One strike against the min-max degree triangulation problem

Klaus Jansen*

Fachbereich IV, Mathematik und Informatik, Universität Trier, Postfach 3825, W-5500 Trier, Germany

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Abstract


In this paper we analyze the computational complexity of the min-max degree triangulation problem. The problem arises in the generation of two-dimensional meshes for plane objects. We show that the problem to triangulate a plane geometric graph with degree at most seven is NP-complete.

1. Introduction

First, we give some definitions. Let $V$ be a set of $n$ points in $\mathbb{R}^2$. An edge is a closed line segment connecting two points of $V$. Let $E$ be a set of edges. Then $G = (V, E)$ is a geometric graph if for every edge $ab \in E$, $ab \cap V = \{a, b\}$. A geometric graph is called plane if for every two edges $ab \neq cd$ in $E$, either $ab \cap cd = \emptyset$ or $ab \cap cd$ is an endpoint of both edges.

The connected components of $\mathbb{R}^2$ minus all points in $V$ and on edges of $E$ are the faces of $G$. If the edges in $E$ are pairwise disjoint, then $G$ is a matching and we have only one unbounded face. If $V$ is fixed and $E$ is maximal such that no two edges cross, then $G$ is a geometric triangulation of the convex hull of $V$. Then, the bounded faces of a triangulation are triangles. A triangulation of a geometric graph $G = (V, E)$ is a plane geometric graph $G' = (V, E')$, where $E \subseteq E'$ and

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where $G'$ is a geometric triangulation. The degree $\delta(v)$ of a point $v$ of a graph is the number of incident edges. Let $\Delta(G)$ be the maximum degree over all points in $G = (V, E)$.

The problem studied in this paper is described as follows. Let $G = (V, E)$ be a plane geometric graph. The problem is to find a triangulation $G'$ of $G$ that minimizes $\Delta(G')$. Clearly, for a triangulation of this form it is not allowed to add new points. The decision problem has the following form.

**Problem.** Min-max degree triangulation

Given. A plane geometric graph $G = (V, E)$ in $\mathbb{R}^2$ with finite point set $V$ and edge set $E$, and an integer $k \in \mathbb{N}$.

Question. Is there a triangulation $G'$ of $G$ with maximum degree $\Delta(G') \leq k$?

This problem was raised as an open problem by Edelsbrunner [4]. We note that for any triangulation $G'$ of a plane geometric graph $G = (V, E)$ with $|V| \geq 5$, $\Delta(G') \geq 4$ holds and that for any integer $n \geq 5$ there exist triangulated graphs $G' = (V, E')$ with $|V| = n$ and $\Delta(G') = 4$. An application of our studied problem in numerical engineering is given by Frey and Field [5].

The problem to find a triangulation $\tilde{G} = (V, \tilde{E})$ as a subset of a geometric graph $G = (V, E)$ with $\tilde{E} \subseteq E$ is studied by Lloyd [9]. Using a reduction from 3-SAT he showed that this triangulation problem is NP-complete. Given a plane geometric graph with or without constraining edges, several optimal triangulation problems have been studied [1-3]. Optimal means that the form of the triangles of the triangulations is optimized. In contrast to polynomial algorithms in [1-3], we give the first negative result for an optimal triangulation problem.

The NP-completeness of a similar problem to triangulate a planar graph while minimizing the maximum degree has been proved by Kant and Bodlaender [7]. One difference in [7] to our considered problem and to the studied triangulation problems in [1-3] is that an embedding of the graph in the plane is not given in the problem instance. The second important difference is that the constructed lines in the triangulation in [7] are not straight lines.

## 2. NP-completeness

In this section we show that the max-degree triangulation problem is NP-complete. We use a reduction from the planar 3-SAT problem which is shown to be NP-complete by Lichtenstein [8]. A general formula $\phi$ in 3-SAT is given by a set of clauses $C = \{c_1, \ldots, c_m\}$ over a set $X = \{x_1, \ldots, x_n\}$ of variables. Each clause $c_i$ is a set of at most three literals from the set $X = \{x_1, \ldots, x_n\}$ and $\overline{X} = \{\overline{x}_1, \ldots, \overline{x}_n\}$. For convenience, a clause will be written as $(a \vee b \vee c)$ instead of $\{a, b, c\}$ and a formula will be written as $(c_1 \wedge \cdots \wedge c_m)$ instead of $\{c_1, \ldots, c_m\}$. A formula $\phi$ is satisfiable if there is a truth assignment to the
variables such that each clause is satisfied. We have a planar 3-SAT formula iff the undirected graph $G_\phi = (X \cup C, \{x, c \mid x \in c \in C \text{ or } \overline{x} \in c \in C\})$ is planar. For our reduction we use a restricted version of planar 3-SAT which is also NP-complete [8].

**Problem.** Restricted planar 3-SAT

*Given.* A formula $\phi$ with a set $C$ of clauses over a set $X$ of variables that satisfies the following three conditions:

(i) Each clause contains at most three and at least two literals.

(ii) Each variable occurs in at most three and at least two clauses, where we count $x$ as well as $\overline{x}$ as an occurrence of $x \in X$.

(iii) The undirected graph $G_\phi$ is planar.

*Question.* Is $\phi$ satisfiable?

Now we give an overview of the reduction. Given a planar graph $G_\phi$ for a formula $\phi$ we compute in the first step of the reduction a rectilinear planar layout with horizontal lines for vertices and vertical lines for edges. Then we grow the lines to rectangles of unit height or width.

In the second step we construct blocks that represent variables, complemented variables and clauses. These are placed inside the corresponding horizontal rectangles. For each variable $x \in X$ we generate a block that allows us to assign a truth value by choosing a triangulation of the block. To get a complemented variable we use an inverter. For each clause $c \in C$ we generate a block that can be triangulated with maximum degree seven or less if and only if at least one of the corresponding literals has the truth value true. For moving the information between a literal and a clause we use a sequence of squares.

We simplify the construction in the second step and give only a part of each block with points of small degree. To understand the function of the block designs, we assume that the used points in the blocks have a larger degree five or six. To get these degrees for the points, we add in the last step of the reduction for each block some new points connected to the old ones. Furthermore, the new points allow us to triangulate the regions between the block designs with degree less than eight.

**Theorem 2.1.** The min-max degree triangulation is NP-complete for $k = 7$.

**Proof.** Clearly, the triangulation problem is in NP. In the following we give the details of the three reduction steps from restricted planar 3-SAT.

**Step 1.** Given the planar graph $G_\phi$ (for short $G$), we compute a rectilinear planar layout. This layout maps vertices of $G$ to horizontal line segments and edges to vertical line segments with all endpoints at positive integer coordinates. Two horizontal line segments are connected by a vertical one iff the corresponding vertices are adjacent in $G$. An example for this transformation is given in Fig. 1.
For a planar graph with \( n \) vertices a layout can be computed in \( O(n) \) time. The height of the layout is at most \( n \) and the width is at most \( 2n - 4 \). For details regarding the algorithm used to obtain a rectilinear planar layout, we refer to [10].

For our reduction we use a modified type of a layout which can be obtained directly from the planar rectilinear layout.

(i) Vertices are mapped to disjoint axes parallel rectangles of height one with integer endpoints.

(ii) Edges are mapped to disjoint axes parallel rectangles of width one with integer endpoints.

(iii) Two horizontal rectangles are connected by a vertical rectangle iff the corresponding vertices are adjacent. In this case the vertical rectangle touches the upper horizontal rectangle at the lower side and the lower horizontal rectangle at the upper side.

(iv) No horizontal line intersects more than one horizontal rectangle.
Such a type of a layout can be obtained by stretching the horizontal and vertical lines and by thickening them to unit height or width rectangles. Clearly, this step can be done in polynomial time. For our example in Fig. 1 we get the layout illustrated in Fig. 2.

**Step 2.** In this step we describe the blocks for connections, variables, inverters, clauses and shifters.

*Connections.* Let us first consider the connections between variables and clauses. Variables and clauses correspond to the horizontal rectangles and the connections to the vertical rectangles. In each of these vertical rectangles we put a sequence or path of squares, as illustrated in Fig. 4.

We see that each point in such a path has degree 3. In Step 3 we will add points to the left and right of a path, and connect these points to the ones of the path, so that the latter have degree 6 each. For now we just assume that all points of a path have degree 6. Under this assumption the squares of the path can be triangulated so that the points have degree at most seven in only two ways, see Fig. 5.

The first way is to take all diagonals from the left lower point to the right upper point, and the other way is to take the opposite diagonals. The choice corresponds to the truth value of the incident literal. In the following we assume that the diagonals of the first form represent *true* and the other *false* (see also Fig. 3).

*Variable block.* Now we consider a variable $x \in X$. If $x$ has degree three we construct a block as illustrated in Fig. 6 or 7. Then, we place this block into the
Fig. 5. Two ways to triangulate a path of squares.

Fig. 6. The variable setting with three variables on one side.

Fig. 7. The variable setting with two variables above and one below.
corresponding horizontal rectangle of the layout. For a variable with degree two we use similar block designs; see Fig. 8. For variables \( x \in X \) with degree three we have to distinguish two cases. In the first case all three clauses \( c \in C \) with \( x \in C \) or \( \bar{x} \in C \) lie above (or below) \( x \) in the rectilinear layout. In this case we use the block design of Fig. 6. For the other case with two clauses above and one below (or two below and one above) \( x \) we take the design of Fig. 7. In both cases, the block is stretched and placed so that the incident vertical paths inside the vertical rectangles connect properly.

To save some redundance we only consider the block design of Fig. 6 and assume that all points of the inner cycle (these are the white points in Fig. 9) and of the three paths have degree six, see Step 3. Moreover, we assume that the region bounded by the inner cycle is triangulated in Step 3. Again there are only two ways to triangulate the rectangles around the inner cycle, so that all points have degree seven or less. Both ways are shown in Fig. 9 and Fig. 10.

**Inverter block.** We see that in each path we have only diagonals of the first
form with truth-value *true*, or only diagonals with truth-value *false*. Since we have
variables as well as complemented variables in a clause we need an *inverter* that
transforms diagonals of the first form into the second and diagonals of the second
form into the first. A block design for an inverter is given in Fig. 11. Similar as for
a path or a variable setting there are only two ways to triangulate the component
with degree of at most seven; we apply the same argument that the points in the
inner cycle and in the paths have degree six (using additional points given in the
third step). Both ways are shown in Fig. 12.

In addition to the block for variables we pack some inverters in the
corresponding horizontal rectangle. Each edge $e = (y, c)$ in the planar graph $G_\phi$
corresponds to a variable $y = x \in X$ with $x \in c$ or to a complemented variable
$y = \bar{x} \in \bar{X}$ with $\bar{x} \in c$. In the second case, we add an inverter at the top or bottom
of the corresponding path for the variable $x$. We assume that any horizontal line
cuts at most one block and at most one inverter.

*Clause block.* The last construction is a block design for a clause. We have to
consider clauses with two or three literals. Moreover, we have to distinguish two
cases with all literals above (or below) the clause or with one or two literals on one side and one on the other. The designs are illustrated in Figs. 13 and 14.

In the following we consider only the left clause construction with three literals in Fig. 13 and assume that the points \( a \), \( b \) and \( c \) have degree five and all other points have degree six. Then, there are at most eight possible triangulations of such a component with maximum degree at most seven. Each triangulation corresponds to a choice of the truth values for the literals in the corresponding three paths which are connected at \( a \), \( b \) and \( c \).

In Fig. 15 three of the possible triangulations are shown. In the leftmost triangulation the middle vertex, \( b \), has degree eight. In the other two and also in the remaining five triangulations all points have degree at most seven. To get a useful clause construction we invert the second signal of the right upper path. This means that the block design for a clause with three literals is extended by one inverter which we add at the top of the right upper path. It is clear that three
false signals give a triangulation with degree eight at $b$ and all other signal combinations generate triangulations with degree at most seven.

**Shifter.** In many cases, it is not possible to connect the paths in the vertical rectangles directly with a clause design. Therefore, it is necessary to break a path and shift one part to the left or right. Such a shift is illustrated in Fig. 16. At most one shift suffices for each path so it can be connected to a clause block. Moreover, we can assume that no horizontal line cuts more than one shift construction. In total, one clause block consists of a design as in Fig. 13, an inverter, and some shifts which we pack within in the corresponding horizontal rectangle.

In Fig. 17 we have given the layout for the formula $\phi = (a \lor b \lor c) \land (\bar{a} \lor d) \land (\bar{b} \lor c)$ and the triangulated paths between the variables $a, b, c, d$ and the clauses 1, 2 and 3. Consider for instance the variable $a$ and the corresponding paths to clauses 1 and 2. Since we have in the first clause the variable $a$ and in the second clause the complemented variable $\bar{a}$, an inverter in the rectangle corresponding to

Fig. 14. Two constructions for a clause with two literals.

Fig. 15. Three of eight configurations for a clause of the first form.
the variable $a$ inverts the signal and we get two different forms of diagonals. The truth values of the variables chosen in Fig. 17 are: $t(a) = t(b) = t(d) = \text{true}$ and $t(c) = \text{false}$.

**Step 3.** The last step adds new points to get the necessary degree five or six at the points in each block design. First, we consider the simplest example: a path of squares inside a vertical rectangle. To get degree six at each point we add to the left and right of the path a sequence of points and connect each old point with three new points. The outside region, now bounded by new points and edges, must eventually be triangulated too. For this reason we place new points in a uniform and symmetric fashion to the left and right of each path. An illustration of the positions of the old points (colored black) and new points (colored white) is given in Fig. 18. Using this construction the old points have degree six and the new points have degree three or four.

A more complicated construction is needed for a variable block. We consider the case with three literals on one side. First, we add eight points in the inner cycle to generate degree six at each old point of the cycle. As before, the old points in Fig. 19 are black and the new points are white. Next, we consider the region between two variable paths. The construction generates degree six at each old point of the paths, see Fig. 20. The white new points have degree three and four and only the both points at the bottom have degree five. Therefore, it is clear that the region between the two paths can be triangulated with degree at most six (take a horizontal line and a diagonal one after another). The last part is the connection of the outside of a variable block to other points to the left and right.

Since there is at most one variable block in every horizontal rectangle, only paths of squares in vertical rectangles can lie to its left and right. Let us assume that there is a path to its left and also to its right. This can be guaranteed if we add two independent paths of squares to the left and to the right side of the entire construction. We can then add a sequence of white points outside the variable block and a sequence outside the paths. To do this we need a fixed number of
rectangles in the vertical paths so that the region in between can be properly triangulated. In Fig. 21 we need six squares where one square in the path is stretched to a rectangle.

Fig. 21 only shows the outside of a variable block and two paths to the left and right using black points. As before the new points are white. All black points at the outside of the variable block have degree six, except for the two black points at the bottom. A simple calculation shows that we can triangulate the region between the white points with degree at most seven.
Fig. 18. Additional points for a path.

Fig. 19. Additional points inside a variable cycle.

Fig. 20. Additional points between two paths at a variable component.

Fig. 21. The left and lower side of a variable construction.
Similar constructions exist for inverters and clause designs. As above, we have to use a fixed number of rectangles in the vertical rectangles passing a horizontal block. The number and the position of the points in these rectangles are given directly by the white points we add outside the components.

Then, using this reduction it follows directly that the formula $\phi$ is satisfiable if and only if the constructed plane geometric graph can be triangulated with degree at most seven. This proves the NP-completeness of the considered min-max degree triangulation problem.

3. Conclusion

In this paper we have solved the min-max degree triangulation problem for a plane geometric graph. We have proved that the problem is NP-complete with maximum degree seven. Using a more complicated construction it might be possible to improve this to degree six. The complexity of the min-max degree problem without any constraining edges remains open.

Another problem is to find a triangulation of a point set with minimizing the sum of the edge distances. The complexity of this problem called the minimum length triangulation problem is one of the remaining open problems of Garey and Johnson [6]. Surprisingly, the triangulation problem considered in this paper is NP-complete, which gives hope that some of the ideas will eventually lead to an NP-completeness proof for the minimum length triangulation problem.

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References