Scalar and Vector Iteration

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Received May 18, 1976

In "Monadic Computation and Iterative Algebraic Theories" Elgot [3] introduced the notion of a vector iterative theory and showed that vector iterative theories have a close connection with several aspects of the theory of computation. In private conversations Elgot formulated the notion of a scalar iterative theory and raised the question whether scalar and vector iterative theories coincide. We answer this question in the affirmative.

1. Introduction

This paper is a supplement to "Monadic Computation and Iterative Algebraic Theories," by Elgot [3], henceforth referred to as MC. Familiarity with the notation and elementary results of MC is assumed. In MC, the notion (repeated below) of a (vector) iterative theory was defined and shown to have a close connection with several aspects of the theory of computation.

In private conversations Elgot formulated the notion of scalar iterative theory and raised the question whether scalar and vector iterative theories coincide. We answer this question in the affirmative. This results in a considerable simplification of the notion of iterative theory, and gives a simpler proof of the second half of the Main Theorem of MC. The result may also enable one to show more easily that a given theory is iterative.

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2. Preliminary Remarks

The notion of an ideal algebraic theory was defined in MC.

**Definition.** An ideal algebraic theory is *(vector) iterative* if for any \( p \geq 0 \), for any ideal morphism \( f : \{s\} \to \{s \cdot p\} \) there is a unique morphism \( f^+ : \{s\} \to \{p\} \) satisfying the equation

\[
f^+ = f \cdot (f^+, 1_p).
\]

An ideal algebraic theory is *scalar iterative* if condition (*) holds with \( s = 1 \).

Our principal result is that this apparently weaker condition is actually as strong as the other, that is, that every scalar iterative theory is *(vector) iterative*, so that the parenthesized qualification is really unnecessary.

We will make use of diagrams similar to those appearing in MC 4.1 and 5.6 for motivation and to facilitate rapid comprehension. Readers may find it useful to draw their own diagrams where they are not provided. Our diagrams differ from those of MC in the treatment of source-tupling. Let \( f : [2] \to [3], g : [3] \to [3] \) be morphisms. We represent \((f, g)\) as in Fig. 1, where the two diagrams are intended to be synonymous. Identities

\[
1_p : \{p\} \to \{p\}
\]

will usually be represented by a simple line (no box), so that the iteration equation (*) is represented by Fig. 2.

Other conventions should become clear in context. It is shown in [1, 4] that these diagrams, properly interpreted, constitute a complete model for iterative theories, so they are really quite a reliable guide to intuition. In the context of the present paper, however, this remains to be proven for scalar iterative theories, so we can use them only as an expository aid.

![Fig. 1. Two equivalent diagrams for the source-tuple \((f, g)\).](image1)

![Fig. 2. \( f^+ = f \cdot (f^+, 1_p) \).](image2)
The starting point of this work, historically, was the observation that the scalar iteration operation \( f: [s] \to [p \vdash 1] \) of MC can be extended to apply to general "scalar-ideal" morphisms \((f: Is \to [p \vdash 1])\) such that \((1_1 \oplus 0_{s-1}) \cdot f\) is ideal. We write the injection \((0_{n-1} \oplus 1_1 \oplus 0_{s-n})\) as \(n\), so \((1_1 \oplus 0_{s-1}) \cdot f \cdot 1f\). Define \(f^{11}\) as the unique solution of the equation

\[
\xi = f \cdot (1\xi, 1_p). \quad (\ast_1)
\]

This breaks down by components into the system

\[
\begin{align*}
1\xi &= 1f \cdot (1\xi, 1_p), \\
2\xi &= 2f \cdot (1\xi, 1_p), \\
\vdots \\
s\xi &= sf \cdot (1\xi, 1_p),
\end{align*}
\]

where the first equation is just \((\ast_1)\) for \(1f\), a scalar morphism for which \((1f)^{11} = 1\xi\) is well defined by the definition of scalar iterative theory. The remaining equations define the rest of \(\xi\) in terms of \(f\) and \(1\xi\), hence \(\xi\) is well defined. In terms of the diagrams, this says that we can put one loop on any scalar-ideal, hence any ideal, morphism. A representation of \(\xi\) is given in Fig. 3.

The first thought is to put on \(s\) loops, one at a time, using the cyclic permutation \(\pi: 1, 2, \ldots, s \to 2, \ldots, s, 1\) to bring the successive entries to first position. This gives the simple picture of Fig. 4, for \(s = 3\), corresponding to \(\pi((\pi((\pi((f^{+1}))^{11}))^{11}))^{11}\). Unfortunately, the proof that this picture is justified, and that the corresponding morphism satisfies \((\ast)\), is less than simple; it is given in [2], and will not be repeated here. Further, it does not seem to lead naturally to a proof that the candidate for \(f^+\) is the unique solution of \((\ast)\). The inductive approach given below gives a more complex form for \(f^+\), but a simpler proof that this form is a solution of \((\ast)\), and an almost equally simple proof that it is the unique solution.
3. MAIN RESULT

THEOREM. Let $T$ be a scalar iterative theory. Then for every ideal morphism $f: [s] \to [s + p]$ there is a unique morphism $f^*: [s] \to [p]$ which satisfies the equation

$$xi = f \cdot (xi, 1_p).$$

Hence every scalar iterative theory is (vector) iterative.

Proof. By induction on $s$. The case $s = 1$ is trivial. Let $f: [1 + s] \to [1 + s + p]$. Define $f_1, f_2$ by

$$f_1: [1] \overset{(11 \oplus 0, 0)}{\rightarrow} [1 + s] \overset{f}{\rightarrow} [1 + s + p],$$

$$f_2: [s] \overset{0, 1}{\rightarrow} [1 + s] \overset{f}{\rightarrow} [1 + s + p].$$

We will show that at least one morphism $g$ satisfying (*) exists, and then that it is unique. $f_1^*: [1] \to [s + p]$ is well defined, since $T$ is scalar iterative. Define

$$h: [s] \overset{f}{\rightarrow} [1 + s + p] \overset{(f_1^*, 1_s, p)}{\rightarrow} [s + p].$$

Then $g_2 = h^*: [s] \to [p]$ also exists (and is unique) by the induction hypothesis. Now define

$$g_1: [1] \overset{f_1^*}{\rightarrow} [s + p] \overset{(h^*, 1_p)}{\rightarrow} [p].$$

We show that $g = (g_1, g_2)$ satisfies (*).

$$g = (f_1^* \cdot (g_2, 1_p), h^*)$$

$$= (f_1 \cdot (f_1^*, 1_s, 1_p) \cdot (g_2, 1_p), f_2 \cdot (f_1^*, 1_s, p) \cdot (h^*, 1_p))$$

$$= (f_1, f_2) \cdot (f_1^*, 1_s, 1_p) \cdot (g_2, 1_p)$$

$$= f \cdot (f_1^* \cdot (g_2, 1_p), g_2, 1_p) = f \cdot (g, 1_p).$$

We recall a special case of Proposition 5.6.3 of MC:

Let $\alpha, \beta$ be morphisms in $T$ such that $\alpha: [1] \to [1 + q]$ is ideal and $\beta: [q] \to [r]$; then $\alpha^* \cdot \beta = [\alpha \cdot (1, 1) \beta]^*$. 

LEMMA. Assume $g = (g_1, g_2)$ and $g_1 = f_1 \cdot (g, 1_p)$. Then $g_1$ is the composition

$$[1] \overset{f_1^*}{\rightarrow} [s + p] \overset{(g_2, 1_p)}{\rightarrow} [p].$$

Proof. Let $\alpha := f_1: [1] \to [1 + s + p]$ and $\beta = (g_2, 1_p): [s + p] \to [p]$. Then by MC 5.6.3, $f_1^* \cdot (g_2, 1_p) = [f_1 \cdot (11 \oplus (g_2, 1_p))]^!$. But $f_1 \cdot (11 \oplus (g_2, 1_p)) \cdot (g_2, 1_p) = f_1 \cdot (g_1, 1_p) = g_1$, and since $f_1 \cdot (11 \oplus (g_2, 1_p)): [1] \to [1 + p]$ has a unique iterate, $g_1 = [f_1 \cdot (11 \oplus (g_2, 1_p))]^! = f_1^* \cdot (g_2, 1_p)$, completing the proof of the lemma.

Now assume that $g$ satisfies (*), i.e., that $g = f(g, 1_p)$, and $g_1, g_2$ are as in the lemma. We will show that $g$ is uniquely determined by $f$. 

We show first that $g_2 \preceq h \cdot (f_2 \cdot (f_1^+ \cdot (g_1^+ \cdot 1_{s+1}))^+)$, and so by the induction hypothesis is unique. It suffices to show $g_2 = h \cdot (g_2, 1_p)$.

$$h \cdot (g_2, 1_p) = f_2 \cdot (f_1^+ \cdot (g_2, 1_p)) \cdot (g_2, 1_p)$$

$$= f_2 \cdot (f_1^+ \cdot (g_2, 1_p), (g_2, 1_p))$$

by the lemma

$$= f_2 \cdot (g_2, 1_p) = g_2.$$

Thus $g_2$ is determined as the unique iterate of $h$, which depends only on $f$. But by the lemma $g_1$ is the composition $f_1^+ \cdot (g_2, 1_p)$, and so is determined as soon as $g_2$ is determined, and we have shown that any $g: [1 + s] \rightarrow [1 + p]$ which satisfies (*) is uniquely determined by $f$. Together with the earlier demonstration that such a $g$ exists, this completes the proof of the theorem.

To illustrate the utility of the diagrams described in the introductory remarks, we will reproduce the proof of the lemma in that form. We believe that, while much more space is required, there will be a substantial gain in clarity and ease of comprehension for many readers.

Assume Fig. 5; then follows Fig. 6, which is just Eq. (*) for the morphism in the dotted box. Since we are in a scalar iterative theory, this gives Fig. 7, the last step being justified by 5.6.3.

![Fig. 5](image)

**Fig. 5.** For the lemma: Assume $g_1 = f_1 \cdot (g_1, 1_p)$.

![Fig. 6](image)

**Fig. 6.** Then $g_1 = f_1 \cdot (g_1, g_2, 1_p)$; hence, simply moving the right side occurrence of $g_1$ to the right, $g_1 = f_1 \cdot (1_s \otimes (g_2, 1_p)) \cdot (g_1, 1_p)$.
4. Final Remarks

The existence part of the main result was obtained by two of us some time ago, and reported in [2]; the uniqueness result was obtained by Ginali more recently, following which we collaborated on the present unified presentation. We wish to thank C. C. Elgot for his interest and useful discussions at later stages, as well as for suggesting the problem.

References