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# Microlocal Properties of Basic Operations in Colombeau Algebras

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The Colombeau algebra of generalized functions allows us to unrestrictedly carry out products of distributions. We analyze this operation from a microlocal point of view, deriving a general inclusion relation for wave front sets of products in the algebra. Furthermore, we give explicit examples showing that the given result is optimal; i.e., its assumptions cannot be weakened. Finally, we discuss the interrelation of these results with the concept of pullback under smooth maps.
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## 1. INTRODUCTION

Algebras of generalized functions in the sense of J. F. Colombeau provide an efficient tool for the treatment of nonlinear problems involving singularities (cf., e.g., [1–3, 9, 14] and the literature cited therein). In particular, unrestricted multiplication (as well as a host of more general nonlinear operations) of distributions can be carried out in Colombeau algebras. Moreover, starting with [14], regularity theory has been introduced into the Colombeau framework and was extended to microlocal analysis with applications to propagation of singularities in [4, 11–13].

In the present paper we study microlocal properties of multiplication of generalized functions as well as of related operations (like pullback) in this setting. Since unlike in the case of intrinsic multiplication of distributions the formation of products in the Colombeau algebra is not subject to regularity conditions (“favorable position of the wave front sets”), new



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effects can (and will) occur. Apart from deriving general results on inclusion relations for wave front sets of products the emphasis of our presentation will be on providing examples illustrating these new effects. At the same time, the examples will demonstrate that the mentioned inclusion relations are optimal in the sense that the assumptions made to derive them cannot be weakened.

Concerning notation and terminology we basically follow [14]. Thus by $\mathscr{A}_0(\mathbb{R})$ we denote the space of test functions on $\mathbb{R}$ with unit integral. For $1 \le q$, $\mathscr{A}_q(\mathbb{R})$ is the subspace of $\mathscr{A}_0(\mathbb{R})$ consisting of those elements whose moments up to order $q$ vanish. For $n \geq 1$, $\mathscr{A}_q(\mathbb{R}^n)$ is the space of $n$-fold tensor products $\phi^{(n)} := \phi \otimes \cdots \otimes \phi$ with $\phi \in \mathscr{A}_0(\mathbb{R})$. If $\phi \in \mathscr{D}(\mathbb{R}^n)$ is any test function we set $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$. Then the basic building blocks of the Colombeau algebra of generalized functions are defined as follows.

$\mathscr{E}_M(\mathbb{R}^n)$ is the set of all maps $R$: $\mathscr{A}_0(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{C}$ which are smooth in $x$ and satisfy $\forall K \Subset \mathbb{R}^n$ $\forall \alpha \in \mathbb{N}_0^n$ $\exists N \in \mathbb{N}$ $\forall \phi \in \mathscr{A}_N(\mathbb{R}^n)$ $\exists c > 0$ $\exists \eta > 0$,

$$\sup_{x\in K} |\partial^\alpha R(\phi_\varepsilon, x)| \le c\varepsilon^{-N} \qquad (0 < \varepsilon < \eta). \tag{1}$$

$\mathscr{N}(\mathbb{R}^n)$ is the subset of $\mathscr{E}_M(\mathbb{R}^n)$ consisting of those $R$ satisfying $\forall K \Subset \mathbb{R}^n$ $\forall \alpha \in \mathbb{N}_0^n$ $\forall q \in \mathbb{N}$ $\exists p \in \mathbb{N}$ $\forall \phi \in \mathscr{A}_p(\mathbb{R}^n)$ $\exists c > 0$ $\exists \eta > 0$,

$$\sup_{x\in K} |\partial^\alpha R(\phi_\varepsilon, x)| \le c\varepsilon^{q} \qquad (0 < \varepsilon < \eta). \tag{2}$$

Then the Colombeau algebra $\mathscr{G}(\mathbb{R}^n)$ is defined as the quotient $\mathscr{E}_M(\mathbb{R}^n)/\mathscr{N}(\mathbb{R}^n)$. We note that to characterize $\mathscr{N}$ as a subspace of $\mathscr{E}_M$ it would suffice to suppose (2) only for $\alpha = 0$ (see [8, Theorem 13.1]). For the definition of $\mathscr{G}(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$ open we refer to [14]. The subalgebra of compactly supported elements of $\mathscr{G}$ will be denoted by $\mathscr{G}_c$.

The respective definitions for the space $\mathscr{G}_\tau = \mathscr{E}_{M,\tau}/\mathscr{N}_\tau$ of tempered Colombeau functions read as follows.

$\mathscr{E}_{M,\tau}(\mathbb{R}^n)$ is the set of all maps $R$: $\mathscr{A}_0(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{C}$ which are smooth in $x$ and satisfy $\forall \alpha \in \mathbb{N}_0^n$ $\exists N \in \mathbb{N}$ $\forall \phi \in \mathscr{A}_N(\mathbb{R}^n)$ $\exists c > 0$ $\exists \eta > 0$,

$$|\partial^\alpha R(\phi_\varepsilon, x)| \le c(1+|x|)^N \varepsilon^{-N} \qquad (x \in \mathbb{R}^n,\ 0 < \varepsilon < \eta) \tag{3}$$

$\mathscr{N}_\tau(\mathbb{R}^n)$ is the subset of $\mathscr{E}_{M,\tau}(\mathbb{R}^n)$ consisting of those $R$ satisfying $\forall \alpha \in \mathbb{N}_0^n$ $\forall q \in \mathbb{N}$ $\exists p \in \mathbb{N}$ $\forall \phi \in \mathscr{A}_p(\mathbb{R}^n)$ $\exists c > 0$ $\exists \eta > 0$,

$$|\partial^\alpha R(\phi_\varepsilon, x)| \le c(1+|x|)^p \varepsilon^{q} \qquad (x \in \mathbb{R}^n,\ 0 < \varepsilon < \eta) \tag{4}$$

The canonical embedding of $\mathscr{D}'$ into $\mathscr{G}$ resp. of $\mathscr{S}'$ into $\mathscr{G}_\tau$ will consistently be denoted by $\iota$. Also, equivalence classes of elements $R$ of $\mathscr{E}_M$ resp. $\mathscr{E}_{M,\tau}$ will be written as class$[(R(\phi, .))_\phi]$.

## 2. BASIC DEFINITIONS, A FIRST EXAMPLE

The starting point for regularity theory and microlocal analysis in Colombeau algebras of generalized functions was the introduction of the subalgebra $\mathcal{G}^\infty$ of $\mathcal{G}$ by Oberguggenberger in [14]. $\mathcal{G}^\infty$ consists of those elements of $\mathcal{G}$ displaying uniform $\varepsilon$-growth in all derivatives. By [14, Theorem 25.2], $\mathcal{G}^\infty \cap \mathcal{D}' = \mathcal{C}^\infty$, an identity on which all further regularity theory is based. The analogous notion $\mathcal{G}_\tau^\infty$ for tempered Colombeau functions was introduced in [11] where it was also shown that $\mathcal{G}_\tau^\infty \cap \mathcal{S}' = \mathcal{O}_M$ [11, Theorem 16]. Here $\mathcal{O}_M$ denotes the space of smooth functions with at most polynomial growth in each derivative.

$U \in \mathcal{G}_c$ is an element of $\mathcal{G}^\infty$ iff its Fourier transform (with respect to any damping measure) is rapidly decreasing [11, Theorem 18]. Based on this observation the concept of wave front set in $\mathcal{G}$, first introduced in [4], has been (equivalently) stated in [11] along the lines of [10, Sect. 8.1]. Thus for $U \in \mathcal{G}_c$ by $\Sigma_g(U)$ we denote the cone (in $\mathbb{R}^n \setminus 0$) which is the complement of those points possessing open conic neighborhoods on which the Fourier transform of $U$ is rapidly decreasing. This notion is again independent of the damping measure used in the definition of Fourier transform in $\mathcal{G}_\tau$. Then for $U \in \mathcal{G}(\Omega)$ and $x_0 \in \Omega$, the cone of irregular directions at $x_0$ is

$$\Sigma_{g,x_0}(U) = \bigcap_{\varphi \in \mathcal{D}(\Omega),\, \varphi(x_0) \neq 0} \Sigma_g(\varphi U). \tag{5}$$

The wave front set of $U$ is given by

$$\mathrm{WF}(U) = \{(x, \xi) \in \Omega \times \mathbb{R}^n \setminus 0 \mid \xi \in \Sigma_{g,x}(U)\}. \tag{6}$$

Finally, we shall make use of the concept of characteristic set of a linear differential operator (introduced in [4] for the case of the special Colombeau algebra):

Definition 2.1. Let $P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$ be a linear differential operator on $\Omega$ with coefficients in $\mathcal{G}(\Omega)$. $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ is not in the characteristic set of $P$ if there exists a neighborhood $V_{x_0}$ of $x_0$, a conic neighborhood $\Gamma_{\xi_0}$ of $\xi_0$, some $r \in \mathbb{R}$, and some $m \in \mathbb{N}_0$ such that $\forall \phi \in \mathcal{A}_m(\mathbb{R}^n)\ \exists \eta > 0\ \exists C > 0$ with

$$|P_m(\phi_\varepsilon, x, \xi)| \geq C\varepsilon^r |\xi|^m, \qquad x \in V_{x_0},\ \xi \in \Gamma_{\xi_0},\ \varepsilon \in (0, \eta). \tag{7}$$

The following example, which was first introduced in [7], gives a first application of these concepts and introduces some methods that will repeatedly be used in the following sections.

Example 2.1. We want to calculate the wave front set of the solution to

$$(\partial_t + a\partial_x)U = 0 \qquad U|_{t=0} = U_0, \tag{8}$$

where $a$ denotes a bounded generalized constant and $U_0$ is allowed to be singular. Thus let $(a_\varepsilon)_{\varepsilon>0}$ be such that $|a_\varepsilon|$ is bounded and denote by $a$ the generalized constant with representative $a(\phi) = a_{d(\phi)}$ (where $d(\phi)$ is the diameter of the support of $\phi$). Then for any $U_0 \in \mathcal{G}(\mathbb{R})$ the solution $U$ of (8) is given by the class of $U(\phi^{(2)}, x, t) = u_0(\phi, x - a(\phi)t)$. Denote by $B$ the set of limit points for $\varepsilon \to 0$ of $(a_\varepsilon)_{\varepsilon>0}$. As was noted in [7],

$$\operatorname{sing\,supp}(U) \subseteq S := \{(x, t) \mid \exists b \in B \text{ s.t. } x - bt \in K\}, \tag{9}$$

where $K = \operatorname{sing\,supp}(U_0)$. In fact, let $(x_0, t_0) \notin S$. Then by differentiating $u_0(\phi_\varepsilon, x - a(\phi_\varepsilon)t)$ and employing the boundedness of $a$ it follows that it suffices to show $x - a_\varepsilon t \notin K$ for $(x, t)$ in a neighborhood of $(x_0, t_0)$ and $\varepsilon$ small which is obviously satisfied.

To begin with, let us determine the wave front set of $U$ for the particular initial value $U_0 = \delta = \operatorname{class}[(\phi)_{\phi\in\mathcal{A}_0(\mathbb{R})}]$, so $K = \{0\}$. Let $(x_0, t_0) \in S$ and let $\psi \in \mathcal{D}(\mathbb{R}^2)$, $\psi(x_0, t_0) \neq 0$. Then

$$\begin{aligned}\big(\psi u(\phi_\varepsilon^{(2)}, .)\big)^\wedge(\xi,\tau) &= \int e^{-i(\xi x+\tau t)}\psi(x,t)\varepsilon^{-1}\phi\left(\frac{x-a_{d(\phi)\varepsilon}t}{\varepsilon}\right)d(x,t)\\ &= \int e^{-i(\xi(\varepsilon x+a_{d(\phi)\varepsilon}t)+\tau t)}\psi\big(\varepsilon x+a_{d(\phi)\varepsilon}t,t\big)\phi(x)\,dx\,dt.\end{aligned}$$

Setting $(\xi,\tau)=\omega(\xi_0,\tau_0)$ this equals

$$\int e^{i\omega f_\varepsilon(x,t)}\psi\big(\varepsilon x+a_{d(\phi)\varepsilon}t,t\big)\phi(x)\,dx\,dt, \tag{10}$$

where $f_\varepsilon(x,t)=-(a_{d(\phi)\varepsilon}\xi_0+\tau_0)t-\varepsilon\xi_0 x$. By [10, Theorem 7.7.1], for any $k\in\mathbb{N}$ we obtain a constant $C$ independent of $\varepsilon$ such that

$$\begin{aligned}\big|\big(\psi u(\phi_\varepsilon^{(2)}, .)\big)^\wedge(\xi,\tau)\big| \le C\omega^{-k}\cdot \sum_{|\alpha|\le k}\sup\Big(\big|D^\alpha\big(\psi(\varepsilon x+a_{d(\phi)\varepsilon}t,t)\\ \times\phi(x)\big)\big|\big|Df_\varepsilon(x,t)\big|^{|\alpha|-2k}\Big).\end{aligned} \tag{11}$$

Here $|Df_\varepsilon(x,t)|^2=\varepsilon^2|\xi_0|^2+|\tau_0+a_{d(\phi)\varepsilon}\xi_0|^2$, which remains bounded away from 0 (uniformly in $\varepsilon$) for $\tau_0\notin -B\xi_0$. Then from (11) we conclude that any such pair $(\xi_0,\tau_0)$ is not contained in $\Sigma_{g,(x_0,t_0)}(U)$. Denoting by $\Gamma_B$ the cone $\{(\xi,\tau)\mid \exists b\in B \text{ with } \tau=-b\xi\}$ we have shown that

$$\mathrm{WF}_g(U)\subseteq S\times\Gamma_B. \tag{12}$$

Conversely, let $\tau_0=-b\xi_0$ for some $b\in B$, fix $\phi\in\mathcal{A}_0(\mathbb{R})$, and choose a sequence $\varepsilon_k\to 0$ with $a_{d(\phi)\varepsilon_k}\to b$. Then from (10) we have $(\psi u(\phi_\varepsilon^{(2)}, .))^\wedge(\xi,\tau) \to \int\psi(bt,t)\,dt$ which for an appropriate choice of $\psi$ is nonzero.

Thus $(\psi u(\phi_\varepsilon^{(2)},.))^\wedge$ is not rapidly decreasing in the direction $(\xi_0,\tau_0)$, so we also obtain the reverse inclusion of (12). Summing up,

$$\mathrm{WF}_g(U)=S\times\Gamma_B. \tag{13}$$

We note that inclusion (12) can even be obtained for general initial data in $\mathcal{G}(\mathbb{R})$ by propagation of singularities: in fact, by Theorem 4 of [4] we have

$$\mathrm{WF}_g U\subseteq \mathrm{Char} P\cup \mathrm{WF}_g P(U) \qquad (U\in\mathcal{G}(\Omega)) \tag{14}$$

for any linear differential operator with coefficients in $\mathcal{G}^\infty(\Omega)$. Since the right hand side of (8) is 0 it therefore remains to determine the characteristic directions of the operator $P=\partial_t+a\partial_x$. Fix $\phi\in\mathcal{A}_0(\mathbb{R})$ and set $P_\varepsilon(x,t;\xi,\tau)=P(\phi_\varepsilon^{(2)},x,t;\xi,\tau)=i(\tau+a_\varepsilon\xi)$ (to simplify notation we assume $d(\phi)=1$). To show that $(x_0,t_0,\xi_0,\tau_0)$ is noncharacteristic it suffices to prove the existence of a neighborhood $V$ of $(x_0,t_0)$, a conic neighborhood of $(\xi_0,\tau_0)$, and constants $r\in\mathbb{R}$ (independent of $\phi$), $C>0$, and $\eta>0$ such that

$$|P_\varepsilon(x,t;\xi,\tau)|\geq C\varepsilon^r(|\xi|+|\tau|) \qquad \big((x,t)\in V,(\xi,\tau)\in\Gamma,0<\varepsilon<\eta\big). \tag{15}$$

Let $\tau_0\notin -B\xi_0$ and suppose $\xi_0\neq 0$ to begin with. Then there exists some $c>0$ such that $|\tau_0/\xi_0+a_\varepsilon|\geq c$ for $\varepsilon$ small. Let

$$\Gamma=\left\{(\xi,\tau)\,|\left|\frac{\tau}{\xi}-\frac{\tau_0}{\xi_0}\right|<\frac{c}{2}\right\}.$$

For $(\xi,\tau)\in\Gamma$ we get

$$|\tau+a_\varepsilon\xi|\geq|\xi|\left(\left|\frac{\tau_0}{\xi_0}+a_\varepsilon\right|-\left|\frac{\tau}{\xi}-\frac{\tau_0}{\xi_0}\right|\right)\geq\frac{c}{2}|\xi|\geq\tilde{c}(|\xi|+|\tau|)$$

so $(\xi,\tau)$ is noncharacteristic. On the other hand, if $\xi_0=0$ then we choose $c$ such that $c|a_\varepsilon|<1/2$ for all $\varepsilon$ and set $\Gamma=\{(\xi,\tau)\,|\,|\xi/\tau|<c\}$. Then again

$$|\tau+a_\varepsilon\xi|\geq|\tau|\left(1-|a_\varepsilon|\frac{|\xi|}{|\tau|}\right)\geq\frac{1}{2}|\tau|\geq\tilde{c}(|\xi|+|\tau|)$$

for $(\xi,\tau)\in\Gamma$. Thus $\mathrm{Char} P\subseteq\mathbb{R}^2\times\Gamma_B$ which by (9) and (14) implies (12).

*Remark* 2.1. In [7] it is shown that inclusion (9)—and consequently also (12)— may be strict.

## 3. THE WAVE FRONT SET OF A PRODUCT

In order to give a concise presentation of the following results we first collect a few facts on cones in $\mathbb{R}^n$ resp. $\mathbb{R}^n \setminus 0$.

LEMMA 3.1. (i) *If $\Sigma_1, \Sigma_2$ are closed cones in $\mathbb{R}^n$ such that $\Sigma_1 \cap \Sigma_2 = \{0\}$ then $\exists \alpha > 0$,*

$$|\xi - \eta| \geq \alpha |\eta| \qquad \forall \xi \in \Sigma_1, \forall \eta \in \Sigma_2.$$

*Let $\Gamma_1, \Gamma_2$ be closed cones in $\mathbb{R}^n \setminus 0$ such that $0 \notin \Gamma_1 + \Gamma_2$. Then*

(ii) $\overline{\Gamma_1 + \Gamma_2}^{\mathbb{R}^n \setminus 0} = (\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$.

(iii) *For any open conic neighborhood $W$ of $\Gamma_1 + \Gamma_2$ in $\mathbb{R}^n \setminus 0$ one can choose open conic neighborhoods $W_1$, $W_2$ in $\mathbb{R}^n \setminus 0$ of $\Gamma_1$, $\Gamma_2$, respectively, such that $W_1 + W_2 \subseteq W$.*

*Remark* 3.1. (i) The second assertion is false (in gen.) if $0 \in \Gamma_1 + \Gamma_2$ by the following example (in $\mathbb{R}^3$) due to M. Grosser: Let $K_1 = \{\lambda \cdot (-1, t, t^2) \mid 0 \leq t \leq 1, \lambda \geq 0\}$, $K_2 = \{\lambda \cdot (1, t, t^2) \mid 0 \leq t \leq 1, \lambda \geq 0\}$. Then $\Gamma_i = K_i \setminus 0$ $(i = 1, 2)$ are closed cones in $\mathbb{R}^3 \setminus 0$. The sequence $\Gamma_1 + \Gamma_2 \ni \xi_n = n \cdot (-1, 1/n, 1/n^2) + n \cdot (1, 1/n, 1/n^2) = (0, 2, 2/n)$ tends to $(0, 2, 0)$ as $n \to \infty$. But $(0, 2, 0) \notin (\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$: first, $(0, 2, 0) = \lambda(-1, t, t^2) + \mu(1, s, s^2)$ implies $\lambda = \mu$, $\lambda \neq 0$ and $s = t = 0$, yielding $0 = 2$ in the second component, so $(0, 2, 0) \notin \Gamma_1 + \Gamma_2$. Also, $(0, 2, 0) \notin \Gamma_i$ $(i = 1, 2)$ by construction.

(ii) The third assertion is false (in gen.) if $0 \in \Gamma_1 + \Gamma_2$ by the following example in $\mathbb{R}^2$: Let $\Gamma_1 = \{(x, 0) \mid x > 0\}$, $\Gamma_2 = \{(x, 0) \mid x < 0\}$. Then the sum of any two open conic neighborhoods of $\Gamma_1$, $\Gamma_2$ is $\mathbb{R}^2$.

*Proof.* (i) Otherwise there would be sequences $\xi_j \in \Sigma_1$ and $\eta_j \in \Sigma_2$ $(j \in \mathbb{N})$ such that $|\xi_j - \eta_j| < |\eta_j|/j$ for all $j \in \mathbb{N}$; this implies $|\xi_j/|\eta_j| - \eta_j/|\eta_j|| < 1/j$ which shows that $\xi_j/|\eta_j|$ has an accumulation point $\xi_0 \in \Sigma_1$ with $|\xi_0| = 1$; but then $\xi_0$ is also an accumulation point of $\eta_j/|\eta_j|$ and therefore an element of $\Sigma_2$—a contradiction.

(ii) See [5, proof of Theorem 1.3.6].

Assume that (iii) does not hold; for each $k$ choose conic neighborhoods $W_j^k$ $(j = 1, 2)$ in $\mathbb{R}^n \setminus 0$ with the following property: $\forall \eta \in W_j^k$ the projection $\eta/|\eta|$ to $S^{n-1}$ has distance less than $1/k$ to the compact set $\Gamma_j \cap S^{n-1}$.

By assumption we can choose $\eta_j^k \in W_j^k$ such that $\eta_1^k + \eta_2^k \notin W$. In particular, $\Gamma = \mathbb{R}^n \setminus W$ is a nonempty closed cone in $\mathbb{R}^n$ and therefore also $\eta^k = (\eta_1^k + \eta_2^k)/\beta_k$ with $\beta_k = |\eta_1^k| + |\eta_2^k|$ is contained in $\Gamma$.

Now we choose $\xi_{j,0}^k \in \Gamma_j \cap S^{n-1}$ such that $|\xi_{j,0}^k - \eta_j^k/|\eta_j^k|| < 1/k$ and set

$$\xi^k = \underbrace{\frac{|\eta_1^k|}{\beta_k}\xi_{1,0}^k}_{\xi_1^k \in \Gamma_1} + \underbrace{\frac{|\eta_2^k|}{\beta_k}\xi_{2,0}^k}_{\xi_2^k \in \Gamma_2} \in \Gamma_1 + \Gamma_2.$$

By (ii) of the current lemma $(\Gamma_1 \cup \{0\}) + (\Gamma_2 \cup \{0\})$ is a closed cone in $\mathbb{R}^n$; it intersects $\Gamma$ only in 0, so we can apply (i): for all $k \in \mathbb{N}$ we have for some $\alpha > 0$

$$\begin{aligned} 0 < \alpha|\xi^k| &\leq |\xi^k - \eta^k| \\ &= \frac{1}{\beta_k}\left| |\eta_1^k|\left(\xi_{1,0}^k - \frac{\eta_1^k}{|\eta_1^k|}\right) + |\eta_2^k|\left(\xi_{2,0}^k - \frac{\eta_2^k}{|\eta_2^k|}\right)\right| < \frac{1}{k}. \end{aligned}$$

Sending $k \to \infty$ we conclude that $\xi^k \to 0$. By construction the summands of $\xi^k$ are bounded: $|\xi_j^k| \leq 1$ $(j = 1, 2)$. If $\xi_1^k$ would tend to 0 then so would $\xi_2^k = \xi^k - \xi_1^k$. But since $\xi_{j,0}^k$ are normalized this would imply that both $|\eta_j^k|/\beta_k$ $(j = 1, 2)$ tend to zero yielding the contradiction $1 = (|\eta_1^k| + |\eta_2^k|)/\beta_k \to 0$. Therefore the norms of (suitable subsequences of) $\xi_j^k$ are bounded away from zero and above. There are subsequences $\xi_j^{k_l}$ $(l \in \mathbb{N})$ such that $\xi_j^{k_l} \to \zeta_j \neq 0$; then $\Gamma_1 + \Gamma_2 \ni \zeta_1 + \zeta_2 = 0$ and therefore $0 \in \Gamma_1 + \Gamma_2$—a contradiction. ∎

An essential new feature of microlocal analysis in the Colombeau setting is the precise quantification of decrease properties in terms of powers of the regularization parameter $\varepsilon$. Recall from [11, Definition 17], that $R \in \mathcal{G}_\tau$ is called rapidly decreasing in a cone $\Gamma$ if $\exists N\ \forall p \in \mathbb{N}_0\ \exists M \in \mathbb{N}_0\ \forall \phi \in \mathcal{A}_M$ $\exists c > 0\ \exists \eta > 0$,

$$|R(\phi_\varepsilon, x)| \leq c\varepsilon^{-N}(1 + |x|)^{-p} \qquad (x \in \Gamma,\ 0 < \varepsilon < \eta). \tag{16}$$

The following lemma shows that on closed cones in the complement of the cone of irregular directions of $U \in \mathcal{G}_c$, the order $N$ in (16) of rapid decrease of the Fourier transform of $U$ can be chosen uniformly.

Lemma 3.2. *Let $U \in \mathcal{G}_c(\Omega)$ and let $\Gamma$ be a closed cone in the complement of $\Sigma_g(U)$. Then* $\exists N\ \forall p \in \mathbb{N}_0\ \exists M \in \mathbb{N}_0\ \forall \phi \in \mathcal{A}_M\ \exists c > 0\ \exists \varepsilon_0 > 0$,

$$|\mathcal{F}(U(\phi_\varepsilon, .))(\xi)| \leq c\varepsilon^{-N}(1 + |\xi|)^{-p} \qquad (\xi \in \Gamma,\ 0 < \varepsilon < \varepsilon_0). \tag{17}$$

*Proof.* For any $\eta \in \Gamma$ there exists an open conic neighborhood $\Gamma(\eta)$ such that $\exists N(\eta)\ \forall p \in \mathbb{N}_0\ \exists M(\eta, p) \in \mathbb{N}_0\ \forall \phi \in \mathscr{A}_M\ \exists c(\eta, p, \phi) > 0$ $\exists \varepsilon_0(\eta, p, \phi) > 0$ such that (17) holds with this set of constants on $\Gamma(\eta)$. The sets $\Gamma(\eta) \cap S^{n-1}$ are open in $S^{n-1}$ and form a covering of the compact set $\Gamma \cap S^{n-1}$. Thus there exist $\eta_1, \ldots, \eta_m \in \Gamma$ such that

$$\Gamma \cap S^{n-1} \subseteq \bigcup_{j=1}^{m} (\Gamma(\eta_j) \cap S^{n-1}).$$

Consequently, $\Gamma$ is contained in the union of the $\Gamma(\eta_j)$ $(1 \le j \le m)$. Now set $N = \max_{1 \le j \le m} N(\eta_j)$ to finish the proof. ∎

Following the terminology of [14] we will say that the wave front sets of two elements $V_1$, $V_2$ of $\mathscr{G}$ are in *favorable position* if $\mathrm{WF}_g(V_1) + \mathrm{WF}_g(V_2)$ does not contain any zero direction (i.e., any element of the form $(x, 0)$). For $V_1 = \iota(v_1)$, $V_2 = \iota(v_2)$ distributions (in which case $\mathrm{WF}(v_i)$ and $\mathrm{WF}_g(\iota(v_i))$ coincide by [11, Corollary 24; 13, Theorem 3.8 ]) this condition ensures that the Fourier product $v_1 v_2$ of $v_1$ and $v_2$ exists in $\mathscr{D}'$ [14, Proposition 6.3]. Also, by [14, Proposition 10.3], $V_1 V_2$ is associated with $v_1 v_2$ in this case. Moreover, the wave front sets of $v_1$, $v_2$ and $v_1 v_1$ are related by (see [10, Theorem 8.2.10])

$$\mathrm{WF}(v_1 v_2) \subseteq (\mathrm{WF}(v_1) + \mathrm{WF}(v_2)) \cup \mathrm{WF}(v_1) \cup \mathrm{WF}(v_2). \tag{18}$$

Our aim in the remainder of this section is to prove the analog of relation (18) for elements of $\mathscr{G}$ whose generalized wave front sets are in favorable position, where the product is to be taken in the algebra $\mathscr{G}$. In the following section it will turn out that the inclusion will in general break down if the assumption of a favorable position of the wave front sets is dropped.

PROPOSITION 3.1. *Let $V_1, V_2 \in \mathscr{G}_c(\mathbb{R}^n)$ and suppose that $0 \notin \Sigma_g(V_1) + \Sigma_g(V_2)$. Then*

$$\begin{aligned} \Sigma_g(V_1 V_2) &\subseteq \overline{\Sigma_g(V_1) + \Sigma_g(V_2)}^{\mathbb{R}^n \setminus 0} \\ &= (\Sigma_g(V_1) + \Sigma_g(V_2)) \cup \Sigma_g(V_1) \cup \Sigma_g(V_2). \end{aligned} \tag{19}$$

*Proof.* Choose representatives $v_1$, $v_2$ with compact support; with the shorthand notation $w_j^\varepsilon(\eta) = \mathscr{F}(v_j(\phi_\varepsilon, .))(\eta)$ we have to estimate

$$\begin{aligned} (2\pi)^n \mathscr{F}\big(v_1(\phi_\varepsilon, .) v_2(\phi_\varepsilon, .)\big)(\xi) &= w_1^\varepsilon * w_2^\varepsilon(\xi) \\ &= \int_{\mathbb{R}^n} w_1^\varepsilon(\xi - \eta) w_2^\varepsilon(\eta)\, d\eta \end{aligned}$$

in a suitable conic neighborhood of any point $\xi_0$ in the complement of the right hand side of (19).

Let $\Gamma_0$ be an open cone containing $\overline{\Sigma_g(V_1) + \Sigma_g(V_2)}^{\mathbb{R}^n \setminus 0}$ such that $\xi_0 \notin \overline{\Gamma_0}$. By Lemma 3.1(iii) there exist open cones $\Gamma_j \supseteq \Sigma_g(V_j)$ $(j = 1, 2)$ such that $\Gamma_1 + \Gamma_2 \subseteq \Gamma_0$. Further, we set $\Gamma = \mathbb{R}^n \setminus \overline{\Gamma_0}$. We claim that $w_1^\varepsilon * w_2^\varepsilon$ is rapidly decreasing in $\Gamma$.

To show this we write

$$w_1^\varepsilon * w_2^\varepsilon(\xi) = \underbrace{\int_{\Gamma_2^c} w_1^\varepsilon(\xi - \eta) w_2^\varepsilon(\eta)\, d\eta}_{I_1^\varepsilon(\xi)} + \underbrace{\int_{\Gamma_2} w_1^\varepsilon(\xi - \eta) w_2^\varepsilon(\eta)\, d\eta}_{I_2^\varepsilon(\xi)}$$

and estimate the summands individually.

Substituting $\eta' = \xi - \eta$, $I_1^\varepsilon$ takes the form

$$I_1^\varepsilon(\xi) = \underbrace{\int_{(\{\xi\}-\Gamma_2^c)\cap\Gamma_1} w_2^\varepsilon(\xi - \eta') w_1^\varepsilon(\eta')\, d\eta'}_{I_{11}^\varepsilon(\xi)} + \underbrace{\int_{(\{\xi\}-\Gamma_2^c)\cap\Gamma_1^c} w_2^\varepsilon(\xi - \eta') w_1^\varepsilon(\eta')\, d\eta'}_{I_{12}^\varepsilon(\xi)}$$

$\boxed{I_{12}^\varepsilon}$ By Lemma 3.2, $\exists N\ \forall p \in \mathbb{N}_0\ \exists M \in \mathbb{N}_0\ \forall \phi \in \mathscr{A}_M\ \exists c > 0\ \exists \varepsilon_0 > 0$,

$$|w_2^\varepsilon(\xi - \eta')| \le c(1 + |\xi - \eta'|^2)^{-p} \varepsilon^{-N} \qquad \big(\eta' \in \{\xi\} - \Gamma_2^c,\ \varepsilon \in (0, \varepsilon_0)\big)$$

and $\exists N'\ \forall p' \in \mathbb{N}_0\ \exists M' \in \mathbb{N}_0\ \forall \phi \in \mathscr{A}_{M'}\ \exists c' > 0\ \exists \varepsilon_0' > 0$,

$$|w_1^\varepsilon(\eta')| \le c'(1 + |\eta'|^2)^{-p'} \varepsilon^{-N'} \qquad (\eta' \in \Gamma_1^c,\ \varepsilon \in (0, \varepsilon_0')).$$

Thus by Peetre's inequality we obtain (for $\varepsilon$ small and $\phi \in \mathscr{A}_{\max(M, M')}$),

$$|I_{12}^\varepsilon(\xi)| \le c'' \varepsilon^{-N-N'} (1 + |\xi|^2)^{-p} \int_{\mathbb{R}^n} (1 + |\eta'|^2)^{p-p'}\, d\eta'.$$

This last integral is convergent for $p' > p + \frac{n}{2}$, so $I_{12}^\varepsilon$ is rapidly decreasing in $\Gamma$.

$\boxed{I_{11}^\varepsilon}$ We abbreviate the domain of integration by $B_\xi = \Gamma_1 \cap (\{\xi\} - \Gamma_2^c)$. For $\eta' \in B_\xi$, $w_1^\varepsilon$ is tempered in $\eta'$ and $w_2^\varepsilon$ is rapidly decreasing in $\xi - \eta'$. (For later use we note here that since $\Gamma_1 \subseteq (\Gamma - \Gamma_2)^c$, the same decrease properties for $w_1^\varepsilon$ and $w_2^\varepsilon$ in fact hold on all of $\Gamma_1$. The following estimates thus remain valid upon replacing $B_\xi$ by $\Gamma_1$.) Hence

$$|I_{11}^\varepsilon(\xi)| \le c\varepsilon^{-N} \int_{B_\xi} (1 + |\xi - \eta'|)^{-p} (1 + |\eta'|)^M\, d\eta'.$$

Supposing $|\xi| \ge 1$ and setting $\xi_0 = \frac{\xi}{|\xi|}$ this equals

$$c\varepsilon^{-N} |\xi|^{M-p} \int_{B_\xi} \left( \frac{1}{|\xi|} + \left| \xi_0 - \frac{\eta'}{|\xi|} \right| \right)^{-p} \left( \frac{1}{|\xi|} + \frac{\eta'}{|\xi|} \right)^M d\eta'$$

$$\le c\varepsilon^{-N} |\xi|^{M+n-p} \int_{\frac{1}{|\xi|} B_\xi} \left( \frac{1}{|\xi|} + |\xi_0 - \eta| \right)^{-p} (1 + |\eta|)^M\, d\eta. \qquad (20)$$

Since $\overline{\Gamma}\cap\overline{\Gamma}_1=\{0\}$, by (i) of Lemma 3.1 we have

$$\exists\alpha>0, \qquad |\xi_0-\eta|\geq\alpha|\eta| \qquad \forall\xi_0\in S^{n-1}\cap\Gamma, \forall\eta\in\Gamma_1 \tag{21}$$

$$\exists\beta>0, \qquad |\xi_0-\eta|\geq\beta|\xi_0|=\beta \qquad \forall\xi_0\in S^{n-1}\cap\Gamma, \forall\eta\in\Gamma_1. \tag{22}$$

We now split the domain of integration in (20) into the parts $B_1=\frac{1}{|\xi|}B_\xi\cap\{|\eta|\leq\frac{1}{2}\}$ and $B_2=\frac{1}{|\xi|}B_\xi\cap\{|\eta|>\frac{1}{2}\}$. Then by (22)

$$\int_{B_1}\left(\frac{1}{|\xi|}+|\xi_0-\eta|\right)^{-p}(1+|\eta|)^M\,d\eta\leq\beta^{-p}\int_{B_1}(1+|\eta|)^M\,d\eta\leq\text{const.}$$

Also, by (21),

$$\int_{B_2}\left(\frac{1}{|\xi|}+|\xi_0-\eta|\right)^{-p}\left(1+|\eta|\right)^M d\eta\leq\alpha^{-p}\int_{B_2}|\eta|^{-p}(1+|\eta|)^M\,d\eta\leq\text{const}$$

for $p>M-n$. It follows that $I_{11}^\varepsilon$ and hence also $I_1^\varepsilon$ is rapidly decreasing in $\Gamma$.

Turning now to $I_2^\varepsilon$, we first note that $w_2^\varepsilon$ is tempered on the domain of integration. Moreover, since $\Gamma_2\subseteq(\Gamma-\Gamma_1)^c$ it follows that $w_1^\varepsilon$ is rapidly decreasing in $\xi-\eta$ in said domain (again by Lemma 3.2). Thus the same reasoning as in the case of $I_{11}^\varepsilon$ (cf. the above remark) shows that $I_2^\varepsilon$ is rapidly decreasing in $\Gamma$ as well, which completes the proof. ∎

THEOREM 3.1. *Let $U_1$, $U_2$ be elements of $\mathcal{G}(\Omega)$ whose wave front sets are in favorable position. Then*

$$\mathrm{WF}_g(U_1U_2)\subseteq(\mathrm{WF}_g(U_1)+\mathrm{WF}_g(U_2))\cup\mathrm{WF}_g(U_1)\cup\mathrm{WF}_g(U_2). \tag{23}$$

*Proof.* Let $(x,\xi)\notin$ *r.h.s.* Then for any $\varphi\in\mathcal{D}(\Omega)$ with $\varphi(x)\neq0$ and support sufficiently close to $x$ we have $\xi\notin\Sigma_g(\varphi U_i)$ $(i=1,2)$. Also, by Lemma 3.1(ii) and (iii), since $\xi$ is not contained in

$$\overline{\Sigma_{g,x}(U_1)+\Sigma_{g,x}(U_2)}^{\mathbb{R}^n\setminus 0}$$

there exist open conic neighborhoods $\Gamma_i$ of $\Sigma_{g,x}(U_i)$ in $\mathbb{R}^n\setminus0$ such that $\xi\notin\Gamma_1+\Gamma_2$ and $0\notin\Gamma_1+\Gamma_2$. Thus, by [11, (13)], if the support of $\varphi$ is close enough to $x$ we also have

$$\xi\notin\Sigma_g(\varphi U_1)+\Sigma_g(\varphi U_2)\subseteq\Gamma_1+\Gamma_2.$$

Since $0\notin\Sigma_g(\varphi U_1)+\Sigma_g(\varphi U_2)$,

$$\begin{aligned}\xi\notin{}&\overline{\Sigma_g(\varphi U_1)+\Sigma_g(\varphi U_2)}^{\mathbb{R}^n\setminus 0}\\ &=(\Sigma_g(\varphi U_1)+\Sigma_g(\varphi U_2))\cup\Sigma_g(\varphi U_1)\cup\Sigma_g(\varphi U_2).\end{aligned}$$

Thus by Proposition 3.1, $\xi\notin\Sigma_g(\varphi^2U_1U_2)$. Again from [11, (13)], the claim follows. ∎

## 4. EXAMPLES

In the previous section we have extended the validity of the wave front inclusion relation (18) to the product in the algebra $\mathcal{G}$, provided that the generalized wave front sets of the factors are in favorable position. Contrary to the distributional situation, however, a favorable position of the wave front sets is of course not a prerequisite for forming the product in the algebra. Thus the question arises whether a further extension of the classical result to arbitrary products in $\mathcal{G}$ is possible. The second example in this section will demonstrate that this is not the case. Before we turn to this matter, we first give an example illustrating some genuinely non-distributional effects in the application of Theorem 3.1.

EXAMPLE 4.1. Denote by $U$ the class in $\mathcal{G}(\mathbb{R}^2)$ of $U(\phi^{(2)},x,y)=(1/d(\phi))\phi(x/d(\phi)-y/\sqrt{d(\phi)})$. As the results of the following calculations are independent of the concrete value of $d(\phi)$ we will for simplicity assume that $d(\phi)=1$ and we will abbreviate $U(\phi_\varepsilon^{(2)},x,y)$ by $u_\varepsilon(x,y)=(1/\varepsilon)\phi(x/\varepsilon-y/\sqrt{\varepsilon})$. As a matter of fact, this assumption effectively transfers the problem into the setting of the special Colombeau algebra. It is easily seen that $U\approx\delta(x)\otimes 1(y)$.

Further, let $A=\iota(\frac{1}{x+i0})\in\mathcal{G}(\mathbb{R})$ and define $B\in\mathcal{G}(\mathbb{R}^2)$ by $B(\phi^{(2)},x,y)=A(\phi,\sqrt{d(\phi)}x+y)$. Employing the same simplification as above we will write $b_\varepsilon(x,y)$ for

$$B(\phi_\varepsilon^{(2)},x,y)=\int_0^\infty \frac{\phi_\varepsilon(\sqrt{\varepsilon}x+y-z)-\phi_\varepsilon(\sqrt{\varepsilon}x+y+z)}{z}\,dz$$
$$-i\pi\phi_\varepsilon(\sqrt{\varepsilon}x+y).$$

Let us first determine $\mathrm{WF}_g(U)$. To begin with, we claim that $\mathrm{supp}(U)=\{0\}\times\mathbb{R}$. Indeed, the inclusion $\subseteq$ is obvious. Conversely, let $(0,a)\in\{0\}\times\mathbb{R}$ and set $x_\varepsilon=\varepsilon+a\sqrt{\varepsilon}$, $y_\varepsilon=\sqrt{\varepsilon}+a$. Then $(x_\varepsilon,y_\varepsilon)$ is the representative of a compactly supported generalized point (cf. [15]) supported in any ball $B_r((0,a))$ ($r>0$ arbitrary). Since $u_\varepsilon(x_\varepsilon,y_\varepsilon)=\frac{1}{\varepsilon}\phi(0)$, the claim follows from [15, Theorem 2.4]. To determine an upper bound for $\mathrm{WF}_g(U)$ we note that setting $P_\varepsilon=\partial_y+\sqrt{\varepsilon}\partial_x$ we have $P_\varepsilon u_\varepsilon=0$. Thus by (14) the set of characteristic directions of $P$ provides such an upper bound. By Example 2.1, $\mathrm{Char}(P)\subseteq\mathbb{R}^2\times\{(\xi,0)\,|\,\xi\neq 0\}$. Next, we show that $\{(\xi,0)\,|\,\xi\neq 0\}\subseteq\Sigma_{g,(0,a)}(U)$ for any $a\in\mathbb{R}$. To this end, let $\psi(x,y)=f(x)g(y)\in\mathcal{D}(\mathbb{R}^2)$, $f(x)\equiv 1$, $g(y)\equiv 1$ near $x=0$ resp. $y=0$, and $g$ positive. For $\varepsilon$ sufficiently small, $\{x\,|\,g(y)\phi_\varepsilon(x-\sqrt{\varepsilon}y)\neq 0\}\subseteq\{x\,|\,f(x)=1\}$, so $\psi u=gu$. Hence

$$\mathcal{F}(\psi u_\varepsilon)(\xi,0)=\int e^{-ix\xi}\phi_\varepsilon(x-\sqrt{\varepsilon}y)g(y)\,dx\,dy=\hat{\phi}_\varepsilon(\xi)\hat{g}(\sqrt{\varepsilon}\xi)$$
$$=\hat{\phi}(\varepsilon\xi)\hat{g}(\sqrt{\varepsilon}\xi)\to\hat{\phi}(0)\hat{g}(0)=\hat{g}(0)\neq 0\qquad(\varepsilon\to 0),$$

which shows that $\mathscr{F}(\psi u_\varepsilon)$ is not rapidly decreasing in the direction $(\xi,0)$. Replacing $g$ by $\tau_a g=g(.-a)$ and setting $\psi_a(x,y)=f(x)\tau_a g(y)$ we obtain $\mathscr{F}(\psi_a u_\varepsilon)=\hat{\phi}(\varepsilon\xi)\hat{g}(\sqrt{\varepsilon}\xi)e^{-ia\xi}$, so the same reasoning gives $\{(\xi,0)\,|\,\xi\neq 0\}\subseteq \Sigma_{g,(0,a)}(U)$. Summing up, we have shown

$$\mathrm{WF}_g(U)=\{0\}\times\mathbb{R}\times\mathbb{R}\setminus 0\times\{0\}. \tag{24}$$

Turning now to $B$, we first show that singsupp$(B)=\{(0,0)\}$. Let $(x,y)\in K\Subset\mathbb{R}^2\setminus 0$. Since $\phi(x/\sqrt{\varepsilon}+(y\pm z)/\varepsilon)\equiv 0$ near $z=0$ for $\varepsilon$ small, we can write

$$b_\varepsilon(x,y)=\frac{1}{\varepsilon}\int_0^\infty \frac{\phi(x/\sqrt{\varepsilon}+(y-z)/\varepsilon)}{z}\,dz-\frac{1}{\varepsilon}\int_0^\infty \frac{\phi(x/\sqrt{\varepsilon}+(y+z)/\varepsilon)}{z}\,dz$$
$$=\begin{cases}\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\frac{\phi(s)ds}{y+\sqrt{\varepsilon}x-\varepsilon s} & \frac{x}{\sqrt{\varepsilon}}+\frac{y}{\varepsilon}>d(\phi)\\ -\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\frac{\phi(s)ds}{\varepsilon s-y-\sqrt{\varepsilon}x} & \frac{x}{\sqrt{\varepsilon}}+\frac{y}{\varepsilon}<-d(\phi).\end{cases}$$

In any case,

$$\partial_y^k\partial_x^l(b_\varepsilon)=\pm\frac{1}{\varepsilon}\varepsilon^{\frac{l}{2}}\int_{-\infty}^{\infty}\frac{\phi(t)dt}{(-\varepsilon t\pm(y+\sqrt{\varepsilon}x))^{k+l+1}}=\mathscr{O}\left(\frac{1}{\varepsilon}\right),$$

so $B$ is an element of $\mathscr{G}^\infty$ off $(x,y)=(0,0)$ (but clearly not in any neighborhood of $(0,0)$ itself). Setting $P_\varepsilon=\partial_x-\sqrt{\varepsilon}\partial_y$ we have $P_\varepsilon b_\varepsilon=0$, so by the same reasoning as above we have $\mathrm{WF}_g(B)\subseteq\mathrm{Char}_g(P)=\mathbb{R}^2\times\{\xi=0\}$. To determine $\Sigma_{g,(0,0)}B$ it therefore remains to estimate (with $\psi$ as above)

$$\begin{aligned}&\mathscr{F}(\psi b_\varepsilon)(0,\eta)\\ &=\int e^{-iy\eta}\int_0^\infty\frac{\phi_\varepsilon(\sqrt{\varepsilon}x+y-z)-\phi_\varepsilon(\sqrt{\varepsilon}x+y+z)}{z}\,dz\\ &\quad\cdot f(x)g(y)dxdy-i\pi\int e^{-iy\eta}\phi_\varepsilon(\sqrt{\varepsilon}x+y)f(x)g(y)dxdy\\ &=\frac{1}{2\pi}\int f(x)\underbrace{\left(\mathscr{F}_{y\to\eta}\left(\left(\left(\mathrm{vp}\left(\frac{1}{x}\right)-i\pi\delta\right)*\phi_\varepsilon\right)(\sqrt{\varepsilon}x+y)\right)*\hat{g}\right)(\eta)}_{=:I^\varepsilon(x,\eta)}\,dx.\end{aligned}$$

Using $\mathscr{F}(\mathrm{vp}(\frac{1}{x})-i\pi\delta)=cH$ (with $H$ the Heaviside function) we have

$$I^\varepsilon(x,\eta)=c\int_0^\infty\hat{g}(\eta-\eta')e^{i\eta'\sqrt{\varepsilon}x}\hat{\phi}(\varepsilon\eta')\,d\eta'.$$

Suppose now that $\eta<0$. Then in the domain of integration of $I^\varepsilon$, $|\eta-\eta'|\geq|\eta|$. Thus for any $l\in\mathbb{N}$ we get

$$|\hat{g}(\eta-\eta')|\leq c_l(1+|\eta-\eta'|)^{-2l}\leq c_l(1+|\eta|)^{-l}(1+|\eta-\eta'|)^{-l}.$$

But then for $l$ sufficiently large $|I^\varepsilon(x,\eta)| \leq c'(1+|\eta|)^{-l}$, implying $\Sigma_{g,(0,0)}(B) \subseteq \{(0,\eta) \mid \eta > 0\}$. Since we have seen above that $\Sigma_{g,(0,0)} \neq \emptyset$, this implies

$$\mathrm{WF}_g(B) = \{(0,0)\} \times \{0\} \times \mathbb{R}^+. \tag{25}$$

By (24) and (25), the wave front sets of $U$ and $B$ are in favorable position, so Theorem 3.1 gives

$$\Sigma_{g,(0,0)}(BU) \subseteq \mathbb{R} \times \mathbb{R}^+. \tag{26}$$

We are now going to establish also the inverse inclusion to (26). With $\psi$ as above, we have to analyze

$$\mathcal{F}(\psi b_\varepsilon u_\varepsilon)(\xi,\eta) = \mathcal{F}_{x\to\xi}(\underbrace{\mathcal{F}_{y\to\eta}(g(y) b_\varepsilon(x,y) u_\varepsilon(x,y))}_{=:J^\varepsilon(x,\eta)} f(x)).$$

Here, $(2\pi)^2 J^\varepsilon(x,\eta) = (\hat{g} * \mathcal{F}_{y\to\eta'}(b_\varepsilon(x,y)) * \mathcal{F}_{y\to\eta'}(u_\varepsilon(x,y)))(\eta)$, and a short calculation gives

$$\mathcal{F}_{y\to\eta'}(b_\varepsilon(x,y))(\eta') = -2\pi i e^{i\eta'\sqrt{\varepsilon}x} H(\eta')\hat{\phi}(\varepsilon\eta')$$
$$\mathcal{F}_{y\to\eta'}(u_\varepsilon(x,y))(\eta') = \frac{1}{\sqrt{\varepsilon}} e^{-i\eta'\frac{x}{\sqrt{\varepsilon}}} \hat{\phi}(-\sqrt{\varepsilon}\eta').$$

Inserting this and substituting $x' = x/\sqrt{\varepsilon}$ we obtain

$$\mathcal{F}(\psi b_\varepsilon u_\varepsilon)(\xi,\eta) = \frac{1}{2\pi i} \iiint_0^\infty e^{-i\xi\sqrt{\varepsilon}x' + ix'(\varepsilon\eta'' - \eta')} \hat{g}(\eta - \eta' - \eta'')$$
$$\cdot \hat{\phi}(\varepsilon\eta'')\hat{\phi}(-\sqrt{\varepsilon}\eta')\, d\eta''\, d\eta'\, f(\sqrt{\varepsilon}x')\, dx'. \tag{27}$$

For $\varepsilon \to 0$ this converges to

$$\frac{1}{2\pi i} \iiint_0^\infty e^{-ix'\eta'} \hat{g}(\eta - \eta' - \eta'')\, d\eta''\, d\eta'\, dx'$$
$$= \frac{1}{2\pi i} \iint_{-\infty}^z \hat{g}(\xi)\, d\xi \int e^{-ix'(\eta - z)}\, dx'\, dz = \frac{1}{2\pi i} \int_{-\infty}^\eta \hat{g}(\xi)\, d\xi.$$

It follows that for $\eta > 0$ and any $\xi$, $\mathcal{F}(\psi b_\varepsilon u_\varepsilon)$ is not rapidly deceasing in the direction $(\xi,\eta)$. Thus, in fact

$$\Sigma_{g,(0,0)}(BU) = \mathbb{R} \times \mathbb{R}^+. \tag{28}$$

The above example demonstrates the usefulness of Theorem 3.1 for determining wave front sets of products of generalized functions even beyond the distributional regime: Provided the wave front sets of the factors are in favorable position, an upper bound for the wave front set of the product is given by (23). However, should the assumption of a favourable position be dropped, despite the fact that the product is still defined in $\mathcal{G}$ the inclusion (23) will break down in general, as is explicitly demonstrated in the following example.

EXAMPLE 4.2. Let $U$ as in Example 4.1 and set $v=u(\phi^{(2)},x,y)=(1/d(\phi))\phi(x/d(\phi)-y/\sqrt{d(\phi)})$. Employing the same notational simplifications as in the previous example we have $v_\varepsilon(x,y)=(1/\varepsilon)\phi(x/\varepsilon+y/\sqrt{\varepsilon})$. Again, $\mathrm{supp}(V)=\{0\}\times\mathbb{R}$, $V\approx\delta(x)\otimes 1(y)$, and

$$\mathrm{WF}_g(V)=\{0\}\times\mathbb{R}\times\mathbb{R}\setminus 0\times\{0\}. \tag{29}$$

Thus the wave front sets of $U$ and $V$ are not in favorable position and we shall demonstrate that in fact the conclusion of Theorem 3.1 is violated for the product $W=UV$.

We first show that $\mathrm{supp}(W)=\{(0,0)\}$. To this end we again utilize [15, Theorem 2.4]. We only have to show that $W$ vanishes in a suitable neighborhood of any point $(0,a)$ with $a\neq 0$. We shall assume $a>0$ (the other case being analogous) and we choose some $r>0$ such that $B_r((0,a))$ does not contain $(0,0)$. Now let $(x_\varepsilon,y_\varepsilon)$ be a representative of any generalized point supported in $B_r((0,a))$. Then if $x_\varepsilon\geq 0$ and $\varepsilon$ is sufficiently small we have

$$\frac{x_\varepsilon}{\varepsilon}+\frac{y_\varepsilon}{\sqrt{\varepsilon}}\geq\frac{a-r}{\sqrt{\varepsilon}}$$

and similarly, for $x_\varepsilon\leq 0, (x_\varepsilon/\varepsilon)-(y_\varepsilon/\sqrt{\varepsilon})\leq-(a-r)/\sqrt{\varepsilon}$. Thus, for small $\varepsilon$, one of the factors of $w_\varepsilon$ always vanishes. This means that $W$ vanishes on all compactly supported points in $B_r((0,a))$, so $W=0$ on $B_r((0,a))$.

It remains to determine $\Sigma_{g,(0,0)}(W)$, to which end we choose $f$, $g$ as in Example 4.1. As above, $f^2(x)g^2(y)u_\varepsilon(x,y)v_\varepsilon(x,y)=g^2(y)u_\varepsilon(x,y)v_\varepsilon(x,y)$ for $\varepsilon$ small. Thus it suffices to consider

$$\begin{aligned}\mathcal{F}(gu_\varepsilon gv_\varepsilon)(\xi,\eta)&=\iint\hat{\phi}_\varepsilon(\xi')\hat{\phi}_\varepsilon(\xi-\xi')\hat{g}(\eta'+\sqrt{\varepsilon}\xi')\\&\qquad\times\hat{g}(\eta-\eta'-\sqrt{\varepsilon}(\xi-\xi'))\,d\xi'\,d\eta'\\&=\int\hat{\phi}(\varepsilon\xi')\hat{\phi}(\varepsilon(\xi-\xi'))\hat{g}*\hat{g}(\eta+2\sqrt{\varepsilon}\xi'-\sqrt{\varepsilon}\xi)\,d\xi'\\&=\frac{1}{\sqrt{\varepsilon}}\int\hat{\phi}(\sqrt{\varepsilon}\xi'')\hat{\phi}(\varepsilon\xi-\sqrt{\varepsilon}\xi'')\hat{g}*\hat{g}(\eta+2\xi''-\sqrt{\varepsilon}\xi)\,d\xi''.\end{aligned}$$

The integral in this equation converges to $\int\hat{g}*\hat{g}(2\xi'')\,d\xi''$, so $\mathcal{F}(gu_\varepsilon gv_\varepsilon)(\xi,\eta)\sim\mathcal{O}(1/\sqrt{\varepsilon})$. In particular, $\mathcal{F}(gu_\varepsilon gv_\varepsilon)$ is not rapidly decreasing in any direction $(\xi,\eta)$. Thus

$$\begin{aligned}\mathrm{WF}_g(UV)&=\{(0,0)\}\times\mathbb{R}^2\setminus 0\not\subseteq\{0\}\times\mathbb{R}\times\mathbb{R}\setminus 0\times\{0\}\\&=(\mathrm{WF}_g(U)+\mathrm{WF}_g(V))\cup\mathrm{WF}_g(U)\cup\mathrm{WF}_g(V).\end{aligned}\tag{30}$$

## 5. CONSEQUENCES FOR MICROLOCAL PROPERTIES OF PULLBACKS

In this final section we are going to compare our previous considerations with an alternative (classical) approach to products of distributions and their microlocal properties [10, Sect. 8.2]. In this approach, one considers the product of two distributions $u$ and $v$ (provided it exists) as the restriction of their tensor product to the diagonal, i.e., as the pullback of $u \otimes v$ under $d$: $x \to (x, x)$. The basic properties of this operation then follow directly from the general theorem about microlocal transformation under composition with smooth maps [10, Theorem 8.2.4].

As a first step, we note that the tensor product of Colombeau generalized functions is "well behaved" from a microlocal point of view. Recall that for open subsets $\Omega \subseteq \mathbb{R}^m$, $\Omega' \subseteq \mathbb{R}^n$ and $U \in \mathcal{G}(\Omega)$, $V \in \mathcal{G}(\Omega')$, $U \otimes V$ is represented by $(\phi^{(m)} \otimes \phi^{(n)}, x, y) \to u(\phi^{(m)}, x) v(\phi^{(n)}, y)$.

LEMMA 5.1.

$$\mathrm{WF}_g(U \otimes V) \subseteq \Big(\mathrm{WF}_g(U) \boxtimes \mathrm{WF}_g(V)\Big) \cup \Big((\operatorname{supp} U \times \{0\}) \boxtimes \mathrm{WF}_g(V)\Big) \cup \Big(\mathrm{WF}_g(U) \boxtimes (\operatorname{supp} V \times \{0\})\Big), \tag{31}$$

*where* $\Gamma_1 \boxtimes \Gamma_2 := \{(x, y, \xi, \eta) \mid (x, \xi) \in \Gamma_1, (y, \eta) \in \Gamma_2\}$ *for arbitrary subsets* $\Gamma_1 \subseteq \Omega \times \mathbb{R}^m$, $\Gamma_2 \subseteq \Omega' \times \mathbb{R}^n$.

*Proof.* This is a straightforward adaptation of the proof of the corresponding distributional result (see, e.g., [6, Theorem 11.2.1]). ∎

If $f: \Omega_1 \to \Omega_2$ is a smooth map between open subsets $\Omega_i \subseteq \mathbb{R}^{n_i}$ and $u \in \mathcal{D}'(\Omega_2)$ then the classical condition ensuring existence of $f^* u$ is

$$\mathrm{WF}(u) \cap N_f = \emptyset,$$

where $N_f = \{(f(x_1), \xi_2) \in \Omega_2 \times \mathbb{R}^{n_2} \mid {}^t f'(x_1) \xi_2 = 0\}$ (cf. [10, Theorem 8.2.4]). Furthermore, in this case

$$\mathrm{WF}(f^* u) \subseteq f^* \mathrm{WF}(u), \tag{32}$$

where $f^* \Gamma = \{(x_1, {}^t f'(x_1) \xi_2 \mid (f(x_1), \xi_2) \in \Gamma\}$ for $\Gamma \subseteq \Omega_2 \times \mathbb{R}^{n_2}$.

Just as the product of generalized functions can be carried out unrestrictedly in the Colombeau algebra, it is also possible to form pullbacks of Colombeau generalized functions under arbitrary smooth maps: For $U \in \mathcal{G}$ and $f$ smooth, $U \circ f$ is defined by componentwise composition. Moreover, for $U$, $V$ in $\mathcal{G}(\Omega)$ we have $UV = (U \otimes V) \circ d$ with $d: \Omega \to \Omega \times \Omega$, $x \mapsto (x, x)$ (simply observe that $UV$ and $(U \otimes V) \circ d$ have identical representatives). We are thus in a position to review our previous examples in this picture:

EXAMPLE 5.1. Let $U$ and $V$ as in Example 4.2 and set $T = U \otimes V$. By Lemma 5.1,

$$\mathrm{WF}_g(T) \subseteq \left\{((0, x_2, 0, x_4), (\xi_1, 0, \xi_3, 0)) \mid x_2, x_4 \in \mathbb{R}, (\xi_1, \xi_3) \in \mathbb{R}^2 \setminus \{(0,0)\}\right\}.$$

In fact, we have equality in the above relation since by using cutoff functions of tensor product form the reasoning leading to (24) can be carried out in parallel in the independent factors corresponding to $U$ and $V$. In the notation of Example 4.2 we have $W = T \circ d$ and $\mathrm{WF}_g(W) = \{(0,0)\} \times \mathbb{R}^2 \setminus 0$. A simple computation shows that $d^*\mathrm{WF}_g(T) = \{((0, r), (\mu, 0)) \mid r, \mu \in \mathbb{R}\}$ which implies that

$$\mathrm{WF}_g(d^*T) \not\subseteq d^*\mathrm{WF}_g(T). \tag{33}$$

Note that $N_d \cap \mathrm{WF}_g(T) \neq \emptyset$ in Example 5.1. We conclude that the validity (32) cannot be extended to arbitrary pullbacks of Colombeau functions under smooth maps.

*Remark* 5.1. We note that a common feature of the examples introduced in Section 4 is that they are formed as pullbacks of (canonical images of) distributions under *generalized* maps. Thus the fact that—contrary to the distributional setting—composition of generalized functions can be carried out in $\mathcal{G}$ (subject to certain growth conditions) can be viewed as one of the causes of the new microlocal effects presented there.

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