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Existence of triple positive solutions for a third-order three-point boundary value problem

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Abstract

In this paper we investigate the existence of triple positive solutions for the nonlinear third-order three-point boundary value problem

$$u'''(t) = a(t)f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1,$$
$$u(0) = \delta u(\eta), \qquad u'(\eta) = 0, \quad u''(1) = 0,$$

where $\delta \in (0, 1)$, $\eta \in [1/2, 1)$ are constants. $f : [0, 1] \times [0, \infty) \times R^2 \to [0, \infty)$, $q : (0, 1) \to [0, \infty)$ are continuous. First, Green's function for the associated linear boundary value problem is constructed, and then, by using a fixed-point theorem due to Avery and Peterson, we establish results on the existence of triple positive solutions to the boundary value problem.
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1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics. In recent years, the existence and multiplicity of positive solutions for nonlinear third-order ordinary differential equations with a three-point boundary value problem (BVP for short) have been studied by several authors. An interest in triple solutions evolved from the Leggett–Williams multiple-fixed-point theorem [11]. And lately, two triple-fixed-point theorems due to Avery [5] and Avery and Peterson [6], have been applied to obtain triple solutions of certain three-point boundary value problems for third-order ordinary differential equations. For example, Anderson [1] proved that there exist at least three positive solutions to the BVP

$$-x'''(t) + f(x(t)) = 0, \quad 0 < t < 1,$$
$$x(0) = x'(t_2) = x''(1) = 0,$$

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where $f : R \to R$ is continuous, f is nonnegative for $x \geq 0$, and $1/2 \leq t_2 < 1$. Bai and Fei [7] obtained the sufficient conditions for the existence of at least three positive solutions for the third-order three-point generalized right focal problem

$$x''' = q(t)f(t, x, x', x'''), \quad t_1 < t < t_3,$$
$$x(t_1) = x'(t_2) = 0, \qquad \eta x(t_3) + \delta x''(t_3) = 0,$$

where $f \in C([t_1, t_3] \times [0, \infty) \times R^2, [0, \infty))$, $q \in C((t_1, t_3), [0, \infty))$ and does not vanish identically on any subinterval of (t_1, t_3). Furthermore, $0 < \int_{t_1}^{t_2} q(s)\mathrm{d}s$, $\int_{t_2}^{t_3} q(s)\mathrm{d}s < +\infty$. $\eta \geq 0$, $\delta > 0$, $k := 2\delta + \eta(t_3 - t_1)(t_3 - 2t_2 + t_1) > 0$, $t_1 < t_2 < t_3$ are real numbers with $t_2 - t_1 > t_3 - t_2$. For other existence results for third-order three-point BVP, one may see [2–4,8–10,12–16,18,19] and the references therein.

Motivated greatly by the above-mentioned works, in this paper we will consider the existence of multiple positive solutions (at least three) to the BVP

$$u'''(t) = a(t)f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = \delta u(\eta), \qquad u'(\eta) = 0, \qquad u''(1) = 0, \tag{1.2}$$

where $\delta \in (0, 1)$, $\eta \in [1/2, 1)$ are constants, $a : (0, 1) \to [0, \infty)$ and $f : [0, 1] \times [0, \infty) \times R \times R \to [0, \infty)$ are continuous. Here, by a positive solution of the BVP we mean a function $u^*(t)$ which is positive on $(0, 1)$ and satisfies differential equation (1.1) and the boundary conditions (1.2). Therefore, our positive solutions are nontrivial ones. The methods used in our work will depend on an application of a fixed-point theorem due to Avery and Peterson [6] which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis is put on the nonlinear term involved with all lower-order derivatives explicitly. The paper is organized as follows. In Section 2, we present some notation and lemmas. In Section 3, we give the main results.

2. Preliminaries

In this section, we present some notation and lemmas that will be used in the proof our main results.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called a *cone* of E if it satisfies the following two conditions:

(1) $x \in K$, $\lambda > 0$ implies $\lambda x \in K$;
(2) $x \in K$, $-x \in P$ implies $x = 0$.

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. Suppose K is a cone in a Banach space E. The map α is a nonnegative continuous concave functional on K provided α: $K \to [0, \infty)$ is continuous and

$$\alpha(rx + (1 - r)y) \geq r\alpha(x) + (1 - r)\alpha(y)$$

for all $x, y \in K$ and $r \in [0, 1]$. Similarly, we say the map β is a nonnegative continuous concave functional on K provided β: $K \to [0, \infty)$ is continuous and

$$\beta(rx + (1 - r)y) \leq r\beta(x) + (1 - r)\beta(y)$$

for all $x, y \in K$ and $r \in [0, 1]$.

Let γ and θ be nonnegative continuous convex functionals on K, α a nonnegative continuous concave functional on K, and φ a nonnegative continuous functional on K.

For positive real numbers a, b, c, and d, we define the following convex sets:

$$P(\gamma, d) = \{x \in K \mid \gamma(x) < d\},$$
$$P(\gamma, \alpha, b, d) = \{x \in K \mid b \leq \alpha(x), \gamma(x) \leq d\},$$
$$P(\gamma, \theta, b, c, d) = \{x \in K \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},$$

and a closed set

$$R(\gamma, \varphi, a, d) = \{x \in K \mid a \leq \varphi(x), \gamma(x) \leq d\}.$$

We shall use the following well-known fixed-point theorem due to Avery and Peterson [6] to search for three positive solutions of the BVP (1.1) and (1.2).

Lemma 2.1 ([6]). *Let E be a real Banach space and $K \subset E$ be a cone in E. Let γ and θ be nonnegative continuous convex functionals on K, α a nonnegative continuous concave functional on K, and φ a nonnegative continuous functional on K satisfying $\varphi(\lambda x) \leq \lambda\varphi(x)$ for $\lambda \in [0, 1]$, such that for some positive numbers M and d,*

$$\alpha(x) \leq \varphi(x) \quad \text{and} \quad \|x\| \leq M\gamma(x), \tag{2.1}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is a completely continuous operator and there exist positive numbers a, b, and c with $a < b$ such that

(C1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$;
(C2) $\alpha(Tx) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$;
(C3) $0 \notin R(\gamma, \varphi, a, d)$ *and* $\varphi(Tx) < a$ *for* $x \in R(\gamma, \varphi, a, d)$ *with* $\varphi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\begin{aligned}
&\gamma(x_i) \leq d \quad \text{for } i = 1, 2, 3;\\
&b < \alpha(x_1);\\
&a < \varphi(x_2) \quad \text{with } \alpha(x_2) < b;\\
&\varphi(x_3) < a.
\end{aligned}$$

We need some preliminary results before proving our main results. First, Green's function for the associated linear BVP is constructed.

Lemma 2.2. *Let $\delta \neq 0$, $h \in C[0, 1]$; then BVP*

$$u'''(t) = h(t), \quad 0 < t < 1, \tag{2.2}$$

$$u(0) = \delta u(\eta), \qquad u'(\eta) = 0, \qquad u''(1) = 0, \tag{2.3}$$

has the unique solution

$$u(t) = \int_0^1 G(t, s)h(s)\mathrm{d}s,$$

where

$$G(t, s) = \begin{cases} \dfrac{s^2}{2(1-\delta)}, & s \leq t,\ s \leq \eta,\\ -\dfrac{1}{2}t^2 + ts + \dfrac{\delta s^2}{2(1-\delta)}, & t \leq s \leq \eta,\\ \dfrac{1}{2}s^2 - ts + \eta t + \dfrac{\delta\eta^2}{2(1-\delta)}, & \eta \leq s \leq t,\\ -\dfrac{1}{2}t^2 + \eta t + \dfrac{\delta\eta^2}{2(1-\delta)}, & \eta \leq s,\ t \leq s. \end{cases} \tag{2.4}$$

Proof. From (2.2) we have

$$u(t) = \frac{1}{2}\int_0^t (t-s)^2 h(s)\mathrm{d}s + At^2 + Bt + C.$$

In particular,

$$
\begin{aligned}
&u(0) = C,\\
&u''(1) = \int_0^1 h(s)\mathrm{d}s + 2A,\\
&u(\eta) = \frac{1}{2}\int_0^\eta (\eta - s)^2 h(s)\mathrm{d}s + A\eta^2 + B\eta + C,\\
&u'(\eta) = \int_0^\eta (\eta - s)h(s)\mathrm{d}s + 2A\eta + B.
\end{aligned}
$$

Combining this with boundary conditions (2.3) we conclude that

$$
\begin{aligned}
&A = -\frac{1}{2}\int_0^1 h(s)\mathrm{d}s,\\
&B = \eta\int_0^1 h(s)\mathrm{d}s - \int_0^\eta (\eta - s)h(s)\mathrm{d}s,\\
&C = \frac{\delta\eta^2}{2(1-\delta)}\int_0^1 h(s)\mathrm{d}s - \frac{\delta}{2(1-\delta)}\int_0^\eta (\eta^2 - s^2)h(s)\mathrm{d}s.
\end{aligned}
$$

Therefore, BVP (2.2) and (2.3) has a unique solution

$$
\begin{aligned}
u(t) &= \frac{1}{2}\int_0^t (t-s)^2 h(s)\mathrm{d}s - \frac{t^2}{2}\int_0^1 h(s)\mathrm{d}s + t\eta\int_0^1 h(s)\mathrm{d}s - t\int_0^\eta (\eta - s)h(s)\mathrm{d}s\\
&\quad + \frac{\delta\eta^2}{2(1-\delta)}\int_0^1 h(s)\mathrm{d}s - \frac{\delta}{2(1-\delta)}\int_0^\eta (\eta^2 - s^2)h(s)\mathrm{d}s\\
&= \frac{1}{2}\int_0^t (t-s)^2 h(s)\mathrm{d}s - \int_0^\eta \left(\frac{\delta(\eta^2 - s^2)}{2(1-\delta)} + t(\eta - s)\right)h(s)\mathrm{d}s\\
&\quad + \int_0^1 \left(-\frac{t^2}{2} + \eta t + \frac{\delta\eta^2}{2(1-\delta)}\right)h(s)\mathrm{d}s\\
&= \begin{cases}
\displaystyle\int_0^t \frac{s^2}{2(1-\delta)}h(s)\mathrm{d}s + \int_t^\eta \left(-\frac{1}{2}t^2 + ts + \frac{\delta s^2}{2(1-\delta)}\right)h(s)\mathrm{d}s & \\
\displaystyle\quad + \int_\eta^1 \left(-\frac{1}{2}t^2 + \eta t + \frac{\delta s^2}{2(1-\delta)}\right)h(s)\mathrm{d}s, & t \le \eta,\\
\displaystyle\int_0^\eta \frac{s^2}{2(1-\delta)}h(s)\mathrm{d}s + \int_\eta^t \left(\frac{1}{2}s^2 - ts + \eta t + \frac{\delta\eta^2}{2(1-\delta)}\right)h(s)\mathrm{d}s & \\
\displaystyle\quad + \int_t^1 \left(-\frac{1}{2}t^2 + \eta t + \frac{\delta\eta^2}{2(1-\delta)}\right)h(s)\mathrm{d}s, & t \ge \eta,
\end{cases}\\
&= \int_0^1 G(t,s)h(s)\mathrm{d}s.
\end{aligned}
$$

This completes the proof. □

Lemma 2.3. *Suppose* $0 < \delta < 1,\ 1/2 \le \eta < 1,\ g(s) = \frac{1}{2(1-\delta)}\min\{s^2, \eta^2\}$. *Then*

$$
\delta g(s) \le G(t,s) \le g(s), \quad t, s \in [0,1].
$$

Proof. For $s \le t, \eta$, the conclusion is obvious. For $t \le s \le \eta$, from (2.4) we know that

$$
G(t,s) = -\frac{1}{2}t^2 + ts + \frac{\delta s^2}{2(1-\delta)} \le \frac{1}{2}s^2 + \frac{\delta s^2}{2(1-\delta)} = \frac{s^2}{2(1-\delta)} = g(s)
$$

and

$$G(t,s) = -\frac{1}{2}t^2 + ts + \frac{\delta s^2}{2(1-\delta)} \geq \frac{\delta s^2}{2(1-\delta)} = \delta g(s).$$

For $\eta \leq s \leq t$, from (2.4) we know that

$$\begin{aligned}G(t,s) &= \frac{1}{2}s^2 - (s-\eta)t + \frac{\delta\eta^2}{2(1-\delta)} = \frac{s^2-\eta^2}{2} - (s-\eta)t + \frac{\eta^2}{2} + \frac{\delta\eta^2}{2(1-\delta)} \\ &= \frac{1}{2}(s-\eta)(s+\eta-2t) + \frac{\eta^2}{2(1-\delta)} \leq \frac{\eta^2}{2(1-\delta)} = g(s)\end{aligned}$$

and

$$\begin{aligned}G(t,s) &= \frac{1}{2}s^2 - (s-\eta)t + \frac{\delta\eta^2}{2(1-\delta)} \geq \frac{1}{2}s^2 - (s-\eta) + \frac{\delta\eta^2}{2(1-\delta)} \\ &\geq \frac{s^2+1}{2} - s + \frac{\delta\eta^2}{2(1-\delta)} \geq \frac{\delta\eta^2}{2(1-\delta)} = \delta g(s).\end{aligned}$$

For $\eta, t \leq s$, from (2.4) we know that

$$G(t,s) = -\frac{1}{2}t^2 + \eta t + \frac{\delta\eta^2}{2(1-\delta)} \leq \frac{1}{2}\eta^2 + \frac{\delta\eta^2}{2(1-\delta)} = g(s)$$

and

$$G(t,s) = -\frac{1}{2}t^2 + \eta t + \frac{\delta\eta^2}{2(1-\delta)} \geq -\frac{1}{2}t^2 + \frac{1}{2}t + \frac{\delta\eta^2}{2(1-\delta)} \geq \frac{\delta\eta^2}{2(1-\delta)} = \delta g(s).$$

The proof is complete. □

From Lemma 2.2 we know that if $0 < \delta < 1$, $1/2 \leq \eta < 1$, then for $h \in C^+[0,1]$, the unique solution $u(t)$ of BVP (2.2) and (2.3) is nonnegative and satisfies

$$\min_{t\in[0,1]} u(t) \geq \delta \max_{t\in[0,1]} u(t). \tag{2.5}$$

In what follows, we shall consider the Banach space $C^2[0,1]$ equipped with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0,1]$, and the maximum norm

$$\|u\| = \max\left\{\max_{0\leq t\leq 1}|u(t)|,\ \max_{0\leq t\leq 1}|u'(t)|,\ \max_{0\leq t\leq 1}|u''(t)|\right\}.$$

Define the cone K by

$$K = \left\{u \in C^+[0,1] : u(0) = \delta u(\eta),\ u'(\eta) = 0,\ u''(1) = 0, u \text{ is concave on } [0,1]\right\}.$$

Define the integral operator $T : K \to C^+[0,1]$ by

$$Tu(t) = \int_0^1 G(t,s)a(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s. \tag{2.6}$$

By Lemma 2.1, BVP (1.1) and (1.2) has a positive solution $u^* = u^*(t)$ if and only if u^* is a fixed point of T.

We adopt the following assumptions:

(H1) $a \in C((0,1),[0,\infty))$ with $0 < \int_0^1 a(s)\mathrm{d}s < \infty$.

(H2) $f \in C([0,1]\times[0,\infty)\times R\times(-\infty,0],[0,\infty))$.

Lemma 2.4. *Assume that* (H_1)–(H_2) *hold. Then* $T : K \to K$ *is completely continuous.*

Proof. From the fact that $u'''(t) = a(t)f(t,u(t),u'(t),u''(t)) \geq 0, u(t) \geq 0$ and Lemma 2.1, we know that u is concave on $[0,1]$. By Lemma 2.2, we know that $T(K) \subset K$. The operator T is completely continuous by an application of the Ascoli–Arzela theorem (see [17]). □

3. Main results

In this section we impose growth conditions on f which allow us to apply Lemma 2.1 to establish the existence of triple positive solutions of BVP (1.1) and (1.2). Let the nonnegative continuous concave functional α, the nonnegative continuous convex functional θ, γ, and the nonnegative continuous functional φ be defined on the cone K by

$$\gamma(u) = \max_{0\le t\le 1} |u''(t)|, \qquad \varphi(u) = \theta(u) = \max_{0\le t\le 1} |u(t)|, \qquad \alpha(u) = \min_{0\le t\le 1} |u(t)|.$$

Note that for $u \in K$, we have

$$\begin{aligned} u(t) &= u(0) + \int_0^t u'(s)\mathrm{d}s = \delta u(\eta) + \int_0^t u'(s)\mathrm{d}s \\ &\le \delta \max_{0\le t\le 1} |u(t)| + t \max_{0\le t\le 1} |u'(t)| \le \delta \max_{0\le t\le 1} |u(t)| + \max_{0\le t\le 1} |u'(t)|, \end{aligned}$$

$$u'(t) = u'(\eta) + \int_\eta^t u''(s)\mathrm{d}s \le \eta \max_{0\le t\le 1} |u''(t)|. \tag{3.1}$$

Therefore,

$$\max_{0\le t\le 1} |u(t)| \le \frac{1}{1-\delta} \max_{0\le t\le 1} |u'(t)| \le \frac{\eta}{1-\delta} \max_{0\le t\le 1} |u''(t)|. \tag{3.2}$$

Consequently, combining with the concavity of u, the functionals defined above satisfy

$$\begin{aligned} &\delta\theta(u) \le \alpha(u) \le \theta(u) = \varphi(u), \\ &\|u\| \le \frac{\eta}{1-\delta}\gamma(u), \end{aligned} \tag{3.3}$$

for all $u \in \overline{P(\gamma, d)}$. Therefore, condition (2.1) is satisfied.

For convenience, in what follows, we denote the constants by

$$B = \min\left\{\int_0^1 G(0,s)a(s)\mathrm{d}s,\ \int_0^1 G(1,s)a(s)\mathrm{d}s\right\},$$

$$M = \int_0^1 a(s)\mathrm{d}s,$$

$$N = \int_0^1 g(s)a(s)\mathrm{d}s.$$

Now we present our main result and proof.

Theorem 3.1. *Suppose* (H_1)–(H_2) *hold. Assume there exist* $0 < a < b \le \frac{\delta\eta}{1-\delta}d$ *such that*

(A1) $f(t,u,v,w) \le \frac{d}{M}$, *for* $(t,u,v,w) \in [0,1] \times [0, \frac{\eta}{1-\delta}d] \times [-\eta d, \eta d] \times [-d, 0]$;
(A2) $f(t,u,v,w) > \frac{b}{B}$, *for* $(t,u,v,w) \in [0,1] \times [b, \frac{1}{\delta}b] \times [-\eta d, \eta d] \times [-d, 0]$;
(A3) $f(t,u,v,w) < \frac{a}{N}$, *for* $(t,u,v,w) \in [0,1] \times [0, a] \times [-\eta d, \eta d] \times [-d, 0]$.

Then BVP (1.1) and (1.2) has at least three positive solutions u_1, u_2, *and* u_3 *satisfying*

$$\max_{0\le t\le 1} |u_i''(t)| \le d \quad \text{for } i = 1,2,3;$$

$$\max_{0\le t\le 1} |u_i'(t)| \le \eta d \quad \text{for } i = 1,2,3;$$

$$b < \min_{0\le t\le 1} |u_1''(t)|;$$

$$a < \max_{0\le t\le 1} |u_2(t)| \le \frac{b}{\delta} \quad \text{with } \min_{0\le t\le 1} |u_2(t)| < b;$$

$$\max_{0\le t\le 1} |u_3(t)| < a.$$

Proof. We now show that all the conditions of Lemma 2.1 are satisfied.

If $u \in \overline{P(\gamma, d)}$, then $\gamma(u) = \max_{0\le t\le 1} |u''(t)| \le d$. From (3.1) and (3.2), one has $\max_{0\le t\le 1} |u'(t)| \le \eta d$, $\max_{0\le t\le 1} |u(t)| \le \frac{\eta}{1-\delta} d$; then assumption (A1) implies $f(t, u, u', u'') \le \frac{d}{M}$. Note that for any $t \in [0, 1]$,

$$(Tu)''(t) = -\int_t^1 a(s) f(s, u(s), u'(s), u''(s))\mathrm{d}s.$$

Therefore,

$$\gamma(u) = \max_{0\le t\le 1} |(Tu)''(t)| = \int_t^1 a(s) f(s, u(s), u'(s), u''(s))\mathrm{d}s \le \frac{\mathrm{d}}{M}\int_0^1 a(s)\mathrm{d}s = d.$$

Hence, $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

To check condition (C1) of Lemma 2.1, we choose $u(t) = b/\delta,\ t \in [0, 1]$. It is easy to see that

$$u(t) = b/\delta \in P(\gamma, \theta, b, b/\delta, d)$$

and

$$\alpha(u) = \alpha(b/\delta) = b/\delta > b,$$

and so

$$\{u \in P(\gamma, \theta, b, b/\delta, d) | \alpha(u) > a\} \neq \emptyset.$$

Hence, if $u \in P(\gamma, \theta, b, b/\delta, d)$, then $b \le u(t) \le b/\delta$, $|u'(t)| \le \eta d$, $|u''(t)| \le d$ for $t \in [0, 1]$. From assumption (A2), we have $f(t, u, u', u'') > \frac{b}{B}$ for $t \in [0, 1]$, and by the conditions of α and the cone K, we have to distinguish two cases, (i) $\alpha(Tu) = Tu(0)$ and (ii) $\alpha(Tu) = Tu(1)$.

In case (i), we have

$$\alpha(Tu) = Tu(0) = \int_0^1 G(0, s)a(s) f(s, u(s), u'(s), u''(s))\mathrm{d}s > \frac{b}{B}\int_0^1 G(0, s)a(s)\mathrm{d}s \ge b.$$

In case (ii), we have

$$\alpha(Tu) = Tu(1) = \int_0^1 G(1, s)a(s) f(s, u(s), u'(s), u''(s))\mathrm{d}s > \frac{b}{B}\int_0^1 G(1, s)a(s)\mathrm{d}s \ge b.$$

i.e.

$$\alpha(Tu) > b \quad \text{for all } u \in P(\gamma, \theta, b, b/\delta, d).$$

This shows that condition (C1) of Lemma 2.1 is satisfied.

Secondly, from (3.3) we have

$$\alpha(Tu) \ge \delta\theta(Tu) > \delta \cdot \frac{b}{\delta} = b,$$

for all $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > \frac{b}{\delta}$. Thus, condition (C2) of Lemma 2.1 is satisfied.

Finally we show that (C3) of Lemma 2.1 holds, too. Clearly, as $\varphi(0) = 0 < a$, we have that $0 \notin R(\gamma, \varphi, a, d)$. Suppose that $u \in R(\gamma, \varphi, a, d)$ with $\varphi(u) = a$. Then, by the assumption (A3),

$$\begin{aligned}\varphi(Tu) &= \max_{0\le t\le 1} |Tu(t)| \\ &= \max_{0\le t\le 1} \int_0^1 G(t, s)a(s) f(s, u(s), u'(s), u''(s))\mathrm{d}s \\ &\le \int_0^1 g(s)a(s) f(s, u(s), u'(s), u''(s))\mathrm{d}s \\ &< \frac{a}{N}\int_0^1 g(s)a(s)\mathrm{d}s = a.\end{aligned}$$

So, condition (C3) of Lemma 2.1 is also satisfied. Therefore, an application of Lemma 2.1 ends the proof. □

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