On a Class of Commutative Power-Associative Nilalgebras∗

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We prove that commutative power associative nilalgebras of nilindex $n$ and dimension $n$ are nilpotent of index $n$. We find a necessary and sufficient condition for such an algebra to be a Jordan algebra and give all corresponding isomorphism classes.

1. INTRODUCTION

Gerstenhaber and Myung [1] have proved that commutative power-associative nilalgebras of dimension 4 over a field of characteristic not 2 are nilpotent. Using this result they prove that Suttles’s dimension 5 counterexample to Albert’s conjecture is the best possible (see Suttles [3]). In particular, Gerstenhaber and Myung prove that a commutative power-associative nilalgebra of nilindex 4 and dimension 4 is nilpotent of index 4.

In this paper we prove that for any $n$ a commutative power-associative nilalgebra of nilindex $n$ and dimension $n$ over a field of characteristic ≠ 2, 3 is nilpotent of index $n$.

Gerstenhaber and Myung prove that in the case $n = 4$ there is a $y \in A - A^2$ such that $yA^2 = 0$. We show that the existence of such an element $y$ is precisely the property that characterizes the subclass of Jordan algebras.

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Throughout, \( \mathcal{A} \) will denote a commutative power-associative nilalgebra of nilindex \( n \) and dimension \( n \), \( n > 2 \) over a field \( K \) with \( \text{Char} \ K \neq 2, 3 \).

We will denote by \( \langle a_1, \ldots, a_j \rangle_K \) the subspace of \( \mathcal{A} \) generated over \( K \) by the elements \( a_1, \ldots, a_j \in \mathcal{A} \).

Linearizing the identity \( x^4 = x^2x^2 \) yields

\[
2((xy)x)x + (x^2y)x + x^3y = 4(xy)x^2, \tag{1}
\]
\[
2((yz)x)x + 2((xy)z)x + 2((xy)x)z + 2((xz)y)x + (x^2y)z
+ 2((xz)x)y + (x^2z)y = 4(yz)x^2 + 8(xy)(xz) \tag{2}
\]

and

\[
(x^2y)x + 2((xy)y)x + 2((xy)x)y + (x^2y)y = 4(xy)^2 + 2x^2y^2. \tag{3}
\]

In what follows we fix an element \( x \) in \( \mathcal{A} \) with \( x^{n-1} \neq 0 \). We remark that \( x^2 \neq 0 \), that the powers \( x, x^2, \ldots, x^{n-1} \) are linearly independent, and that the subspace \( X = \langle x, x^2, \ldots, x^{n-1} \rangle_K \) is an associative subalgebra of \( \mathcal{A} \) of dimension \( n-1 \).

2. NILPOTENCE

Following Gerstenhaber and Myung [1] we obtain the following result:

**Theorem 1.** \( \mathcal{A} \) is nilpotent of index \( n \).

**Proof.** The idea of the proof is to prove that \( \mathcal{A}^s = X^s \) for any \( s > 1 \).

For \( s = 2 \) we only need to check that \( y^2 \in X^2 \) if \( y \in \mathcal{A} - X \). Let \( y \in \mathcal{A} - X \) and let \( Y = \langle y, y^2, \ldots, y^{n-1} \rangle_K \). We have \( n = \dim(X + Y) = n - 1 + \dim Y = \dim X \cap Y \). Hence, since \( \dim Y \leq \dim X \), we have

\[
\dim X \cap Y = \dim Y - 1 = \dim \langle y^2, \ldots, y^{n-1} \rangle_K < \dim X.
\]

Hence, \( X \cap Y \) is a proper subalgebra of \( X \) and is therefore a subalgebra of \( X^2 \). In fact, if we assume the contrary, then there exists an element \( a = a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} \) in \( X \cap Y \) with \( a_2 \neq 0 \). Therefore

\[
\dim \langle a, a^2, \ldots, a^{n-1} \rangle_K = n - 1 = \dim X
\]

which is a contradiction. By the same argument we can prove that \( X \cap Y \) is a subalgebra of \( Y^2 \). Since \( \dim X \cap Y = \dim Y^2 \), we get that \( Y^2 = X \cap Y \) and therefore \( Y^2 \subset X^2 \). It follows that \( y^2 \in X^2 \) and so \( \mathcal{A}^2 = X^2 \).

Since \( X \) is an associative subalgebra of \( \mathcal{A} \), for all elements \( y \in \mathcal{A} \) and \( k \geq 1 \) we have that \( ((xy)x^k)x = (xy)x^{k+1}, (xy)x^k = (xy)x^{k+1} \), and
\[(x^k y) x = (x^k y) x^2,\] thus if we replace \(z\) by \(x^k\) in (2) we obtain
\[
3x^{k+2} y = 4(xy)x^{k+1} + 2(x^k y)x^2 - 2(x^{k+1} y)x - (x^2 y)x^k. \tag{4}
\]

It is now easy to obtain inductively that \(x^s y \in X^s\) for all \(s \geq 1\) (and therefore \(X^s\) is an ideal of \(A\) for every \(s \geq 1\)). We will prove that actually \(x^s y \in X^{s+1}\). In fact, since \(x^2 y \in X^2\) there are \(a_2, a_3, \ldots, a_{n-1} \in K\) such that
\[
x^2 y = a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1}.
\]
If \(a_2 \neq 0\) we can assume that \(a_2 = 1\). Hence, \(x^2 (y + a_3 x + \cdots + a_{n-1} x^{n-3}) = x^2.\) If we let \(z = y + a_3 x + \cdots + a_{n-1} x^{n-3}\) then we have that \(x^2 z = x^2.\) Then straightforward calculations show that
\[
0 = (z + x^2)^n \equiv 2x^2 (\text{mod } X^4).
\]

This is a contradiction. Therefore \(a_2 = 0\) and \(x^2 y \in X^3.\) Using (4) it is easy to obtain inductively that \(x^s y \in X^{s+1}\) and \(X^s = X^s\) for any \(s \geq 2.\) In particular \(A^s = X^s = 0\), which proves the theorem.

3. JORDAN ALGEBRAS

We remark that we can always find \(y \in A - X\) such that \(x^2 y = 0.\) In fact, if \(y' \in A - X\) with \(x^2 y' \neq 0\) and if \(x^2 y' = \alpha_3 x^3 + \cdots + a_{n-1} x^{n-1}\), then \(y = y' - (\alpha_3 x + \cdots + a_{n-1} x^{n-3})\) satisfies \(x^2 y = 0\) which proves the claim.

**Theorem 2.** If \(n \geq 4\), then \(A\) is a Jordan algebra if and only if there is \(y \in A - A^2\) such that \(yA^2 = 0.\)

**Proof.** Assume \(A\) is a Jordan algebra. Let \(y \in A - X\) be such that \(x^2 y = 0.\) If \(xy = \alpha_2 x^2 + \cdots + a_{n-1} x^{n-1}\), we have
\[
0 = (x^2 y)x = x^2 (yx) = \alpha_2 x^4 + \cdots + a_{n-3} x^{n-1}.
\]
It follows that \(a_2 = \cdots = a_{n-3} = 0\) and \(xy = a_{n-2} x^{n-2} + a_{n-1} x^{n-1}\). Now from (1) we have \(x^2 y = 2x^2 (xy) = 2a_{n-2} x^n + 2a_{n-1} x^{n+1} = 0.\) Replacing \(z\) by \(x^k\) in (2) we obtain
\[
2(x^{k+1} y)x + 3x^{k+2} y = 4(xy)x^{k+1} + 2(x^k y)x^2
\]
for any \(k.\) Therefore \(x^s y = 0\) for all \(s \geq 2,\) whence \(yA^2 = 0.\)

We suppose now that there is \(y \in A - A^2\) such that \(yA^2 = 0.\) We will prove that \(y \in A - X.\) In fact, if \(y \in X\) then \(y = \alpha_1 x + \cdots + a_{n-1} x^{n-1}\) and \(0 = x^2 y = \alpha_1 x^3 + \cdots + a_{n-1} x^{n-1}.\) Whence, \(\alpha_1 = \cdots = a_{n-3} = 0\) and \(y = a_{n-2} x^{n-2} + a_{n-1} x^{n-1} \in A^2\), which is a contradiction. Therefore \(y \in A - X.\)
Now from (1) we obtain $2(xy)x^2 = 0$ and therefore $xy \in X^{n-2}$. From (3) we obtain $x^2y^2 = 0$ and therefore $y^2 \in X^{n-2}$.

If $x_0 = \alpha_0 y + p$ and $y_0 = \beta_0 y + q$ with $p, q \in X$ arbitrary elements in $A$, we have $(x_0^2y_0)x_0 - x_0^2(y_0x_0) = (p^2q)p - p^2(qp)$. Since $X$ is an associative subalgebra of $A$ the result follows. This proves the theorem.

**Corollary.** If $n$ is 3 or 4 then $A$ is a Jordan algebra if and only if it is a power-associative algebra.

**Proof.** It is known that a commutative nilalgebra of nilindex 3 is a Jordan algebra and that any Jordan algebra is power-associative (see [4, 2]). Let $n = 4$. If $A$ is a power-associative algebra and $y \in A - X$ is such that $x^2y = 0$, then, since $x^2y \in X^4 = 0$ we obtain that $A$ is a Jordan algebra from Theorem 2. This proves the corollary.

Dimension 4 is the highest dimension for which the Corollary is valid, as the following example shows:

**Example.** Let $B$ be a commutative algebra with basis $y, x, x^2, x^3, x^4$ and nonzero multiplication given by $xy = x^2, x^3y = 2x^4$, and $x^i x^j = x^{i+j}$ for $2 \leq i + j \leq 4$. $B$ is a power-associative nilalgebra of nilindex 5 which is not Jordan since $0 = (x^2y)x \neq x^2(xy) = x^4$.

4. **Classification**

Throughout this section the dimension will be always assumed to be $\geq 4$. If $y \in A - A^2$ is such that $yA^2 = 0$, then from the proof of the Theorem 2 we conclude that $xy = \alpha x^{n-2} + \delta x^{n-1}$ and $y^2 = \beta x^{n-2} + \gamma x^{n-1}$. If we let $y' = y - \delta x^{n-2}$, then we have $xy' = xy - \delta x^{n-1} = \alpha x^{n-2} + \gamma x^{n-1}$. Hence we can assume that $\delta = 0$. In order to obtain a classification theorem for these algebras it is only necessary to know the products $xy$ and $y^2$.

For Theorem 3 we will need additionally that Char $K$ not be divisor of $n - 2$.

**Theorem 3.** If $A$ is a Jordan algebra, then there is $y \in A - A^2$ such that $yA^2 = 0$, and the products $xy$ and $y^2$ are given by one of the following list:

1. $xy = 0$, $y^2 = 0$.
2. $xy = 0$ and $y^2 = \gamma x^{n-1}$ with $\gamma \neq 0$.
3. $xy = 0$ and $y^2 = \beta x^{n-2}$ with $\beta \neq 0$.
4. $xy = x^{n-2}$ and $y^2 = 0$.
5. $xy = x^{n-2}$ and $y^2 = \gamma x^{n-1}$ with $\gamma \neq 0$.
6. If $n = 4$, $xy = x^2 = y^2$. 


Proof. Let $xy = \alpha x^{n-2}$ and $y^2 = \beta x^{n-2} + \gamma x^{n-1}$.

Case $\alpha = 0$. If $\beta = \gamma = 0$ we obtain the algebra (I). If $\beta = 0$ and $\gamma \neq 0$ we obtain (II). If $\beta \neq 0$, we have $y^2 = \beta(x^n + (\gamma/\beta(n-2))x^2)$. Therefore, letting $x' = x + (\gamma/\beta(n-2))x^2$ we obtain $x'y = 0$ and $y^2 = \beta x^{n-2}$. Hence we may assume $\gamma = 0$, which gives (III).

Case $\alpha \neq 0$. If $y' = (1/\alpha)y$ we have that $xy' = x^{n-2}$ and $y^2 = \beta'x^{n-2} + \gamma'x^{n-1}$. We may assume that $\alpha = 1$. Therefore $xy = x^{n-2}$ and $y^2 = \beta x^{n-2} + \gamma x^{n-1}$. If $y = 0$ we obtain (IV). If $\beta = 0$ and $\gamma \neq 0$ we obtain (V). If $\beta \neq 0$, let $x' = x + (\gamma/\beta(n-2))x^2$ and $y' = y + (\gamma/\beta)x^n$. Then $y^2 = \beta x^{n-2}$ and $x'y = x^{n-2}$. Consequently, we may assume that $\gamma = 0$. Thus $xy = x^{n-2}$ and $y^2 = \beta x^{n-2}$ with $\beta \neq 0$. We note that $y(y - \beta x) = 0, (y - \beta x)^3 = -\beta^2 x^3 + \beta^2 x^n$, and for all $k \geq 4, (y - \beta x)^k = (-1)^k(\beta x)^k$. Hence for $n \geq 5, (y - \beta x)^n = (-1)^n \beta x^{n-1} \neq 0$. In such case, if $x' = y - \beta x$, we have $x'y = 0$ and the algebra is one of the above.

Now, if $n = 4$, we have $(y - \beta x)^3 = \beta^2(1 - \beta)x^3$. If $\beta^2(1 - \beta) \neq 0$, the algebra is one of the above. If $\beta^2(1 - \beta) = 0$, then $\beta = 1$ and $y^2 = xy = x^2$, and we obtain (VI).

If $C = \{a \in A: aA^2 = 0\} = Ky + A^{n-2}$, then $C^2 = 0$ in cases (I) and (IV), $0 \neq C^2 \neq A^{n-1}$ in (II) and (VI), and $C^2 = A^{n-1}$ in (I) and (V), and also $CA = A^{n-1}$ in (I) and (II) and $CA = A^{n-2}$ otherwise. Hence only the cases (II) and (VI) have to be separated in dimension 4. But in (II), \( \{a \in A: aC \subset A^{n-1}\} = X \), where there are elements of order $n$, and in (VI) \( \{a \in A: aC \subset A^{n-1}\} = K(x - y) + A^2 \), where any element has order $\leq 3$. This shows that algebras from different classes (I), (II), ..., (VI) are not isomorphic.

We observe that if $y$ and $y'$ are elements in $A - A^2$ with $yA^2 = 0$, $y'A^2 = 0$, then $C = KY + A^{n-2} = K'y' + A^{n-2}$. Thus $y' = \delta y + \epsilon x^{n-2} + \lambda x^{n-1}$, where $\delta \neq 0$. Therefore $y'^2 = \delta^2 y^2$.

Now, we will obtain the isomorphism classes in cases (I), (III), and (V). In the following we will assume that $n \geq 5$. The case $n = 4$ was covered by Gerstenhaber and Myung in [1].

Let $B$ be another commutative Jordan nilalgebra of nilindex $n$ and dimension $n$ over $K$, and $y_0, x_0, x_0^2, \ldots, x_0^{n-1}$ a basis of $B$ such that $y_0 \in B - B^2$, $x_0 B^2 = 0$, $x_0 y_0 = \alpha_0 x_0^{n-2}$, and $y_0^2 = \beta_0 x_0^{n-2} + \gamma_0 x_0^{n-1}$.

We observe that if $\varphi: B \to A$ is an isomorphism, then $\varphi(y_0) \in A - A^2$ and $\varphi(y_0)A^2 = 0$. Hence $\varphi(y_0) = \delta y + \epsilon x^{n-2} + \lambda x^{n-1}$ where $\delta, \epsilon, \lambda \in K$ and $\delta \neq 0$. Hence, $\varphi(y_0^2) = \varphi(y_0)^2 = \delta^2 y^2$. Now, if $\varphi(x_0) = \alpha_0 y + \alpha_1 x + \cdots + \alpha_n x^n$, then we have that $\varphi(x_0^{n-2}) = \alpha_1 x^{n-2} + F(\alpha_0, \alpha_1, \alpha_2) x^{n-1}$ and $\varphi(x_0^{n-1}) = \alpha_1^{n-1} x^n - 1$, where $F(\alpha_0, \alpha_1, \alpha_2)$ is a certain polynomial expression in $\alpha_0, \alpha_1, \alpha_2$. 
Case (II). If \( n \) is an even number, letting \( x' = \gamma x \) and \( y' = \gamma^{(n-3)/2} y \), we obtain \( x'y' = 0 \) and \( y'^2 = x'^{n-1} \). So in this case there is a unique algebra. If \( n \) is an odd number and \( \gamma_0 \neq 0 \in K \), \( x_0y_0 = 0 \), and \( y_0^2 = \gamma_0 x_0^{n-1} \), we have that if \( \varphi : B \to A \) is an isomorphism, then \( \varphi(y_0^2) = \delta^2 y^2 \). Hence we obtain \( \delta^2 y = \gamma_0 \alpha_3^{n-1} \), that is, \( \gamma_0/y \in (k^*)^2 \). Conversely, if \( \gamma_0/y = k^2 \in (k^*)^2 \) the linear function \( \varphi : B \to A \) such that \( \varphi(y_0) = ky \) and \( \varphi(x_0^k) = x^i \) for \( i = 1, 2, \ldots, n \) is an isomorphism of algebras. Thus in this case we have found one family of algebras parametrized by \( k^*/(k^*)^2 \).

Case (III). If \( n \) is an odd number, letting \( x' = \beta x \) and \( y' = \beta^{(n-3)/2} y \), we obtain \( x'y' = 0 \) and \( y'^2 = x'^{n-2} \). So in this case we have again a unique algebra. Now, if \( n \) is an even number, \( \beta_0 \neq 0 \), \( x_0y_0 = 0 \), \( y_0^2 = \beta_0 x_0^{n-2} \), and \( \varphi : B \to A \) is an isomorphism, then we have \( \delta^3 \beta = \beta_0 \alpha_5^{n-2} \). Hence \( \beta_0/\beta \in (k^*)^2 \). Conversely, if \( \beta_0/\beta = k^2 \in (k^*)^2 \), the linear function \( \varphi : B \to A \) such that \( \varphi(y_0) = ky \), \( \varphi(x_0^k) = x^i \), \( i = 1, 2, \ldots, n \), is an isomorphism. Thus in this case we have found another family of algebras parametrized by \( k^*/(k^*)^2 \).

Case (V). Let \( \gamma_0 \neq 0 \), \( x_0y_0 = x_0^{n-2} \), and \( y_0^2 = \gamma_0 x_0^{n-1} \). If \( \varphi : B \to A \) is an isomorphism we obtain \( \delta^2 y = \gamma_0 \alpha_3^{n-1} \). Since \( \varphi(x_0y_0) = \varphi(x_0)\varphi(y_0) \), we have \( \alpha_3 \delta = \alpha_3^{n-2} \). Thus \( \delta = \alpha_3^{n-3} \). Therefore \( \alpha_3^{n-3} y = \gamma_0 \). Now, if \( n = 5 \), \( A \) and \( B \) are isomorphic if and only if \( \gamma = \gamma_0 \). If \( n > 5 \), and there is a \( k \in K \) such that \( k^{n-5} y = \gamma_0 \), then the linear function \( \varphi : B \to A \) such that \( \varphi(y_0) = k^{n-2} y \), \( \varphi(x_0^k) = (kx)^i \), \( i = 1, 2, \ldots, n \) is an isomorphism of algebras. This completes our classification.

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