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Oscillation of solutions of second-order nonlinear differential equations of Euler type

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Abstract

We consider the nonlinear Euler differential equation $t^2x'' + g(x) = 0$. Here $g(x)$ satisfies $xg(x) > 0$ for $x \neq 0$, but is not assumed to be sublinear or superlinear. We present implicit necessary and sufficient condition for all nontrivial solutions of this system to be oscillatory or nonoscillatory. Also we prove that solutions of this system are all oscillatory or all nonoscillatory and cannot be both. We derive explicit conditions and improve the results presented in the previous literature. We extend our results to the extended equation $t^2x'' + a(t)g(x) = 0$.

Keywords: Oscillation; Nonlinear differential equations; Liénard system

1. Introduction

Consider the second-order nonlinear differential equation of Euler type

$$t^2x'' + g(x) = 0, \quad t > 0, \tag{1.1}$$

where $' = \frac{d}{dt}$, $g(x)$ is continuous on $\mathbf{R}$, and

$$xg(x) > 0, \quad \text{for } x \neq 0. \tag{1.2}$$

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We assume that uniqueness is guaranteed for the solutions of (1.1) to the initial value problem. In [16] the authors have proved that all solutions of (1.1) are continuable in the future. Hence, it is worthwhile to discuss whether solutions of (1.1) are oscillatory or not.

A solution $x(t)$ of (1.1) is said to be oscillatory if there exists a sequence t_n tending to ∞ such that $x(t_n) = 0$. Otherwise, $x(t)$ is said to be nonoscillatory. For brevity, Eq. (1.1) is called oscillatory (respectively nonoscillatory) in case all nontrivial solutions are oscillatory (respectively nonoscillatory).

In the case $g(x) = \lambda x$, Eq. (1.1) is called the Euler differential equation and it is well known that all nontrivial solutions are oscillatory if $\lambda > \frac{1}{4}$ and are nonoscillatory if $\lambda \leqslant \frac{1}{4}$. In this case, the number $\frac{1}{4}$ is called the oscillation constant.

A considerable number of studies have been made on the oscillation and nonoscillation of solutions of (1.1) (or (1.5) below) for a long time. Those results can be found in [2–14,18,20–22] and references cited therein.

Sugie and Hara [16] proved that if

$$\frac{g(x)}{x} \geqslant \frac{1}{4} + \frac{\lambda}{\log|x|}, \tag{1.3}$$

for some $\lambda > 0$ and $|x|$ sufficiently large then (1.1) is oscillatory, and if

$$\frac{g(x)}{x} \leqslant \frac{1}{4} + \left(\frac{\lambda}{\log|x|}\right)^2, \tag{1.4}$$

for some λ with $0 < \lambda < \frac{1}{4}$, and $x > 0$ or $x < 0$ with $|x|$ sufficiently large, then (1.1) is nonoscillatory.

Wong [22] studied the equation

$$x'' + a(t)g(x) = 0, \quad t > 0, \tag{1.5}$$

which includes the Emden–Fowler differential equation (as for the Emden–Fowler differential equation, for example, see [3–15,21,22]). Using Sturm's comparison method, he improved the results presented by Sugie and Hara [16]. In fact he proved that if $t^2a(t) \geqslant 1$ for t sufficiently large and there exists $\lambda > \frac{1}{4}$ such that

$$\frac{g(x)}{x} \geqslant \frac{1}{4} + \frac{\lambda}{(\log|x|)^2}, \tag{1.6}$$

for $|x|$ sufficiently large, then (1.5) is oscillatory. Also he proved that if $0 \leqslant t^2a(t) \leqslant 1$ for t sufficiently large,

$$A(t) := \frac{a'(t)}{2a^{\frac{3}{2}}(t)} + 1 = O(1), \quad \text{as } t \to \infty,$$

$A(t) \leqslant 0$, and there exists λ with $0 < \lambda \leqslant \frac{1}{16}$ such that

$$\frac{g(x)}{x} \leqslant \frac{1}{4} + \frac{\lambda}{(\log|x|)^2}, \tag{1.7}$$

for $x > 0$ or $x < 0$, $|x|$ sufficiently large, then (1.5) is nonoscillatory.

Wong [22] expected that if $\frac{1}{16} < \lambda \leqslant \frac{1}{4}$, then Eq. (1.1) with

$$\frac{g(x)}{x} = \frac{1}{4} + \frac{\lambda}{(\log|x|)^2},$$

has both oscillatory and nonoscillatory solutions. But Sugie and Kita in [17] showed that this conjecture is not true and if (1.6) holds for $\lambda > \frac{1}{16}$, then Eq. (1.1) is oscillatory. However, yet they expected that Eq. (1.1) can have both oscillatory and nonoscillatory solutions. In Section 4 we will show that the solutions of (1.1) are either all oscillatory or all nonoscillatory and cannot be both.

In [17] Sugie and Kita considered Eq. (1.5) and proved that if $t^2 a(t) \geqslant 1$ for t sufficiently large and there exists λ with $\lambda > \frac{1}{16}$ such that (1.6) holds for $|x|$ sufficiently large, then (1.5) is oscillatory. Also they proved that if

$$\int_0^x g(\xi)\, d\xi \leqslant \frac{1}{2}x^2, \quad \text{for } x \in \mathbf{R}, \tag{1.8}$$

$0 \leqslant t^2 a(t) \leqslant 1$ for t sufficiently large, and (1.7) holds for $\lambda = \frac{1}{16}$, then (1.5) is nonoscillatory.

Sugie and Yamaoka [19] continued the investigation of oscillation properties of Eq. (1.1). Using phase analysis of a certain Liénard-type system associated with (1.1), they proved the following theorem.

Theorem A. *Let the functions L_n, l_n be defined respectively by $l_1(x) = 2\log x$, $l_{n+1} = \log(l_n(x))$, $L_1 = 1$, $L_{n+1}(x) = L_n(x) l_n(x)$, and let $S_n(x) = \sum_{k=1}^{n} \frac{1}{L_k(x)^2}$. If there exists an integer n such that $\frac{g(x)}{x} \leqslant \frac{1}{4} S_n(|x|)$ for $x > 0$ or $x < 0$, $|x|$ sufficiently large, then all nontrivial solutions of* (1.1) *are nonoscillatory.*

Also in [20] the authors proved that if there exists λ with $\lambda > \frac{1}{4}$ and $n \in N$ such that

$$\frac{g(x)}{x} \geqslant \frac{1}{4} S_{n-1}\big(|x|\big) + \frac{\lambda}{\{l_n(|x|)\}^2}, \tag{1.9}$$

for $|x|$ sufficiently large, then all nontrivial solutions of (1.1) are oscillatory.

In this paper, we present an implicit necessary and sufficient condition for all nontrivial solutions of (1.1) to be oscillatory or nonoscillatory. Then we use our implicit condition to derive explicit conditions and improve the results presented in the above easily.

2. Reduction to the equivalent Liénard system

The change of variable $t = e^s$ reduces Eq. (1.1) to the equation

$$\ddot{x} - \dot{x} + g(x) = 0, \quad s \in \mathbf{R},$$

where $\dot{} = \frac{d}{ds}$. This equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y + x, \\ \dot{y} &= -g(x), \end{aligned} \tag{2.1}$$

which is of Liénard type. Hereafter we denote s by t again.

We say that system (2.1) has property (X^+) in the right half-plane (respectively in the left half-plane), if for every point (x_0, y_0) with $y_0 > -x_0$ and $x_0 \geqslant 0$ (respectively $y_0 < -x_0$ and $x_0 \leqslant 0$), the positive semitrajectory of (2.1) passing through (x_0, y_0) crosses the vertical isocline $y = -x$. Recently, in [1] the authors have presented some sufficient conditions for the property (X^+) in the Liénard systems.

In order to study the oscillation problem for (1.1) the significant point is to find conditions for deciding whether all orbits intersect the vertical isocline $y = -x$ in the equivalent Liénard system (2.1) or not. The following theorem gives a necessary and sufficient condition for the system (2.1) to have property (X^+).

Theorem 2.1. *System* (2.1) *fails to have property* (X^+) *in the right half-plane if and only if there exist a constant* $b \geqslant 0$ *and a function* $\varphi \in C^1(R)$ *such that* $\varphi(x) > 0$, *and*

$$g(x) \leqslant \varphi(x)\big(1 - \varphi'(x)\big), \quad \textit{for } x \geqslant b. \tag{2.2}$$

Proof. *Sufficiency*. Consider the positive semitrajectory of (2.1) starting at a point (b, y_0) with $y_0 > \varphi(b) - b$. This trajectory is considered as a solution $y(x)$ of

$$\frac{dy}{dx} = -\frac{g(x)}{y + x}, \tag{2.3}$$

with $y(b) = y_0$. Suppose that the positive semitrajectory $y(x)$ crosses the vertical isocline $y = -x$. Then it also meets the curve $y = \varphi(x) - x$, at a point $(s, \varphi(s) - s)$ with $s > b$ such that

$$\frac{dy}{dx}(s) = \frac{-g(s)}{\varphi(s)} < \frac{d}{dx}\big(\varphi(x) - x\big)\bigg|_{x=s}.$$

Thus,

$$\frac{-g(s)}{\varphi(s)} < \varphi'(s) - 1.$$

Therefore,

$$g(s) > \varphi(s)\big(1 - \varphi'(s)\big).$$

This is a contradiction. Thus, the trajectory $y(x)$ does not cross the vertical isocline, and therefore system (2.1) fails to have property (X^+).

Necessity. Suppose that system (2.1) fails to have property (X^+) in the right half-plane. Then there exists a positive semitrajectory of (2.1) starting at a point (x_0, y_0) with $x_0 > 0$ and $y_0 > -x_0$, which does not meet the vertical isocline $y = -x$. This trajectory can be regarded as the graph of a function $\psi(x)$ which is a solution of (2.3). Let $(b, \psi(b))$ be a point on the trajectory $\psi(x)$ with $b \geqslant x_0$, and $\varphi(x) = \psi(x) + x$. Then it is clear that $\varphi(x) > 0$, and

$$g(x) = \varphi(x)\big(1 - \varphi'(x)\big),$$

for $x \geqslant b$. Hence, (2.2) is verified and the proof is complete. □

Similarly, the following analogous result is obtained with respect to property (X^+) in the left half-plane.

Theorem 2.2. *System* (2.1) *fails to have property* (X^+) *in the left half-plane if and only if there exist a constant* $b \leqslant 0$ *and a function* $\varphi \in C^1(R)$, *such that* $\varphi(x) > 0$ *for* $x \leqslant b$ *and*

$$g(x) \geqslant -\varphi(x)\big(1 + \varphi'(x)\big), \quad \textit{for } x \leqslant b. \tag{2.4}$$

Corollary 2.1. *Suppose that system* (2.1) *with* g_1 *has* (*fails to have*) *property* (X^+) *in the right half-plane. If*

$$g_2(x) \geqslant g_1(x) \quad \big(g_2(x) \leqslant g_1(x)\big), \quad \text{for } x > R,$$

with sufficiently large R, *then the system* (2.1) *with* g_2 *has* (*fails to have*) *property* (X^+) *in the right half-plane.*

Corollary 2.2. *Suppose that the system* (2.1) *with* g_1 *has* (*fails to have*) *property* (X^+) *in the left half-plane. If*

$$g_2(x) \leqslant g_1(x) \quad \big(g_2(x) \geqslant g_1(x)\big), \quad \text{for } x < -R,$$

with sufficiently large R, *then the system* (2.1) *with* g_2 *has* (*fails to have*) *property* (X^+) *in the left half-plane.*

3. Some lemmas

In this section, we present some lemmas that we will need in Section 4.

Definition 3.1. Let $f_1(x)$ and $f_2(x)$ be real valued functions. By $f_1(x) \succcurlyeq f_2(x)$ we mean that $f_1(x) \geqslant f_2(x)$ for x sufficiently large.

Lemma 3.1. *Let* $\lambda > \frac{1}{16}$ *and* $k > 0$. *Then does not exist a function* $\varphi(x)$ *such that* $\varphi \in C^1(R)$, $\varphi(x) > 0$ *for* x *sufficiently large, and*

$$\varphi(x)\big(1 - \varphi'(x)\big) \succcurlyeq \frac{1}{4}x + \frac{\lambda x}{\log^2(kx)}. \tag{3.1}$$

Proof. Suppose that there exists λ with $\lambda > \frac{1}{16}$ such that

$$\varphi(x)\big(1 - \varphi'(x)\big) \succcurlyeq \frac{1}{4}x + \frac{\lambda x}{\log^2(kx)}. \tag{3.2}$$

Let $\varphi(x) = \frac{1}{2}x + \beta(x)$. From (3.2) we have

$$\frac{1}{2}\beta(x) - \frac{1}{2}x\beta'(x) - \beta(x)\beta'(x) \succcurlyeq \frac{\lambda x}{\log^2(kx)}.$$

Assume $\{x_n\}$ tends to infinity and $\beta(x_n) = 0$, then there exists a sequence $\{c_n\}$ which tends to infinity, $\beta'(c_n) = 0$, and $\beta(c_n) \leqslant 0$. This contradicts with the last inequality. Thus, $\beta(x)$ is eventually positive or negative. Hence, we can let $\varphi(x) = \frac{1}{2}x + \frac{x}{f(x)}$, for $x \geqslant c$ with c sufficiently large, where $f \in C^1(\mathbf{R})$, and $f(x) \neq 0$, for x sufficiently large.

Here we show that $\frac{f(x)}{\log kx}$ is bounded for $x \geqslant c$ with c sufficiently large. We have

$$\varphi(x)\big(1 - \varphi'(x)\big) = \frac{1}{4}x + \frac{x}{f^2(x)}\left(xf'(x)\left(\frac{1}{2} + \frac{1}{f(x)}\right) - 1\right). \tag{3.3}$$

Therefore, by (3.2),

$$f'(x)\left(\frac{1}{2} + \frac{1}{f(x)}\right) \succcurlyeq \frac{1}{x}.$$

Thus,

$$f(x) + 2\log\big(\big|f(x)\big|\big) \geqslant 2\log(kx),$$

for x sufficiently large. Hence, $f(x) \succcurlyeq \lambda \log(kx)$ for every $\lambda < 2$. On the other hand, (3.2) and (3.3) imply

$$\frac{1}{f^2(x)}\left(xf'(x)\left(\frac{1}{2} + \frac{1}{f(x)}\right) - 1\right) \succcurlyeq \frac{1}{16(\log kx)^2}, \tag{3.4}$$

or

$$f'(x)\left(\frac{1}{2f^2(x)} + \frac{1}{f^3(x)}\right) \succcurlyeq \frac{1}{16x(\log kx)^2} + \frac{1}{xf^2(x)}.$$

Hence, for every $y > x \geqslant c$ with c sufficiently large, we have

$$\frac{1}{2f(x)} + \frac{1}{2f^2(x)} - \frac{1}{2f(y)} - \frac{1}{2f^2(y)} \geqslant \frac{1}{16\log kx} - \frac{1}{16\log ky} + \int_x^y \frac{1}{\eta f^2(\eta)}\,d\eta.$$

Since $f(y)$ tends to infinity as $y \to \infty$,

$$\mu\log(kx) \succcurlyeq f(x), \quad \text{for every } \mu > 8.$$

Therefore, $h(x) = \frac{f(x)}{\log(kx)}$ is bounded for $x \geqslant c$ with c sufficiently large.

Now we show that $\lim_{x\to\infty} h(x) = \frac{f(x)}{\log(kx)}$ exists. From (3.4) we have

$$\begin{aligned}&\frac{x}{h^2(x)}h'(x)\log(kx)\left(\frac{1}{2} + \frac{1}{h(x)\log(kx)}\right)\\ &\quad + \frac{1}{h(x)}\left(\frac{1}{2} + \frac{1}{h(x)\log(kx)}\right) - \frac{1}{h^2(x)} \succcurlyeq \frac{1}{16}.\end{aligned} \tag{3.5}$$

Suppose that sequence x_n tends to infinity and $h'(x_n) = 0$. Then from (3.5) we have

$$\left(1 - \frac{h(x_n)}{4}\right)^2 \leqslant \frac{1}{\log(kx_n)},$$

for n sufficiently large. Thus,

$$\lim_{n\to\infty} h(x_n) = 4.$$

Since h' vanishes in the extremum points, if $h(x)$ is not eventually increasing or decreasing,

$$\limsup_{x\to\infty} h(x) = \liminf_{x\to\infty} h(x) = 4.$$

Therefore, $\lim_{x\to\infty} h(x)$ exists and we can let $f(x) = \alpha\log(kx) + p(x)$, where $\lim_{x\to\infty} \frac{p(x)}{\log(kx)} = 0$ and $\alpha \in [2, 8]$.

Let

$$\psi(x) = \frac{1}{2}x + \frac{x}{\alpha\log(kx)}. \tag{3.6}$$

Then we have

$$\begin{aligned}k(x) &= \varphi(x)\big(1-\varphi'(x)\big) - \psi(x)\big(1-\psi'(x)\big)\\ &= \left(\frac{1}{2}x + \frac{x}{\alpha\log(kx)+p(x)}\right)\left(\frac{1}{2} - \left(\frac{x}{\alpha\log(kx)+p(x)}\right)'\right)\\ &\quad - \left(\frac{1}{2}x + \frac{x}{\alpha\log(kx)}\right)\left(\frac{1}{2} - \left(\frac{x}{\alpha\log(kx)}\right)'\right)\\ &= xp'(x)\left(\frac{1}{(\alpha\log(kx)+p(x))^3} + \frac{1}{2(\alpha\log(kx)+p(x))^2}\right) + O\left(\frac{x}{\log^2(kx)}\right).\end{aligned}$$

Since $\lim_{x\to\infty}\frac{p(x)}{\log(kx)} = 0$, there exists a sequence $\{x_n\}$ which tends to infinity and $x_n p'(x_n) \leqslant 1$. Thus, for every $\lambda > 0$,

$$k(x_n) < \frac{\lambda x_n}{\log^2(kx_n)},$$

for n sufficiently large. To complete the proof it is sufficient to show that for every $\lambda > \frac{1}{16}$ and $\alpha \in [2, 8]$,

$$\psi(x)\big(1-\psi'(x)\big) \preccurlyeq \frac{1}{4}x + \frac{\lambda x}{\log^2(kx)},$$

where $\psi(x)$ is defined by (3.6). We have

$$\begin{aligned}\psi(x)\big(1-\psi'(x)\big) &= \left(\frac{1}{2}x + \frac{x}{\alpha\log(kx)}\right)\left(\frac{1}{2} - \left(\frac{x}{\alpha\log(kx)}\right)'\right)\\ &= \frac{1}{4}x + \left(\frac{1}{2\alpha} - \frac{1}{\alpha^2}\right)\frac{x}{\log^2(kx)} + \frac{x}{\alpha^2\log^3(kx)}\\ &\leqslant \frac{1}{4}x + \frac{x}{16\log^2(kx)} + \frac{x}{\alpha^2\log^3(kx)}\\ &\preccurlyeq \frac{1}{4}x + \frac{\lambda x}{\log^2(kx)}, \quad \text{for every } \lambda > \frac{1}{16}.\end{aligned}$$

This contradicts with (3.2) and the proof is complete. □

In [16] the authors went into details about asymptotic behavior of (2.1) and proved the following lemmas.

Lemma 3.2. *For each point $P = (p, -p)$ with $p > 0$, the positive semitrajectory of* (2.1) *crosses the negative y-axis.*

Lemma 3.3. *For each point $P = (-p, p)$ with $p > 0$, the positive semitrajectory of* (2.1) *crosses the positive y-axis.*

4. Oscillation theorems

In this section, we present necessary and sufficient conditions for all nontrivial solutions of (1.1) to be oscillatory or nonoscillatory. The main theorem is as follows.

Theorem 4.1. *All nontrivial solutions of* (1.1) *are nonoscillatory if there exist a constant* $R > 0$ *and a function* $\varphi \in C^1(\mathbf{R})$ *such that*

$$\varphi\big(|x|\big) > 0 \quad \text{and} \quad \frac{g(x)}{x} \leqslant \frac{\varphi(|x|)(1-\varphi'(|x|))}{|x|}, \tag{4.1}$$

for $x > R$ *or* $x < -R$. *Otherwise, all nontrivial solutions of* (1.1) *are oscillatory. Therefore, the solutions of second-order nonlinear differential equation* (1.1) *are all oscillatory or all nonoscillatory and cannot be both.*

Proof. First suppose that (4.1) does not hold. Then the system (2.1) which is equivalent to (1.1) has property (X^+) in the right and left half-plane. Thus, it follows from Lemmas 3.1 and 3.2 that every solution of (2.1) keeps on rotating around the origin except the zero solution. Hence, all nontrivial solutions of (1.1) are oscillatory.

Now suppose that (4.1) holds. Then by Theorem 2.1 or 2.2 the system (2.1) fails to have property (X^+) in the right or left half-plane. We consider only the case that (2.1) fails to have property (X^+) in the right half-plane, because the other case is carried out in the same way. Suppose that the system (2.1) fails to have property (X^+) in the right half-plane. Hence, there exists a point $P_0(x_0, y_0)$ with $x_0 \geqslant 0$ and $y_0 > -x_0$ such that the positive semitrajectory of (2.1) runs to infinity without intersecting the curve $y = -x$. With the same argument in [16, Proof of Theorem 2.1] we can conclude that the system (2.1) cannot have any oscillatory solution. The proof is complete. □

In [16] the authors, judging from the oscillation result for Euler's equation, expected that all nontrivial solutions of (1.1) have a tendency to be oscillatory as $g(x)$ grows larger in some sense. The following corollary of Theorem 4.1 gives a mathematical proof for their statement.

Corollary 4.1. *Suppose that all nontrivial solutions of the system* (1.1) *with* g_1 *are oscillatory. If*

$$\frac{g_2(x)}{x} > \frac{g_1(x)}{x},$$

for $|x| > R$ *with a sufficiently large* R, *then all nontrivial solutions of* (1.1) *with* g_2 *are oscillatory.*

Corollary 4.2. *Suppose that all nontrivial solutions of the system* (1.1) *with* g_1 *are nonoscillatory. If*

$$\frac{g_2(x)}{x} < \frac{g_1(x)}{x},$$

for $|x| > R$ *with a sufficiently large* R, *then all nontrivial solutions of* (1.1) *with* g_2 *are nonoscillatory.*

The following theorem gives an explicit sufficient condition for all nontrivial solutions of (1.1) to be nonoscillatory.

Theorem 4.2. *For* $n \in N$, $k > 0$, *and* x *sufficiently large, let*

$$T_n^k(x) = 1 + \frac{1}{4}\sum_{j=1}^{n-1} \frac{1}{(\log kx \times \log\log kx \times \cdots \times \underbrace{\log\log\cdots\log}_{j\text{-times}} kx)^2}.$$

If there exist an integer n and $k > 0$ such that $\frac{g(x)}{x} \leqslant \frac{1}{4} T_n^k(|x|)$, for $x > 0$ or $x < 0$, $|x|$ sufficiently large, then all nontrivial solutions of Eq. (1.1) *are nonoscillatory.*

Proof. Let

$$\varphi_n(x) = \frac{1}{2}x + \frac{1}{4}\sum_{j=1}^{n-1} \frac{x}{\log kx \times \log\log kx \times \cdots \times \underbrace{\log\log\cdots\log}_{j\text{-times}} kx}. \tag{4.2}$$

Then we have

$$1 - \varphi_n'(x) = \frac{1}{2} - \frac{1}{4}\underbrace{\sum_{j=1}^{n-1} \frac{1}{\log kx \times \log\log kx \times \cdots \times \underbrace{\log\log\cdots\log}_{j\text{-times}} kx}}_{N(x)}$$

$$+\frac{1}{4}\underbrace{\sum_{j=1}^{n-1}\sum_{i=1}^{j} \frac{1}{(\log kx \times \log\log kx \times \cdots \times \underbrace{\log\log\cdots\log}_{i\text{-times}} kx)^2 \times \underbrace{\log\log\cdots\log}_{(i+1)\text{-times}} kx \times \cdots \times \underbrace{\log\log\cdots\log}_{j\text{-times}} kx}}_{M(x)}.$$

It is easy to verify that

$$T_n^k(x) = 1 + \frac{1}{4}\big(2M(x) - \big(N(x)\big)^2\big).$$

Thus,

$$\begin{aligned}\frac{\varphi_n(|x|)}{|x|}\big(1 - \varphi_n'\big(|x|\big)\big) &= \frac{1}{4} - \frac{1}{16}\big(N\big(|x|\big)\big)^2 + \frac{1}{8}M\big(|x|\big) + \frac{1}{16}N\big(|x|\big)M\big(|x|\big) \\ &= \frac{1}{4}T_n^k\big(|x|\big) + \frac{1}{16}N\big(|x|\big)M\big(|x|\big) > \frac{1}{4}T_n^k\big(|x|\big) \geqslant \frac{g(x)}{x},\end{aligned} \tag{4.3}$$

for $|x|$ sufficiently large. Hence, by Theorem 4.1 all nontrivial solutions of Eq. (1.1) are nonoscillatory. □

Remark 4.1. Let the functions W_n, w_n be defined respectively by $w_1(x) = 2\log x$, $w_{n+1} = \log(w_n(x))$, $W_1 = 1$, $W_{n+1}(x) = W_n(x)w_n(x)$. Define $S_n(x) = \sum_{k=1}^{n} \frac{1}{W_k(x)^2}$. Then we have

$$S_n\big(|x|\big) < T_n^1\big(|x|\big),$$

for $|x|$ sufficiently large, thus, the main theorem in [19] follows from Theorem 4.2.

Let

$$L_n(x) = \log kx \times \log\log kx \times \cdots \times \underbrace{\log\log\cdots\log}_{(n-1)\text{-times}} kx, \quad \text{for } n > 1. \tag{4.4}$$

Notice that

$$L_n'(x) = \sum_{j=1}^{n-1} \frac{L_n(x)}{x(\log kx \times \log\log kx \times \cdots \times \underbrace{\log\log\cdots\log}_{j\text{-times}} kx)}$$

and

$$\lim_{x\to\infty} xL'_n(x)N(x) = 0.$$

We will need the following lemma in the proof of Theorem 4.3.

Lemma 4.1. *Let* $f \in C^1(\mathbf{R})$. *Suppose that* $\lim_{x\to\infty}|f(x)| = \infty$ *and*

$$-\frac{1}{f(x)}\left(\frac{N(x)}{2} + \frac{1}{f(x)}\right) + \frac{f'(x)x}{f^2(x)}\left(\frac{1}{2} + \frac{N(x)}{4} + \frac{1}{f(x)}\right) \succcurlyeq \frac{\lambda}{(L_{n+1}(x))^2}, \quad \lambda > \frac{1}{16}. \tag{4.5}$$

Then $\lim_{x\to\infty} \frac{f(x)}{L_{n+1}(x)} = 4$.

Proof. Let $g(x) = \frac{f(x)}{L_{n+1}(x)}$. Then from (4.5) we have

$$\frac{x(g'(x)L_{n+1}(x) + g(x)L'_{n+1}(x))}{g^2(x)}\left(\frac{1}{4}N(x) + \frac{1}{g(x)L_{n+1}(x)}\right) + \frac{xg'(x)L_{n+1}(x)}{2g^2(x)} \succcurlyeq \lambda + \frac{1}{g^2(x)} - \frac{1}{2g(x)}. \tag{4.6}$$

Suppose that $\{x_n\}$ tends to infinity and $g'(x_n) \leqslant 0$, then from (4.6) we have

$$x_n g(x_n)L'_{n+1}(x_n)\left(\frac{1}{4}N(x_n) + \frac{1}{g(x_n)L_{n+1}(x_n)}\right) \succcurlyeq \left(1 - \frac{g(x_n)}{4}\right)^2.$$

It is easy to check that $\limsup_{n\to\infty}|g(x_n)| < \infty$. Thus, the left-hand side of the last inequality tends to 0 as $n \to \infty$. Therefore, if $g(x)$ is not eventually increasing, then with the same argument as in the proof of Lemma 3.1 we have

$$\limsup_{x\to\infty} g(x) = \liminf_{x\to\infty} g(x) = 4.$$

Now assume $g'(x) > 0$, for x sufficiently large. Since $N(x)$ and $\frac{1}{g(x)L_{n+1}(x)} = \frac{1}{f(x)}$ tend to 0, as $x \to \infty$, from (4.6) we have

$$xg'(x)L_{n+1}(x) \geqslant \left(\frac{g(x)}{4} - 1\right)^2 - xg(x)L'_{n+1}(x)\left(\frac{1}{4}N(x) + \frac{1}{g(x)L_{n+1}(x)}\right), \tag{4.7}$$

for x sufficiently large. Assume $\lim_{x\to+\infty} g(x) \neq 4$. Since

$$\lim_{x\to\infty}\left(xL'_{n+1}(x)\left(\frac{1}{4}N(x) + \frac{1}{g(x)L_{n+1}(x)}\right)\right) = 0,$$

from (4.7) we conclude that there exists c with $0 < c < 1$ such that

$$\frac{g'(x)}{\left(\frac{g(x)}{4} - 1\right)^2} \geqslant \frac{c}{xL_{n+1}(x)},$$

for x sufficiently large. Thus,

$$4\left(\frac{1}{1 - \frac{1}{4}g(x)} - \frac{1}{1 - \frac{1}{4}g(y)}\right) \geqslant \int_y^x \frac{c}{\xi L_{n+1}(\xi)}\,d\xi = \underbrace{\log\log\cdots\log}_{(n+1)\text{-times}} kx - \underbrace{\log\log\cdots\log}_{(n+1)\text{-times}} ky,$$

for every $x > y$ with y sufficiently large. Since the right-hand side of the last inequality tends to infinity as $x \to \infty$, $\lim_{x\to\infty} g(x) = 4$. This is a contradiction and the proof is complete. □

The following theorem gives a sufficient condition for all nontrivial solutions of the system (1.1) to be oscillatory. Notice that this theorem and Theorem 4.2 present very sharp conditions for the oscillation of solutions of the system (1.1).

Theorem 4.3. *Let* (1.2) *holds. Suppose that there exist* λ *with* $\lambda > \frac{1}{16}$, $k > 0$, *and* $n \in N$ *such that*

$$\frac{g(x)}{x} \geqslant \frac{1}{4} T_{n-1}^k(x) + \frac{\lambda}{(L_n(x))^2} \tag{4.8}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of* (1.1) *are oscillatory.*

Proof. Suppose that there exist a function $\varphi \in C^1(\mathbf{R})$ with $\varphi(x) > 0$ for x sufficiently large, and $\lambda > \frac{1}{16}$ such that

$$\frac{\varphi(x)(1-\varphi'(x))}{x} \geqslant \frac{1}{4} T_{n-1}^k(x) + \frac{\lambda}{(L_n(x))^2}. \tag{4.9}$$

We claim that if (4.9) holds, then $\varphi(x) = \varphi_{n-1}(x) + \frac{x}{f(x)}$, where $\varphi_{n-1}(x)$ is defined by (4.2), and

$$\lim_{x\to\infty} \frac{f(x)}{L_n(x)} = 4. \tag{4.10}$$

By Lemma 3.1 our claim is true for $n = 1$. Suppose the claim is true for n, and (4.9) holds for $n+1$. Let $\varphi(x) = \varphi_n(x) + \beta(x)$. Using the same argument as in the beginning of the proof of Lemma 3.1 we conclude that $\beta(x) \neq 0$, for x sufficiently large. Thus, we can let $\varphi(x) = \varphi_n(x) + \frac{x}{f(x)}$, where $f \in C^1(\mathbf{R})$, and $f(x) \neq 0$, for x sufficiently large. From (4.3) we have

$$\begin{aligned}\frac{\varphi(x)(1-\varphi'(x))}{x} &= \frac{1}{4} T_n^k(x) + \frac{1}{16} N(x) M(x) \\ &\quad - \frac{1}{f(x)}\left(\frac{N(x)}{2} - \frac{M(x)}{4} + \frac{1}{f(x)}\right) + \frac{f'(x)x}{f^2(x)}\left(\frac{1}{2} + \frac{N(x)}{4} + \frac{1}{f(x)}\right) \\ &\geqslant \frac{1}{4} T_n^k(x) + \frac{\lambda}{(L_{n+1}(x))^2}, \quad \lambda > \frac{1}{16}.\end{aligned} \tag{4.11}$$

Hence,

$$\begin{aligned}&-\frac{1}{f(x)}\left(\frac{N(x)}{2} + \frac{1}{f(x)}\right) + \frac{f'(x)x}{f^2(x)}\left(\frac{1}{2} + \frac{N(x)}{4} + \frac{1}{f(x)}\right) \\ &\quad > \frac{\lambda'}{(L_{n+1}(x))^2}, \quad \lambda > \lambda' > \frac{1}{16},\end{aligned}$$

for x sufficiently large. We have

$$\varphi(x) = \varphi_{n-1}(x) + \frac{x}{4L_n(x)} + \frac{x}{f(x)} = \varphi_{n-1}(x) + \frac{x}{h(x)},$$

where

$$h(x) = \frac{4f(x)L_n(x)}{f(x) + 4L_n(x)}.$$

Since the claim is true for n,

$$\lim_{x\to\infty}\frac{f(x)}{f(x)+4L_n(x)}=\lim_{x\to\infty}\frac{h(x)}{4L_n(x)}=1.$$

Hence, $\lim_{x\to\infty}|f(x)|=\infty$, and by Lemma 4.1 $\lim_{x\to\infty}\frac{f(x)}{L_{n+1}(x)}=4$. Therefore, $f(x)=4L_{n+1}(x)+p(x)$, where

$$\lim_{x\to\infty}\frac{p(x)}{L_{n+1}(x)}=0.$$

With the same argument as in Lemma 3.1, we conclude that for every $\lambda>0$ there exists a sequence $\{x_n\}$ which tends to infinity and

$$\frac{\varphi(x_n)(1-\varphi'(x_n))}{x_n}-\frac{\varphi_{n+1}(x_n)(1-\varphi'_{n+1}(x_n))}{x_n}<\frac{\lambda}{(L_{n+1}(x_n))^2}. \tag{4.12}$$

From (4.3) we conclude that (4.12) contradicts with (4.9). Thus, by Theorem 4.1 all nontrivial solutions of (1.1) are oscillatory. □

Example 4.1. Suppose that $xg(x)>0$, for $x\neq 0$, and

$$\frac{g(x)}{x}=\frac{1}{4}+k\cos(x)-\frac{1}{x},\quad |k|<\frac{1}{4},$$

for $x>0$, x sufficiently large. Let

$$\varphi(x)=\frac{1}{2}x-2k\sin(x).$$

Then

$$\frac{g(x)}{x}\leqslant\frac{\varphi(x)(1-\varphi'(x))}{x},$$

and by Theorem 4.1, all nontrivial solutions of Eq. (1.1) are nonoscillatory.

Notice, in this example

$$\liminf_{|x|\to\infty}\frac{g(x)}{x}=\frac{1}{4}-|k|<\frac{1}{4}<\limsup_{|x|\to\infty}\frac{g(x)}{x}=\frac{1}{4}+|k|,$$

and all of the results presented in the previous literature are inapplicable to this example.

5. Extension to general second-order nonlinear differential equations

Here we consider the equation

$$x''+a(t)g(x)=0, \tag{5.1}$$

and give oscillation and nonoscillation theorems which extend Theorem 4.1 for the system (5.1). Here we assume that $a(t)$ and $g(x)$ satisfy suitable smoothness conditions for the uniqueness of solutions of the initial value problem and

$$xg(x)>0,\quad \text{for } x\neq 0. \tag{5.2}$$

As is well known, the uniqueness of solutions of (5.1) is guaranteed if $a(t)$ is continuous and $g(x)$ is locally Lipschitz continuous. We also assume that every solution of (5.1) exists in the future.

Using the Sturm's comparison method we can easily extend Theorem 4.1. Sugie and Yamaoka [20] proved that if $t^2a(t) \geqslant 1$ for t sufficiently large, then Eq. (5.1) is oscillatory if and only if Eq. (1.1) is oscillatory. Thus, we have the following theorem.

Theorem 5.1. *Suppose* (5.2) *holds,* $a(t)$ *satisfies*

$$t^2a(t) \geqslant 1,$$

for t *sufficiently large, and does not exist function* $\varphi \in C^1(\mathbf{R})$ *such that* $\varphi(|x|) > 0$, *and*

$$\frac{g(x)}{x} \leqslant \frac{\varphi(|x|)(1-\varphi'(|x|))}{|x|}, \tag{5.3}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of* (5.1) *are oscillatory.*

Comparing solutions of (5.1) and (1.1), and using the same argument as in [17, Theorem 5.2] we can prove the following nonoscillation theorem for Eq. (5.1).

Theorem 5.2. *Assume* (5.2) *and* (1.8) *hold. Suppose* $a(t)$ *satisfies*

$$0 \leqslant t^2a(t) \leqslant 1,$$

for t *sufficiently large, and there exists a function* $\varphi \in C^1(\mathbf{R})$ *such that* $\varphi(|x|) > 0$, *and*

$$\frac{g(x)}{x} \leqslant \frac{\varphi(|x|)(1-\varphi'(|x|))}{|x|}, \tag{5.4}$$

for $x < 0$ *or* $x > 0$, $|x|$ *sufficiently large. Then all nontrivial solutions of* (5.1) *are nonoscillatory.*

Example 5.1. Suppose that $a(t)$ is a continuously differentiable function such that $a(t) = t^\alpha$ for t sufficiently large and $\alpha \leqslant -2$. Also assume that $xg(x) > 0$, for $x \neq 0$ and

$$\frac{g(x)}{x} = \frac{1}{4} + k\sin(x) - \frac{1}{x}, \quad |k| < \frac{1}{4},$$

for $x > 0$, x sufficiently large. Let $\varphi(x) = \frac{1}{2}x + 2k\cos(x)$. It is easy to check that (5.4) holds for $x > 0$ sufficiently large. Hence, by Theorem 5.2 all nontrivial solutions of (5.1) are nonoscillatory.

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