiv 0 \pmod 4$.*
(b) *If there exists an* HSD$(4^n 12^1)$*, then there exists an* SQ$(4n + 12, k)$ *for any feasible $k$.*
(c) *If there exists an* HSD$(4^n 1^1)$*, then there exists an* SQ$(4n + 1, k)$ *for any feasible $k \equiv 1 \pmod 4$.*
(d) *If there exist an* HSD$(4^{n+2} 1^1)$ *and an* HSD$(4^n 9^1)$*, then there exists an* SQ$(4n + 9, k)$ *for any feasible $k$.*

**Proof.** (a) Let $k = 4t$ where $0 \le t \le n$. We may fill in $t$ holes of size 4 with an SQ$(4, 4)$ and $n - t$ holes of size 4 with an SQ$(4, 0)$ to get $4t$ idempotent elements.
(b) As in the proof of (a), we may get $4t$ idempotent elements, where $0 \le t \le n$, from $n$ holes of size 4. Then we may fill in an SQ$(12, s)$, where $s = 0, 2, 4, 6, 8$, and 12, as needed to have $k$ idempotent elements in SQ$(4n + 12, k)$.
(c) Analogous to that of (b) and we fill one in the hole of size one to get $4t + 1$ idempotent elements.

(d) Similarly, from an HSD($4^{n+2}1^1$), we may get $4t+1$ idempotent elements where $0 \le t \le n+2$. If $k \equiv 3 \pmod 4$, we need to use an HSD($4^n 9^1$) where we get $4t$ idempotent elements, $0 \le t \le n$, from $n$ holes of size 4, and 3 idempotent elements from the hole of size 9. □

## 4. SQ($v, k$) for $v \equiv 0 \pmod 4$

**Lemma 4.1.** *There exists an* SQ(16, $k$) *for any feasible* $k$.

**Proof.** If $k \equiv 0 \pmod 4$, the result comes from Construction 3.10(a) because HSD($4^4$) exists (Theorem 1.4). If $k \equiv 2 \pmod 4$, the result comes from Lemma 2.8. □

Since an HSD($4^n$) exists for $n \ge 4$, by Construction 3.10(a), we have an SQ($4n, k$) for $k \equiv 0 \pmod 4$. For $k \equiv 2 \pmod 4$, we use Construction 3.10(b).

**Theorem 4.2.** *An* HSD($4^n 12^1$) *exists if and only if* $n \ge 7$.

**Proof.** The necessary condition comes from Lemma 1.3.
For $n \ge 7$, there exists an FSOLS($1^n 3^1$). Using Construction 3.6, we have an HSD($4^n 12^1$). □

**Theorem 4.3.** *For* $v \equiv 0 \pmod 4$, *an* SQ($v, n$) *exists if* $n \le v$, $n \ne v-2$, *and* $v-n$ *is even, except for* $(v, n) \in \{(8, 2), (8, 4)\}$, *and except possibly for* $(v, n)$, *where* $v \in \{20, 24, 28, 32, 36\}$ *and* $n \equiv 2 \pmod 4$.

**Proof.** For $v \le 16$, the result comes from Lemmas 2.2, 2.3 and 4.1. For $16 < v \le 36$ and $n \equiv 0 \pmod 4$, they come from Construction 3.10(a) and Theorem 1.4. For $v > 36$, the result comes from Construction 3.10(b) and Theorem 4.2. □

## 5. SQ($v, k$) for $v \equiv 1 \pmod 4$

In light of Construction 3.10, we are interested in the existence of HSDs of type $4^n u^1$ for $u = 1$ and 9. We commence with the following useful lemma.

**Lemma 5.1.** *For any* $n \ge 6$, *there exists an* HSD($4^n u^1$) *whenever* $n \equiv 0 \pmod 3$, $0 \le u \le 2n-2$, *and* $u \equiv 1 \pmod 3$.

**Proof.** There exist 4-GDDs of the same type (see, for example, [8]). So we can give all points of this GDD weight one to get the desired HSD($4^n u^1$). □

**Theorem 5.2.** *An* HSD($4^n 1^1$) *exists if and only if* $n \ge 4$.

**Proof.** The necessary condition comes from Lemma 1.3.
Lemma 2.13 covers the case when $n < 14$ and $n \not\equiv 0 \pmod 3$, Lemma 5.1 covers $n = 6, 9, 12, 15, 18$, and Appendix A.6 covers $n = 14, 17$, and 19, Lemma 3.7 covers $n = 16$. And now we assume $n \ge 20$.
When $n = 4m, 4m+5, 4m+6$ or $4m+7$ for any $m \ge 4$, we have HSDs of types $4^{4m}1^1$, $4^{4m}21^1$, $4^{4m}25^1$ and $4^{4m}29^1$ by applying Lemma 3.7 with $u = 0$ and $k = 1$, $u = 5$ and $k = 1, 5$, $u = 6$ and $k = 5$. By filling in the holes of sizes 21, 25 and 29 with HSDs of types $4^5 1^1$, $4^6 1^1$ and $4^7 1^1$, respectively, we get an HSD of type $4^n 1^1$ as desired. This completes the proof of the theorem. □

**Lemma 5.3.** *There exists an* SQ(17, $k$) *for any feasible* $k$.

**Proof.** If $k \equiv 1 \pmod 4$, the result comes from Construction 3.10(c) because HSD($4^4 1^1$) exists. If $k \equiv 3 \pmod 4$, the result comes from Lemma 2.9. □

**Lemma 5.4.** *There exists an* HSD($4^n 9^1$) *for* $6 \le n \le 18$.

**Proof.** An HSD($4^{16} 9^1$) comes from Lemma 3.7. The starter blocks of the other cases are given in Appendix A.5. □

**Theorem 5.5.** *An* HSD($4^n 9^1$) *exists if and only if* $n \ge 6$.

**Proof.** The necessary condition comes from Lemma 1.3.
The previous lemma covers the case when $n < 19$. Now we assume $n \ge 19$. When $n = 4m, 4m+6$ or $4m+7$ for any $m \ge 5$, we have HSDs of types $4^{4m}9^1$, $4^{4m}33^1$ and $4^{4m}37^1$ by applying Lemma 3.7 with $u = 2$ and $k = 1$, and with $u = 8$ and $k = 1, 5$. By filling in the holes of sizes 33 and 37, with HSDs of types $4^6 9^1$ and $4^7 9^1$, respectively, we get an HSD of type $4^n 9^1$.
When $n = 4m+9$ for any $m \ge 6$, we have an HSD of type $4^{4m}45^1$ by applying Lemma 3.7 with $u = 10$ and $k = 5$. By filling in the holes of size 45, with an HSD of type $4^9 9^1$, we get an HSD of type $4^n 9^1$.
The above covers $n = 20, 24, 26, 27, 28$ and all $n \ge 30$. For $n = 21, 25$ and 29, we apply Lemma 3.9 with $m = 5, 7$, $k = 1, 5$ and $u = 9$.
For $n = 19, 22$, first of all, we start with a TD(8, 8): For $n = 19$, in the first four groups of this TD we give all of the points a weight of two. In the fifth, sixth and seventh groups, we give two points weight two and the other points weight zero. For $n = 22$, in the first five groups of this TD we give all of the points a weight of two. In the sixth and seventh groups, we give

two points weight two and the other points weight zero. For both $n = 19$ and 22, in the last group, we give seven points a weight of one and one point of weight two for a total weight of 9. Since we have HSDs of types $2^n$ for $n = 5, 6, 7, 8$ and $2^n1^1$ for $n = 4, 5, 6, 7$ [4], we get HSDs of types $16^4 4^3 9^1$ and $16^5 4^2 9^1$ by Construction 3.1. By filling in the holes of size 16 with an HSD($4^4$), the resulting designs are HSD($4^n 9^1$) for $n = 19, 22$.

For $n = 23$, we start with a TD(6, 5). In the first four groups of this TD we give all of the points a weight of four. In the fifth group, we give three points weight four and the other points weight zero. In the last group, we give all the points a weight of one. Since we have HSDs of types $4^n 1^1$ for $n = 4, 5$, we get an HSD of type $20^4 12^1 5^1$. By adjoining four points to this HSD and filling in the holes of sizes 12 and 20 with HSDs of types $4^4$ and $4^6$, respectively, the resulting design is an HSD($4^{23} 9^1$). $\square$

**Theorem 5.6.** *For $v \equiv 1 \pmod 4$, an SQ$(v, n)$ exists if $n \le v$, $n \ne v - 2$, and $v - n$ is even, except for $(v, n) \in \{(5, 1), (5, 5), (9, 1), (9, 5)\}$, and except possibly for $(v, n)$, where $v \in \{21, 25, 29\}$ and $n \equiv 3 \pmod 4$.*

**Proof.** For $v \le 17$, the result comes from Lemmas 2.2, 2.6 and 5.3. For $17 < v \le 29$ and $n \equiv 1 \pmod 4$, the result comes from Construction 3.10(c) and Theorem 5.2. For $v > 29$, the result comes from Construction 3.10(d) and Theorem 5.5. $\square$

## 6. Conclusions

We have used the concept of holey Schröder designs and specifically investigated the existence of HSD($4^n u^1$) mainly for $u = 1, 9$ and 12. We proved that such HSDs exist for $u = 1, 9$, and 12 and $n \ge max\{(u+2)/2, 4\}$. The results have provided an application to the construction of Schröder quasigroups with a specified number of idempotent elements. Most recursive constructions used in this paper are standard in combinatorial designs and many of the direct constructions of HSDs in this paper are carried out by a computer search. Apart from a handful of possible exceptions, which remain under investigation, we have been able to provide fairly conclusive results. From the previous sections, we obtain the main theorem of this paper:

**Theorem 6.1.** *An SQ$(v, n)$ exists if and only if $v \equiv 0, 1 \pmod 4$, $0 \le n \le v$, $n \ne v - 2$, and $v - n$ is even, except for $(v, n) \in \{(5, 1), (5, 5), (8, 2), (8, 4), (9, 1), (9, 5)\}$, and except possibly for $(v, n)$, where $v \in \{20, 21, 24, 25, 28, 29, 32, 36\}$ and $n \equiv 2 \pmod 4$, when $v$ is even, and $n \equiv 3 \pmod 4$, when $v$ is odd.*

**Proof.** The necessary conditions come from [10], and sufficiency comes from Theorems 4.3 and 5.6. $\square$

We define an $n^2 \times k$ *orthogonal array* based on an $n$-set $S$ to be a rectangular array of $n^2$ rows and $k$ columns where, for any two distinct columns, the set of ordered pairs occurring in the $n^2$ rows of these two columns is precisely the set of all $n^2$ distinct ordered pairs from $S$. Evidently, any quasigroup $(Q, *)$ of order $n$ is equivalent to an $n^2 \times 3$ orthogonal array, where $(x, y, z)$ is a row of the array if and only if $x * y = z$. In the statement of Theorem 6.1 for every integer $v > 1$, where $v \equiv 0, 1 \pmod 4$ except for $v = 5$, we have presented a large variety of non-idempotent Schröder quasigroups of order $v$. For each Schröder quasigroup $(Q, *)$, we can define a $|Q|^2 \times 4$ array where the rows are $\{(x, y, x * y, y * x) : x, y \in Q\}$. It is perhaps worth mentioning that all of these non-idempotent models of Schröder quasigroups of order $v$ correspond to $v^2 \times 4$ orthogonal arrays that have the Klein 4-group as conjugate invariant subgroup. For more details on this association, the interested reader is referred to [10]. We can now formally state the following theorem.

**Theorem 6.2.** *For all orders $v > 1$, where $v \equiv 0, 1 \pmod 4$ except for $v = 5$, there exists a variety of non-idempotent Schröder quasigroups of order $v$, all of which correspond to $v^2 \times 4$ orthogonal arrays that have the Klein 4-group as conjugate invariant subgroup.*

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## Appendix

### *A.1.* SQ(12, $n$)

The following are SQ(12, $k$) for $k = 0, 2, 4, 6, 8$, in that order.

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 6 | 11 | 5 | 10 | 4 | 2 | 9 | 1 | 7 | 3 | 8 | 0 |
| 1 | 0 | 9 | 11 | 1 | 10 | 8 | 3 | 5 | 6 | 7 | 4 | 2 |
| 2 | 10 | 8 | 4 | 11 | 7 | 1 | 2 | 0 | 5 | 6 | 3 | 9 |
| 3 | 3 | 6 | 1 | 7 | 11 | 9 | 10 | 2 | 4 | 8 | 0 | 5 |
| 4 | 11 | 7 | 3 | 5 | 2 | 6 | 1 | 9 | 8 | 0 | 10 | 4 |
| 5 | 8 | 5 | 7 | 0 | 9 | 10 | 11 | 6 | 1 | 4 | 2 | 3 |
| 6 | 4 | 1 | 9 | 2 | 8 | 3 | 0 | 11 | 10 | 5 | 6 | 7 |
| 7 | 9 | 2 | 8 | 4 | 5 | 11 | 7 | 3 | 0 | 10 | 1 | 6 |
| 8 | 2 | 4 | 0 | 8 | 3 | 5 | 6 | 10 | 11 | 9 | 7 | 1 |
| 9 | 5 | 0 | 2 | 9 | 6 | 7 | 4 | 8 | 3 | 1 | 11 | 10 |
| 10 | 7 | 10 | 6 | 3 | 1 | 0 | 8 | 4 | 9 | 2 | 5 | 11 |
| 11 | 1 | 3 | 10 | 6 | 0 | 4 | 5 | 7 | 2 | 11 | 9 | 8 |

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 10 | 11 | 9 | 6 | 2 | 5 | 7 | 8 | 0 | 1 | 3 | 4 |
| 1 | 9 | 7 | 11 | 10 | 4 | 0 | 1 | 5 | 6 | 8 | 2 | 3 |
| 2 | 7 | 8 | 2 | 11 | 5 | 1 | 10 | 6 | 3 | 4 | 9 | 0 |
| 3 | 4 | 0 | 1 | 9 | 11 | 7 | 2 | 3 | 8 | 10 | 5 | 6 |
| 4 | 11 | 1 | 8 | 0 | 6 | 10 | 3 | 2 | 9 | 7 | 4 | 5 |
| 5 | 0 | 9 | 10 | 1 | 3 | 8 | 11 | 4 | 2 | 5 | 6 | 7 |
| 6 | 2 | 6 | 5 | 8 | 0 | 3 | 4 | 11 | 7 | 9 | 1 | 10 |
| 7 | 6 | 3 | 0 | 7 | 9 | 11 | 5 | 1 | 4 | 2 | 10 | 8 |
| 8 | 8 | 10 | 6 | 3 | 1 | 4 | 0 | 9 | 5 | 11 | 7 | 2 |
| 9 | 5 | 4 | 7 | 2 | 8 | 9 | 6 | 0 | 10 | 3 | 11 | 1 |
| 10 | 1 | 5 | 3 | 4 | 10 | 2 | 8 | 7 | 11 | 6 | 0 | 9 |
| 11 | 3 | 2 | 4 | 5 | 7 | 6 | 9 | 10 | 1 | 0 | 8 | 11 |

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 11 | 1 | 10 | 2 | 8 | 7 | 5 | 6 | 4 | 9 | 3 |
| 1 | 9 | 7 | 11 | 6 | 0 | 1 | 8 | 2 | 3 | 5 | 10 | 4 |
| 2 | 4 | 8 | 5 | 11 | 6 | 10 | 1 | 3 | 7 | 9 | 2 | 0 |
| 3 | 7 | 9 | 0 | 3 | 11 | 4 | 10 | 8 | 5 | 2 | 1 | 6 |
| 4 | 11 | 2 | 8 | 1 | 10 | 9 | 0 | 7 | 4 | 3 | 6 | 5 |
| 5 | 3 | 5 | 6 | 9 | 1 | 2 | 11 | 4 | 8 | 10 | 0 | 7 |
| 6 | 8 | 0 | 7 | 4 | 9 | 3 | 6 | 11 | 2 | 1 | 5 | 10 |
| 7 | 10 | 6 | 9 | 2 | 4 | 11 | 5 | 1 | 0 | 7 | 3 | 8 |
| 8 | 1 | 10 | 3 | 0 | 8 | 5 | 4 | 6 | 9 | 11 | 7 | 2 |
| 9 | 6 | 4 | 2 | 7 | 5 | 0 | 3 | 9 | 10 | 8 | 11 | 1 |
| 10 | 5 | 1 | 10 | 8 | 3 | 7 | 2 | 0 | 11 | 6 | 4 | 9 |
| 11 | 2 | 3 | 4 | 5 | 7 | 6 | 9 | 10 | 1 | 0 | 8 | 11 |

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 7 | 11 | 5 | 6 | 2 | 1 | 10 | 9 | 8 | 4 | 0 | 3 |
| 1 | 9 | 1 | 11 | 10 | 6 | 7 | 3 | 8 | 0 | 2 | 5 | 4 |
| 2 | 4 | 8 | 2 | 11 | 3 | 10 | 7 | 1 | 6 | 5 | 9 | 0 |
| 3 | 1 | 4 | 0 | 8 | 11 | 9 | 2 | 3 | 5 | 7 | 10 | 6 |
| 4 | 11 | 9 | 6 | 1 | 10 | 2 | 0 | 7 | 4 | 8 | 3 | 5 |
| 5 | 8 | 10 | 1 | 2 | 0 | 5 | 11 | 4 | 9 | 3 | 6 | 7 |
| 6 | 5 | 0 | 8 | 4 | 9 | 3 | 6 | 11 | 7 | 1 | 2 | 10 |
| 7 | 3 | 6 | 9 | 7 | 4 | 11 | 5 | 0 | 2 | 10 | 1 | 8 |
| 8 | 0 | 5 | 10 | 9 | 8 | 4 | 1 | 6 | 3 | 11 | 7 | 2 |
| 9 | 6 | 7 | 3 | 0 | 5 | 8 | 4 | 2 | 10 | 9 | 11 | 1 |
| 10 | 10 | 2 | 7 | 3 | 1 | 0 | 8 | 5 | 11 | 6 | 4 | 9 |
| 11 | 2 | 3 | 4 | 5 | 7 | 6 | 9 | 10 | 1 | 0 | 8 | 11 |

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 11 | 7 | 1 | 3 | 10 | 8 | 6 | 4 | 5 | 9 | 2 |
| 1 | 9 | 1 | 11 | 8 | 10 | 3 | 2 | 5 | 7 | 4 | 0 | 6 |
| 2 | 8 | 0 | 10 | 11 | 2 | 7 | 6 | 3 | 9 | 1 | 5 | 4 |
| 3 | 10 | 6 | 8 | 3 | 11 | 4 | 7 | 9 | 5 | 2 | 1 | 0 |
| 4 | 11 | 7 | 4 | 2 | 6 | 9 | 0 | 8 | 1 | 3 | 10 | 5 |
| 5 | 2 | 10 | 1 | 9 | 0 | 5 | 11 | 4 | 6 | 8 | 3 | 7 |
| 6 | 5 | 9 | 2 | 0 | 8 | 1 | 4 | 11 | 3 | 7 | 6 | 10 |
| 7 | 3 | 2 | 9 | 6 | 1 | 11 | 5 | 7 | 0 | 10 | 4 | 8 |
| 8 | 6 | 4 | 5 | 10 | 9 | 0 | 1 | 2 | 8 | 11 | 7 | 3 |
| 9 | 4 | 8 | 6 | 7 | 5 | 2 | 3 | 0 | 10 | 9 | 11 | 1 |
| 10 | 7 | 3 | 0 | 5 | 4 | 8 | 10 | 1 | 11 | 6 | 2 | 9 |
| 11 | 1 | 5 | 3 | 4 | 7 | 6 | 9 | 10 | 2 | 0 | 8 | 11 |

### *A.2.* SQ(13, $n$)

The following are SQ(13, $k$) for $k = 1, 3, 5, 7$, in that order.

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 7 | 12 | 10 | 11 | 8 | 1 | 3 | 5 | 6 | 0 | 2 | 9 | 4 |
| 1 | 6 | 9 | 12 | 5 | 11 | 0 | 10 | 1 | 7 | 2 | 4 | 3 | 8 |
| 2 | 12 | 4 | 8 | 1 | 6 | 11 | 7 | 2 | 10 | 5 | 9 | 0 | 3 |
| 3 | 2 | 0 | 11 | 10 | 12 | 9 | 4 | 8 | 3 | 6 | 5 | 1 | 7 |
| 4 | 4 | 3 | 9 | 2 | 5 | 12 | 6 | 11 | 0 | 8 | 7 | 10 | 1 |
| 5 | 3 | 7 | 5 | 12 | 0 | 4 | 8 | 9 | 11 | 10 | 1 | 2 | 6 |
| 6 | 9 | 5 | 2 | 6 | 3 | 10 | 11 | 12 | 1 | 4 | 8 | 7 | 0 |
| 7 | 1 | 8 | 6 | 3 | 7 | 2 | 5 | 0 | 12 | 11 | 10 | 4 | 9 |
| 8 | 10 | 1 | 0 | 7 | 4 | 8 | 12 | 3 | 2 | 9 | 6 | 5 | 11 |
| 9 | 11 | 10 | 7 | 0 | 9 | 3 | 2 | 6 | 4 | 1 | 12 | 8 | 5 |
| 10 | 8 | 11 | 1 | 9 | 10 | 6 | 0 | 4 | 5 | 7 | 3 | 12 | 2 |
| 11 | 0 | 2 | 3 | 4 | 1 | 5 | 9 | 7 | 8 | 12 | 11 | 6 | 10 |
| 12 | 5 | 6 | 4 | 8 | 2 | 7 | 1 | 10 | 9 | 3 | 0 | 11 | 12 |

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 12 | 2 | 11 | 8 | 4 | 10 | 9 | 7 | 6 | 1 | 3 | 5 |
| 1 | 9 | 4 | 12 | 5 | 11 | 1 | 6 | 0 | 10 | 2 | 3 | 7 | 8 |
| 2 | 12 | 5 | 8 | 0 | 3 | 11 | 1 | 7 | 6 | 10 | 9 | 4 | 2 |
| 3 | 5 | 1 | 11 | 9 | 12 | 0 | 8 | 2 | 3 | 4 | 6 | 10 | 7 |
| 4 | 4 | 7 | 9 | 10 | 1 | 12 | 3 | 11 | 0 | 8 | 5 | 2 | 6 |
| 5 | 10 | 3 | 4 | 12 | 6 | 7 | 9 | 8 | 11 | 5 | 2 | 0 | 1 |
| 6 | 7 | 2 | 6 | 3 | 10 | 5 | 11 | 12 | 1 | 0 | 8 | 9 | 4 |
| 7 | 6 | 8 | 3 | 7 | 2 | 10 | 4 | 5 | 12 | 11 | 0 | 1 | 9 |
| 8 | 1 | 0 | 10 | 6 | 4 | 8 | 12 | 3 | 2 | 9 | 7 | 5 | 11 |
| 9 | 11 | 10 | 5 | 2 | 9 | 6 | 7 | 1 | 4 | 3 | 12 | 8 | 0 |
| 10 | 8 | 11 | 1 | 4 | 0 | 9 | 2 | 6 | 5 | 7 | 10 | 12 | 3 |
| 11 | 2 | 9 | 7 | 1 | 5 | 3 | 0 | 4 | 8 | 12 | 11 | 6 | 10 |
| 12 | 3 | 6 | 0 | 8 | 7 | 2 | 5 | 10 | 9 | 1 | 4 | 11 | 12 |

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 5 | 12 | 6 | 11 | 8 | 7 | 10 | 2 | 0 | 3 | 9 | 4 | 1 |
| 1 | 9 | 4 | 12 | 5 | 11 | 2 | 3 | 1 | 7 | 10 | 6 | 0 | 8 |
| 2 | 12 | 0 | 8 | 2 | 10 | 11 | 7 | 9 | 6 | 5 | 1 | 3 | 4 |
| 3 | 1 | 10 | 11 | 3 | 12 | 5 | 8 | 6 | 4 | 2 | 0 | 9 | 7 |
| 4 | 0 | 5 | 4 | 10 | 1 | 12 | 9 | 11 | 3 | 8 | 2 | 7 | 6 |
| 5 | 10 | 9 | 7 | 12 | 2 | 0 | 6 | 8 | 11 | 4 | 3 | 1 | 5 |
| 6 | 3 | 7 | 0 | 4 | 5 | 9 | 11 | 12 | 1 | 6 | 8 | 10 | 2 |
| 7 | 6 | 8 | 3 | 1 | 0 | 10 | 4 | 7 | 12 | 11 | 5 | 2 | 9 |
| 8 | 4 | 1 | 10 | 9 | 6 | 8 | 12 | 3 | 2 | 0 | 7 | 5 | 11 |
| 9 | 11 | 2 | 1 | 7 | 3 | 6 | 5 | 4 | 10 | 9 | 12 | 8 | 0 |
| 10 | 8 | 11 | 9 | 6 | 4 | 1 | 2 | 0 | 5 | 7 | 10 | 12 | 3 |
| 11 | 7 | 3 | 2 | 0 | 9 | 4 | 1 | 5 | 8 | 12 | 11 | 6 | 10 |
| 12 | 2 | 6 | 5 | 8 | 7 | 3 | 0 | 10 | 9 | 1 | 4 | 11 | 12 |

| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 12 | 10 | 11 | 8 | 9 | 5 | 1 | 4 | 7 | 3 | 2 | 6 |
| 1 | 2 | 1 | 12 | 7 | 11 | 3 | 10 | 9 | 6 | 0 | 5 | 4 | 8 |
| 2 | 12 | 4 | 8 | 3 | 9 | 11 | 2 | 10 | 5 | 6 | 7 | 1 | 0 |
| 3 | 4 | 0 | 11 | 9 | 12 | 6 | 8 | 2 | 10 | 5 | 1 | 3 | 7 |
| 4 | 10 | 2 | 6 | 5 | 4 | 12 | 7 | 11 | 3 | 8 | 9 | 0 | 1 |
| 5 | 1 | 9 | 0 | 12 | 7 | 5 | 6 | 8 | 11 | 4 | 2 | 10 | 3 |
| 6 | 6 | 3 | 9 | 4 | 10 | 0 | 11 | 12 | 1 | 2 | 8 | 7 | 5 |
| 7 | 3 | 8 | 5 | 10 | 1 | 2 | 0 | 7 | 12 | 11 | 6 | 9 | 4 |
| 8 | 9 | 10 | 7 | 0 | 6 | 8 | 12 | 3 | 2 | 1 | 4 | 5 | 11 |
| 9 | 11 | 5 | 2 | 1 | 0 | 10 | 4 | 6 | 7 | 3 | 12 | 8 | 9 |
| 10 | 8 | 11 | 3 | 6 | 5 | 7 | 1 | 4 | 0 | 9 | 10 | 12 | 2 |
| 11 | 5 | 7 | 4 | 2 | 3 | 1 | 9 | 0 | 8 | 12 | 11 | 6 | 10 |
| 12 | 7 | 6 | 1 | 8 | 2 | 4 | 3 | 5 | 9 | 10 | 0 | 11 | 12 |

### *A.3.* HSD($4^n 1^1$)

Here we list some HSDs which are used in the previous sections. All of them are obtained by computer. In the following list, the point set of an HSD($4^n u^1$) consists of $Z_{4n}$ and $u$ infinite points which are denoted by alphabet. For simplicity, we only list the starter blocks or the corresponding Latin square. We also use the $+2$ method or the $+4$ method to develop blocks, which means that we add two or four (mod $4n$) to each point of the starter blocks to obtain all blocks.

```
n = 5 (+2 mod 20):
 {0,  1, 13, 17}, {0,  2,  1, 18}, {0,  4, 17,  3}, {0,  6, 12,  4}, {0,  7,  9,  1},
 {0, 11,  7,  8}, {0, 13,  6,  x}, {0, 17, 11,  9}, {1, 12,  3,  x}

n = 7 (+2 mod 28):
 {0,  1,  4, 24}, {0,  2, 19, 27}, {0,  3,  1, 20}, {0,  4, 13, 22}, {0,  5,  2, 18},
 {0, 10, 23, 26}, {0, 11,  3,  1}, {0, 13,  9,  5}, {0, 15, 20, 19}, {0, 17, 27, 15},
 {0, 22,  6,  x}, {0, 23, 17, 11}, {1, 19,  7,  x}

n = 8 (+2 mod 32):
 {0,  1, 14, 13}, {0,  2, 31, 12}, {0,  3, 20, 10}, {0,  4,  6, 17}, {0,  5, 30,  4},
 {0,  7,  4, 25}, {0,  9, 23, 26}, {0, 12, 17, 31}, {0, 13, 11,  9}, {0, 14,  3, 23},
 {0, 17,  7, 11}, {0, 23, 27,  1}, {0, 25,  5, 27}, {0, 27, 21,  x}, {1, 18,  4,  x}

n = 10 (+2 mod 40):
 {0,  1, 32,  3}, {0,  2, 14, 33}, {0,  3,  7, 25}, {0,  4, 19,  7}, {0,  5, 13, 16},
 {0,  6, 35, 22}, {0,  7, 23, 31}, {0,  8,  6,  1}, {0,  9, 36,  8}, {0, 14, 18, 39},
 {0, 16,  2, 27}, {0, 17,  5, 21}, {0, 18, 15, 17}, {0, 23, 37,  x}, {0, 27, 21, 35},
 {0, 29, 27, 23}, {0, 31, 12,  5}, {0, 39, 17, 11}, {1, 26, 32,  x}

n = 11 (+2 mod 44)
 {0,  1, 41, 18}, {0,  2, 19, 28}, {0,  3, 28,  x}, {0,  4, 35,  6}, {0,  5, 36, 12},
 {0,  6, 15, 35}, {0,  7, 25,  8}, {0,  8, 31, 21}, {0,  9, 17,  3}, {0, 10,  6, 20},
 {0, 12,  5,  7}, {0, 13, 23, 41}, {0, 15, 20,  1}, {0, 16,  1,  2}, {0, 18,  4, 25},
 {0, 19,  3, 10}, {0, 31, 32, 27}, {0, 41, 21,  5}, {1,  5,  3, 39}, {1,  7, 21, 33},
 {1, 18, 33,  x}

n = 13 (+2 mod 52):
 {0,  1, 38,  5}, {0,  2, 21, 36}, {0,  3, 44,  9}, {0,  4, 29, 31}, {0,  5, 50, 42},
 {0,  6, 41, 51}, {0,  7, 22,  2}, {0,  9, 37, 21}, {0, 10, 48, 41}, {0, 11, 47, 43},
 {0, 12, 20, 45}, {0, 14, 43, 20}, {0, 15,  7,  1}, {0, 16, 10, 28}, {0, 21, 35,  4},
 {0, 22, 19, 47}, {0, 24, 36, 33}, {0, 29, 11, 30}, {0, 35, 23, 34}, {0, 43, 49, 27},
 {0, 47, 24,  x}, {0, 51,  3, 17}, {1,  9, 11, 31}, {1, 13, 45, 11}, {1, 26, 29,  x}
```

For $n = 14, 17, 19$, please see Appendix A.6.

### *A.4.* HSD($4^n u^1$) *for* $n = 4, 5$

For $u = 0, 1, 4$ the HSDs come from Theorem 1.4 and Lemma 2.13. For $n = 4$ and $u = 2$, see Example 2.11.

```
n = 4, u = 3 (+2 mod, 16):
 {0,  2, 15,  9}, {0,  3,  2, x3}, {0,  6,  5,  7}, {0,  7,  9, 10}, {0, 11,  6, x2},
 {0, 13,  3, x1}, {1,  0,  3, x3}, {1,  8,  6, x1}, {1, 12,  7, x2}

n = 4, u = 5 (+2 mod, 16):
 {0,  1,  3, x5}, {0,  2,  5, x4}, {0,  3, 10,  1}, {0,  9, 14, x1}, {0, 10,  9, x2},
 {0, 13,  7, x3}, {1,  2,  4, x5}, {1,  3,  6, x4}, {1,  6, 11, x1}, {1,  7,  0, x2},
 {1, 12,  2, x3},

n = 4, u = 6 (+1 mod, 16):
 {0,  1,  2, x6}, {0,  3,  6, x5}, {0,  5,  7, x4}, {0,  9,  3, x2}, {0, 10, 15,  x},
 {0, 14,  5, x3}

n = 5, u = 2 (+1 mod, 20):
 {0,  2, 14,  6}, {0,  3, 12,  1}, {0,  4, 11, 17}, {0,  7,  3, x1}, {0, 19,  1, x2},

n = 5, u = 3 (+2 mod, 20):
 {0,  2, 14, 13}, {0,  3,  2, x3}, {0,  6, 12, 19}, {0,  8, 17,  4}, {0,  9,  3, 17},
 {0, 16,  7, x2}, {0, 17,  1, x1}, {1,  0,  3, x3}, {1,  3,  5,  9}, {1,  9,  2, x2},
 {1, 10, 12, x1}

n = 5, u = 5 (+2 mod, 20):
 {0,  1,  3, x5}, {0,  2,  6, x4}, {0,  4, 12, x3}, {0,  6,  7, 19}, {0,  7, 16, 13},
 {0,  8, 11,  7}, {0,  9,  1, x1}, {0, 11,  2, x2}, {1,  2,  4, x5}, {1,  7, 13, x4},
 {1,  8,  5, x2}, {1, 18, 12, x1}, {1, 19,  3, x3}

n = 5, u = 6 (+1 mod, 20):
 {0,  1,  2, x6}, {0,  4, 11,  3}, {0,  6,  8, x4}, {0,  9, 13, x2}, {0, 13,  4, x1},
 {0, 17,  3, x5}, {0, 18,  6, x3}
```

*A.5.* $\mathrm{HSD}(4^n9^1)$ *for* $6 \le n \le 18$

$\mathrm{HSD}(4^{16}9^1)$ comes from Lemma 3.7.

```
n = 6 (+2 mod 24):
 {0,  1,  3, x9}, {0,  2,  5, x8}, {0,  4,  8, x7}, {0,  5, 13, x5}, {0,  9, 20, 17},
 {0, 10,  7, x4}, {0, 13,  2, x2}, {0, 16,  1, x3}, {0, 17, 10, x1}, {0, 19,  9, x6},
 {1,  2,  4, x9}, {1,  5,  6, x8}, {1,  9, 14, x4}, {1, 10, 15, x2}, {1, 11, 10, x3},
 {1, 14,  5, x1}, {1, 18,  2, x6}, {1, 22,  8, x5}, {1, 23,  3, x7}

n = 7 (+2 mod 28):
 {0,  1,  3, x9}, {0,  2,  5, x8}, {0,  3,  2, 13}, {0,  4,  9, x6}, {0,  6, 10, x5},
 {0,  9, 12, x4}, {0, 10, 22,  6}, {0, 13,  1, x2}, {0, 15, 19, x7}, {0, 19,  8, x1},
 {0, 20, 11, x3}, {1,  2,  4, x9}, {1,  3, 16, x3}, {1,  4, 12, x7}, {1,  5,  6, x8},
 {1,  6, 23, x1}, {1,  7, 13, x5}, {1,  9, 14, x6}, {1, 11, 21,  9}, {1, 12,  2, x2},
 {1, 24,  5, x4}

n = 8 (+2 mod 32):
 {0,  1,  3, x9}, {0,  2,  5, x8}, {0,  4, 21, x6}, {0,  5,  4, x5}, {0,  6, 19, 26},
 {0,  7, 10, x4}, {0,  9, 13, x7}, {0, 13, 26, 12}, {0, 17, 23, 11}, {0, 20,  2, x3},
 {0, 21,  7, 10}, {0, 22, 15, x1}, {0, 23, 14, x2}, {0, 27, 17,  7}, {1,  2,  4, x9},
 {1,  3,  8, x6}, {1,  5,  6, x8}, {1,  7, 19, x3}, {1, 14,  3, x4}, {1, 15, 24, x1},
 {1, 18, 23, x5}, {1, 22,  5, x2}, {1, 30,  2, x7}

n = 9 (+2 mod 36):
 {0,  1,  3, x9}, {0,  2,  5, x8}, {0,  4, 16, 30}, {0,  5, 21, x5}, {0,  6,  2, x7},
 {0,  8, 19, 22}, {0, 10, 15, x6}, {0, 11, 28, 13}, {0, 12,  7, x4}, {0, 13,  1, 17},
 {0, 15, 30, 25}, {0, 17, 25, x3}, {0, 19,  4, x1}, {0, 20, 12, x2}, {0, 23, 17, 29},
 {0, 25, 11, 33}, {1,  2,  4, x9}, {1,  3,  2, x6}, {1,  5,  6, x8}, {1,  8, 15, x1},
 {1,  9, 13, x7}, {1, 11, 24, x4}, {1, 30, 14, x3}, {1, 31, 21, x2}, {1, 34,  8, x5}
```

n = 10 (+2 mod 40):
{0, 1, 16, 24}, {0, 2, 36, x9}, {0, 4, 21, 23}, {0, 5, 11, x8}, {0, 6, 34, 5},
{0, 7, 14, 1}, {0, 12, 23, 19}, {0, 13, 25, 37}, {0, 14, 3, 9}, {0, 15, 31, x5},
{0, 18, 9, x6}, {0, 19, 33, x7}, {0, 21, 8, x3}, {0, 24, 38, x1}, {0, 29, 37, 15},
{0, 31, 28, x4}, {0, 33, 29, 32}, {0, 39, 18, x2}, {1, 6, 33, x3}, {1, 9, 14, x6},
{1, 15, 13, x9}, {1, 16, 23, x4}, {1, 17, 39, x1}, {1, 18, 36, x8}, {1, 24, 26, x7},
{1, 32, 28, x5}, {1, 38, 37, x2}

n = 11 (+2 mod 44):
{0, 2, 39, 27}, {0, 3, 42, x1}, {0, 4, 12, 35}, {0, 5, 26, 34}, {0, 6, 5, x2},
{0, 7, 19, 13}, {0, 9, 43, 6}, {0, 10, 8, 7}, {0, 12, 29, 39}, {0, 13, 28, 4},
{0, 14, 38, x9}, {0, 15, 7, x3}, {0, 16, 30, x8}, {0, 17, 37, x4}, {0, 18, 34, x5},
{0, 19, 21, 3}, {0, 21, 20, x7}, {0, 27, 40, x6}, {0, 39, 1, 41}, {0, 41, 23, 15},
{1, 0, 9, x7}, {1, 3, 33, x9}, {1, 10, 35, x6}, {1, 14, 32, x3}, {1, 16, 28, x4},
{1, 20, 41, x1}, {1, 21, 25, x5}, {1, 29, 10, x2}, {1, 31, 15, x8}

n = 12 (+2 mod 48):
{0, 3, 31, 25}, {0, 4, 43, 8}, {0, 6, 16, x4}, {0, 7, 42, 16}, {0, 8, 41, x9},
{0, 9, 25, x1}, {0, 10, 28, 21}, {0, 14, 8, 47}, {0, 15, 13, 17}, {0, 16, 2, 35},
{0, 17, 10, 31}, {0, 18, 47, 4}, {0, 19, 27, x7}, {0, 20, 35, x6}, {0, 23, 30, x8},
{0, 25, 15, x2}, {0, 27, 45, 43}, {0, 29, 23, 9}, {0, 31, 5, 27}, {0, 37, 3, 19},
{0, 45, 22, x3}, {0, 46, 1, x5}, {1, 0, 28, x1}, {1, 2, 39, x3}, {1, 29, 32, x5},
{1, 31, 20, x9}, {1, 36, 41, x8}, {1, 38, 12, x7}, {1, 39, 43, x4}, {1, 41, 40, x6},
{1, 44, 42, x2}

n = 13 (+2 mod 52):
{0, 1, 49, 43}, {0, 2, 36, x1}, {0, 4, 24, 42}, {0, 5, 20, x6}, {0, 6, 44, 16},
{0, 8, 2, 12}, {0, 11, 45, x3}, {0, 14, 47, 51}, {0, 15, 7, 23}, {0, 16, 41, x9},
{0, 17, 22, 45}, {0, 19, 12, 44}, {0, 21, 9, 30}, {0, 22, 31, x2}, {0, 25, 11, 19},
{0, 27, 17, x7}, {0, 29, 51, 49}, {0, 37, 1, 35}, {0, 40, 19, x5}, {0, 41, 35, 5},
{0, 43, 23, x4}, {0, 45, 21, 41}, {0, 47, 18, 17}, {0, 49, 46, x8}, {1, 11, 13, x1},
{1, 13, 38, x5}, {1, 18, 39, x8}, {1, 20, 35, x6}, {1, 25, 24, x2}, {1, 39, 28, x9},
{1, 44, 20, x3}, {1, 46, 50, x7}, {1, 50, 48, x4}

n = 14 (+2 mod 56):
{0, 1, 27, 36}, {0, 3, 54, 2}, {0, 5, 7, 27}, {0, 6, 29, 39}, {0, 7, 41, x9},
{0, 8, 46, 45}, {0, 9, 52, 41}, {0, 10, 53, 32}, {0, 11, 30, x8}, {0, 12, 11, 15},
{0, 13, 17, x5}, {0, 15, 40, x2}, {0, 16, 12, x7}, {0, 17, 55, 12}, {0, 18, 15, 9},
{0, 19, 25, 40}, {0, 20, 13, 51}, {0, 23, 6, 33}, {0, 25, 45, 6}, {0, 26, 51, 19},
{0, 29, 38, x3}, {0, 32, 39, x6}, {0, 34, 8, x4}, {0, 37, 36, 29}, {0, 54, 32, x1},
{1, 3, 19, x1}, {1, 4, 7, x8}, {1, 6, 27, x2}, {1, 9, 17, x4}, {1, 13, 3, 25},
{1, 22, 14, x5}, {1, 24, 34, x9}, {1, 26, 37, x3}, {1, 27, 48, x6}, {1, 41, 53, x7}

n = 15 (+2 mod 60):
{0, 1, 40, 59}, {0, 2, 28, 57}, {0, 3, 29, 54}, {0, 4, 36, 2}, {0, 5, 7, 24},
{0, 6, 49, 16}, {0, 7, 23, 49}, {0, 11, 18, 55}, {0, 13, 22, x9}, {0, 14, 34, 27},
{0, 16, 17, x8}, {0, 18, 5, 10}, {0, 20, 33, 8}, {0, 21, 16, 7}, {0, 22, 31, 19},
{0, 24, 1, 11}, {0, 27, 4, 35}, {0, 32, 14, x7}, {0, 39, 43, x6}, {0, 41, 59, 17},
{0, 47, 39, x1}, {0, 48, 19, x5}, {0, 49, 35, 39}, {0, 50, 53, x2}, {0, 52, 6, x3},
{0, 57, 51, x4}, {1, 2, 24, x1}, {1, 7, 29, 13}, {1, 9, 36, x2}, {1, 15, 35, x3},
{1, 21, 49, 25}, {1, 33, 23, x7}, {1, 38, 50, x4}, {1, 39, 58, x8}, {1, 44, 40, x6},
{1, 52, 21, x9}, {1, 59, 34, x5}

n = 17 (+2 mod 68):
{0, 2, 44, 14}, {0, 3, 58, 27}, {0, 4, 60, 39}, {0, 5, 30, 40}, {0, 6, 54, 65},
{0, 7, 52, 29}, {0, 12, 3, x1}, {0, 13, 23, x2}, {0, 14, 18, 50}, {0, 15, 31, 6},
{0, 16, 48, x6}, {0, 18, 19, 63}, {0, 19, 62, x3}, {0, 20, 42, x4}, {0, 21, 2, 61},
{0, 22, 11, x8}, {0, 23, 46, 33}, {0, 24, 39, 9}, {0, 26, 67, 5}, {0, 27, 40, 11},
{0, 28, 4, 53}, {0, 29, 65, 2}, {0, 31, 49, 35}, {0, 33, 55, x5}, {0, 41, 37, x7},
{0, 43, 29, 49}, {0, 57, 13, 21}, {0, 60, 63, x9}, {0, 61, 21, 31}, {0, 65, 57, 58},

```
 {1,  0,  8, x5}, {1,  3, 51, 63}, {1,  5, 26, x8}, {1, 16,  0, x2}, {1, 17, 15, 43},
 {1, 23, 54, x1}, {1, 27, 28, x9}, {1, 34, 64, x7}, {1, 37, 49, x6}, {1, 51, 13, x4},
 {1, 60, 67, x3}

n = 18,  (+2 mod 72):
 {0,  1, 31, 63}, {0,  2, 52, 35}, {0,  3, 44, 30}, {0,  4,  2, 55}, {0,  6, 65, 56},
 {0,  7, 66, 15}, {0,  8, 19, x6}, {0, 11, 12, 60}, {0, 13, 63,  4}, {0, 15, 46, 65},
 {0, 17, 69,  6}, {0, 20, 10,  5}, {0, 22, 61, 14}, {0, 23, 34, 69}, {0, 25, 59,  8},
 {0, 27, 41,  7}, {0, 28, 47, 67}, {0, 29, 25, x4}, {0, 30, 15, 46}, {0, 32, 49, 53},
 {0, 34, 67, 37}, {0, 39, 30, 27}, {0, 43, 40, x2}, {0, 45, 43, 44}, {0, 46, 14, x9},
 {0, 49,  4, x8}, {0, 56,  7, x5}, {0, 57, 29,  1}, {0, 60, 37, x7}, {0, 61, 35, x1},
 {0, 62, 24, x3}, {0, 65, 71, 21}, {1,  7, 31, 41}, {1, 13,  3, 17}, {1, 17, 46, x5},
 {1, 36, 60, x4}, {1, 40, 20, x1}, {1, 42, 67, x8}, {1, 47, 68, x7}, {1, 49, 41, x3},
 {1, 65, 18, x6}, {1, 68, 25, x2}, {1, 71, 59, x9}
```

*A.6. Miscellaneous* HSD($4^n u^1$)

An HSD($4^{14}1^1$) can be obtained from HSD($4^{10}17^1$) by filling an HSD($4^4 1^1$) in the hole of size 17. For $n = 17$ and 19, an HSD($4^n 1^1$) can be obtained from HSD($4^{n-5}21^1$) by filling an HSD($4^5 1^1$) in the hole of size 21.

```
n = 10, u = 17,  (+2 mod 40):
 {0,  2, 24, x8}, {0,  4, 19, y6}, {0,  7, 33, y4}, {0, 11, 28, y2}, {0, 12, 37, x3},
 {0, 15, 17, y7}, {0, 16, 13, x6}, {0, 18, 14, x5}, {0, 19,  6, x4}, {0, 23,  4, y0},
 {0, 25,  9, x9}, {0, 26,  7, y5}, {0, 27, 18, y3}, {0, 29,  1,  8}, {0, 32, 38, x2},
 {0, 34, 25, y1}, {0, 37,  3, x7}, {0, 39, 32, x1}, {1,  0, 12, y7}, {1,  3, 14, x3},
 {1,  6,  5, y3}, {1, 10, 26, x7}, {1, 13, 10, x6}, {1, 17, 39, x5}, {1, 19, 15, x8},
 {1, 20, 25, y0}, {1, 24, 35, y2}, {1, 27, 19, x2}, {1, 28, 30, y4}, {1, 32, 18, x9},
 {1, 33,  6, y1}, {1, 35,  2, y6}, {1, 36, 13, x4}, {1, 37, 36, y5}, {1, 38, 33, x1}

n = 12, u = 21 (+2 mod 48)
 {0,  1, 17, z1}, {0,  3, 43, z0}, {0,  4, 20, x8}, {0,  5, 18, x2}, {0,  8, 38, y7},
 {0, 10, 19,  1}, {0, 15,  8, y6}, {0, 16,  1, y2}, {0, 21, 23, y0}, {0, 22, 44, y1},
 {0, 23, 33, y9}, {0, 25, 34, y4}, {0, 27,  5, x4}, {0, 28, 21, x5}, {0, 30, 47, x6},
 {0, 33,  6, x1}, {0, 34, 31, y8}, {0, 37, 32, x3}, {0, 39, 11, y3}, {0, 42, 46, x9},
 {0, 43, 22, x7}, {0, 46,  9, y5}, {1,  2, 22, y0}, {1,  3, 36, y8}, {1,  4, 14, x4},
 {1,  7,  3, x9}, {1,  8, 42, y3}, {1,  9, 15, y1}, {1, 14, 12, z0}, {1, 15, 20, y2},
 {1, 18, 24, y9}, {1, 20, 19, x7}, {1, 27,  8, x6}, {1, 29, 46, x5}, {1, 30, 43, x3},
 {1, 32, 21, y6}, {1, 33,  4, y5}, {1, 36, 11, y4}, {1, 38, 41, x1}, {1, 39,  5, x8},
 {1, 40, 17, x2}, {1, 42, 34, z1}, {1, 45, 27, y7}

n = 14, u = 21 (+2 mod 56):
 {0,  1, 36, x7}, {0,  4, 35, y6}, {0,  6, 30, x9}, {0, 10, 18, y9}, {0, 13, 37, y0},
 {0, 15, 54, y7}, {0, 16, 31, 11}, {0, 17, 12, x1}, {0, 18, 40, y5}, {0, 19, 39, 12},
 {0, 20, 55, 29}, {0, 21, 23, 30}, {0, 22, 13, z0}, {0, 23,  4, y1}, {0, 24, 19, 13},
 {0, 30, 34, x3}, {0, 33, 50, x4}, {0, 35, 32, x5}, {0, 37, 11, z1}, {0, 39, 29, x2},
 {0, 44, 21, x6}, {0, 45, 46, x8}, {0, 48, 49, y3}, {0, 51, 17, y8}, {0, 53,  3, y4},
 {0, 54,  8, y2}, {1,  2,  0, z1}, {1,  5, 49, x3}, {1,  9, 16, z0}, {1, 10,  4, x2},
 {1, 13, 42, x6}, {1, 14, 11, x4}, {1, 16,  5, y7}, {1, 19, 27, y9}, {1, 23, 39, y5},
 {1, 26, 33, x7}, {1, 28, 13, x5}, {1, 32, 14, y0}, {1, 33, 52, y3}, {1, 41, 23, y2},
 {1, 46, 55, x1}, {1, 47, 24, y6}, {1, 48, 32, y4}, {1, 50, 30, y8}, {1, 52, 21, x8},
 {1, 54, 41, y1}, {1, 55, 51, x9}
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## References


[1] R.J. Abel, A.E. Brouwer, C.J. Colbourn, J.H. Dinitz, in: C.J. Colbourn, J.H. Dinitz (Eds.), Mutually Orthogonal Latin Squares (MOLS), 2nd ed., in: The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 2007, pp. 160–193.
[2] R.D. Baker, Quasigroups and tactical systems, Aequationes Math. 18 (1978) 296–303.
[3] F.E. Bennett, R. Wei, The existence of Schröder designs with equal-sized holes, Discrete Math. 170 (1997) 15–28.
[4] F.E. Bennett, R. Wei, H. Zhang, Holey Schröder designs of type $2^n u^1$, J. Combin. Des. 6 (1998) 131–150.
[5] F.E. Bennett, R. Wei, L. Zhu, Incomplete idempotent Schröder quasigroups and related packing designs, Aequationes Math. 51 (1996) 100–114.
[6] A.E. Brouwer, A. Schrijver, H. Hanani, Group divisible designs with block size 4, Discrete Math. 20 (1977) 1–10.
[7] C.J. Colbourn, D.R. Stinson, Edge-colored designs with block size four, Aequationes Math. 36 (1988) 230–245.

[8] G. Ge, in: C.J. Colbourn, J.H. Dinitz (Eds.), Group Divisible Designs, 2nd ed., in: The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 2007, pp. 255–260.
[9] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975) 225–369.
[10] C.C. Lindner, N.S. Mendelsohn, S.R. Sun, On the construction of Schröder quasigroups, Discrete Math. 32 (1980) 271–280.
[11] N.S. Mendelsohn, Self-orthogonal Weisner designs, Ann. NY Acad. Sci. 319 (1979) 391–398.
[12] R.M. Wilson, Constructions and uses of pairwise balanced designs, Math. Centre Tracts 55 (1974) 18–41.
[13] Xu Yunqing, H. Zhang, Zhu Lie, Existence of frame SOLS of type $a^n b^1$, Discrete Math. 250 (2002) 211–230.
[14] H. Zhang, Specifying Latin squares in propositional logic, in: R. Veroff (Ed.), Automated Reasoning and its Applications, MIT Press, 1997, Essays in honor of Larry Wos, Chapter 6.
[15] H. Zhang, Combinatorial designs by SAT solvers, in: Toby Walsh (Ed.), Handbook of Satisfiability, IOS Press, 2008, Chapter 17.
[16] H. Zhang, S. Ziewacz, On holey Schröder designs of type $2^n u^1$, preprint.