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Hyperbolic manifolds with geodesic boundary which are determined by their fundamental group

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Abstract

Let M_1 and M_2 be n-dimensional connected orientable finite-volume hyperbolic manifolds with geodesic boundary, and let $\varphi : \pi_1(M_1) \to \pi_1(M_2)$ be a given group isomorphism. We study the problem whether there exists an isometry $\psi : M_1 \to M_2$ such that $\psi_* = \varphi$. We show that this is always the case if $n \geqslant 4$, while in the 3-dimensional case the existence of ψ is proved under some (necessary) additional conditions on φ. Such conditions are trivially satisfied if ∂M_1 and ∂M_2 are both compact.

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Let M_1 and M_2 be connected orientable finite-volume hyperbolic n-manifolds with geodesic boundary. Suppose $n \geqslant 3$ and let $\varphi : \pi_1(M_1) \to \pi_1(M_2)$be an isomorphism of abstract groups. We determine necessary and sufficient conditions for φ to be induced by an isometry $\psi : M_1 \to M_2$. When this is the case, we say that φ is *geometric* (see Section 1 for a more detailed definition). Mostow–Prasad's rigidity theorem ensures geometricity of φ whenever the boundary of M_i is empty for $i = 1, 2$. Building on classical results in the theory of Kleinian groups, we will extend Mostow–Prasad's result to the non-empty

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boundary case, following slightly different strategies according to the dimension of the manifolds involved.

If M_1 and M_2 are 3-dimensional hyperbolic manifolds with non-empty geodesic boundary, applying Mostow–Prasad's rigidity theorem to their doubles, i.e., to the manifolds obtained by mirroring M_1 and M_2 in their boundary, we will show that φ is geometric provided it is induced by a homeomorphism, rather than an isometry. A result of Marden and Maskit [8] will then be applied to relate the existence of a homeomorphism inducing φ to the behaviour of φ with respect to the *peripheral* subgroups of $\pi_1(M_1)$ and $\pi_1(M_2)$ (see below for a definition).

If $\dim(M_1) = \dim(M_2) \geqslant 4$, the existence of an isometry $\psi : M_1 \to M_2$ such that $\psi_* = \varphi$ will be proved by a more direct argument using results from [1,12].

1. Preliminaries and statement

In this section we list some preliminary facts about the topology and geometry of orientable finite-volume hyperbolic n-manifolds with geodesic boundary and we state our main theorem and its corollaries. From now on we will always suppose $n \geqslant 3$. Moreover, all manifolds will be connected and orientable. We omit all proofs about the basic material addressing the reader to [3,6,7].

Before going into the real matter, we devote the first paragraph to give a formal definition of the notion of *geometric isomorphism* between fundamental groups of hyperbolic manifolds. To this aim we will need to spell out in detail some well-known elementary results in the theory of fundamental groups.

1.1. Homomorphisms between fundamental groups

If $\varphi, \varphi' : G \to H$ are group homomorphisms, we say that φ' is conjugated to φ if there exists $h \in H$ such that $\varphi'(g) = h\varphi(g)h^{-1}$ for every $g \in G$. Let X be a manifold and x_0, x_1 be points in X. Then there exists an isomorphism $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ which is canonical up to conjugacy. It follows that an abstract group $\pi_1(X)$ is well-defined and for any $x_0 \in X$ there exists a preferred conjugacy class of isomorphisms between $\pi_1(X)$ and $\pi_1(X, x_0)$.

If $f : X \to Y$ is a continuous map between manifolds, then f determines a well-defined conjugacy class of homomorphisms $f_* \in \mathrm{Hom}(\pi_1(X), \pi(Y))/\pi_1(Y)$. If a homomorphism $\varphi : \pi_1(X) \to \pi_1(Y)$ is given, we say that φ is induced by f if φ belongs to f_*; if so, with an abuse we will write $\varphi = f_*$, rather than $[\varphi] = f_*$.

Definition 1.1. Let M_1 and M_2 by hyperbolic manifolds with geodesic boundary and $\varphi : \pi_1(M_1) \to \pi_1(M_2)$ be a group isomorphism. Then φ is geometric if $\varphi = \psi_*$ for some isometry $\psi : M_1 \to M_2$.

1.2. Natural compactification of hyperbolic manifolds

Let N be an orientable complete finite-volume hyperbolic n-manifold with (possibly empty) geodesic boundary (from now on we will summarize all this information saying just that N is hyperbolic). Then ∂N, endowed with the Riemannian metric it inherits from N, is a hyperbolic $(n-1)$-manifold without boundary (completeness of ∂N is obvious, and the volume of ∂N is proved to be finite in [6]). Moreover, if $D(N)$ is the *double* of N, i.e., the manifold obtained by mirroring N along its geodesic boundary, then $D(N)$ admits an obvious complete finite-volume hyperbolic structure. Also observe that $D(N)$ has no boundary, and that there exist natural embeddings $\partial N \hookrightarrow N \hookrightarrow D(N)$, so we can think of ∂N and N as of subsets of $D(N)$. It is well-known [1,9] that $D(N)$ consists of a compact portion together with some cusps based on Euclidean $(n-1)$-manifolds, and it is easily seen that the ends of N can be obtained by intersecting N with the ends of $D(N)$. So also N consists of a compact portion together with some cusps of the form $T \times [0,\infty)$, where T is a compact Euclidean $(n-1)$-manifold with (possibly empty) geodesic boundary such that $(T \times [0,\infty)) \cap \partial N = \partial T \times [0,\infty)$. A cusp based on a closed Euclidean $(n-1)$-manifold is therefore disjoint from ∂N and is called *internal*, while a cusp based on a Euclidean $(n-1)$-manifold with non-empty boundary intersects ∂N in one or two internal cusps of ∂N, and is called a *boundary cusp*. This description of the ends of N easily implies that N admits a natural compactification $\overline{N}$ obtained by adding a closed Euclidean $(n-1)$-manifold for each internal cusp and a compact Euclidean $(n-1)$-manifold with non-empty geodesic boundary for each boundary cusp. Some more details on the structure of the ends of N will be given in the last two paragraphs of this section.

When $n = 3$, $\overline{N}$ is obtained by adding to N some tori and some closed annuli. In this case we denote by $\mathcal{A}_N$ the family of added closed annuli, and we observe that no annulus in $\mathcal{A}_N$ lies on a torus in $\partial \overline{N}$. Note also that $\mathcal{A}_N = \emptyset$ if ∂N is compact. A loop $\gamma \in \pi_1(N)$ will be called an *annular cusp loop* if it is freely-homotopic to a loop in some annulus of $\mathcal{A}_N$.

1.3. Main result

We are now ready to state our main result.

Theorem 1.2. *Let N_1 and N_2 be hyperbolic n-manifolds, and let $\varphi : \pi_1(N_1) \to \pi_1(N_2)$ be a group isomorphism. If $n = 3$, suppose also that the following condition holds*:

- *$\varphi(\gamma)$ is an annular cusp loop in $\pi_1(N_2)$ if and only if γ is an annular cusp loop in $\pi_1(N_1)$.*

Then φ is geometric.

Theorem 1.2 readily implies the following corollaries:

Corollary 1.3. *Let N_1 and N_2 be hyperbolic 3-manifolds with compact geodesic boundary and let $\varphi : \pi_1(N_1) \to \pi_1(N_2)$ be an isomorphism. Then φ is geometric.*

Corollary 1.4. *Let N be a hyperbolic n-manifold, let* $\mathrm{Isom}(N)$ *be the group of isometries of N and let* $\mathrm{Out}(\pi_1(N)) := \mathrm{Aut}(\pi_1(N))/\pi_1(N)$ *be the group of the outer isomorphisms of* $\pi_1(N)$. *If $n = 3$, suppose also that the boundary of N is compact. Then there is a natural isomorphism* $\mathrm{Isom}(N) \cong \mathrm{Out}(\pi_1(N))$.

Proof. Let $h : \mathrm{Isom}(N) \to \mathrm{Out}(\pi_1(N))$ be the map defined by $h(\psi) = \psi_*$. Then h is a well-defined homomorphism. Injectivity of h is a well-known fact, while surjectivity of h is an immediate consequence of Theorem 1.2 and Corollary 1.3. □

1.4. Universal covering and action at infinity

Let N be a n-dimensional hyperbolic manifold and let $\pi : \widetilde{N} \to N$ be the universal covering of N. By developing $\widetilde{N}$ in $\mathbb{H}^n$ we can identify $\widetilde{N}$ with a convex polyhedron of $\mathbb{H}^n$ bounded by a countable number of disjoint geodesic hyperplanes S_i, $i \in \mathbb{N}$. For any $i \in \mathbb{N}$ let S_i^+ denote the closed half-space of $\mathbb{H}^n$ bounded by S_i and containing $\widetilde{N}$, let S_i^- be the closed half-space of $\mathbb{H}^n$ opposite to S_i^+ and let Δ_i be the internal part of the closure at infinity of S_i^-. Of course we have $\widetilde{N} = \bigcap_{i \in \mathbb{N}} S_i^+$, so denoting by $\widetilde{N}_\infty$ the closure at infinity of $\widetilde{N}$ we obtain $\widetilde{N}_\infty = \partial\mathbb{H}^n \setminus \bigcup_{i \in \mathbb{N}} \Delta_i$.

The group of the automorphisms of the covering $\pi : \widetilde{N} \to N$ can be identified in a natural way with a discrete torsion-free subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}^n)$ such that $\gamma(\widetilde{N}) = \widetilde{N}$ for any $\gamma \in \Gamma$ and $N \cong \widetilde{N}/\Gamma$. Also recall that there exists an isomorphism $\pi_1(N) \cong \Gamma$, which is canonical up to conjugacy. Let $\Lambda(\Gamma)$ denote the limit set of Γ and let $\Omega(\Gamma) = \partial\mathbb{H}^n \setminus \Lambda(\Gamma)$. Kojima has shown in [6] that $\Lambda(\Gamma) = \widetilde{N}_\infty$, so the round balls Δ_i, $i \in N$ previously defined actually are the connected components of $\Omega(\Gamma)$. A subgroup of Γ is called *peripheral* if it is equal to the stabilizer of one of the Δ_i's.

Since $\widetilde{N}_\infty = \Lambda(\Gamma)$, we have that $\widetilde{N}$ is the intersection of $\mathbb{H}^n$ with the convex hull of $\Lambda(\Gamma)$, so N is the convex core (see [1,9]) of the hyperbolic manifold $\mathbb{H}^n/\Gamma$. This implies that N uniquely determines Γ up to conjugation by elements in $\mathrm{Isom}^+(\mathbb{H}^n)$, that Γ is geometrically finite and that N is homeomorphic to the manifold $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$.

1.5. Parabolic subgroups of Γ

Let Γ' be a subgroup of Γ. We say that Γ' is *maximal parabolic* if it is parabolic (i.e., all its non-trivial elements are parabolic) and it is maximal with respect to inclusion among parabolic subgroups of Γ. If Γ' is a maximal parabolic subgroup of Γ, then there exists a point $q \in \partial\mathbb{H}^n$ such that Γ' equals the stabilizer of q in Γ. Then Γ' can be naturally identified with a discrete subgroup of $\mathrm{Isom}^+(\mathbb{E}^{n-1})$, so by Bierbebach's Theorem [9] Γ' contains an Abelian subgroup H of finite index. If k is the rank of H, we say that Γ' is a rank-k parabolic subgroup of Γ. Now it is shown in [6] that if $i \neq j$, then $\overline{\Delta}_i \cap \overline{\Delta}_j$ is either empty or consists of one point p whose stabilizer is a rank-$(n-2)$ parabolic subgroup of Γ. Moreover, any maximal rank-$(n-2)$ parabolic subgroup of Γ is the stabilizer of a point p which lies on the boundary of two different Δ_i's. On the other hand, the intersection of $\widetilde{N}$ with a horoball centered at a point with rank-$(n-2)$ parabolic stabilizer projects onto a boundary cusp of N, and any boundary cusp of N lifts to the intersection of $\widetilde{N}$ with a horoball centered at a point with rank-$(n-2)$ parabolic stabilizer. It follows that there is a natural correspondence between the boundary cusps of N and the conjugacy classes of rank-$(n-2)$ maximal parabolic subgroups of Γ.

We shall see that rank-1 maximal parabolic subgroups of Γ play a special role in the proof of our main theorem. Since any parabolic subgroup of Γ corresponds to a cusp of N, we have that if $n \geqslant 4$ then Γ does not contain rank-1 maximal parabolic subgroups, while when $n = 3$ the elements of rank-1 maximal parabolic subgroups of Γ correspond to the annular cusp loops previously defined.

Proposition 1.5. *Let γ be an element of Γ. Then γ is parabolic if and only if one of the following conditions holds*:

(1) *γ belongs to a rank-*1 *maximal parabolic subgroup of Γ, or*
(2) *γ belongs to a subgroup Γ' of Γ which contains a finite-index Abelian subgroup of rank* $\geqslant 2$.

Proof. The "only if" part of the statement is an immediate consequence of the discussion above, so we concentrate on the "if" part. Let Γ' a subgroup of Γ as in condition (2). A standard result in the theory of Kleinian groups implies that Γ' is elementary, i.e., it consists either of parabolic elements having all the same fixed point or of hyperbolic elements having all the same axe. However, in the latter case Γ' should be isomorphic to $\mathbb{Z}$, a contradiction. □

For later purpose we point out the following:

Remark 1.6. For any $k \in \mathbb{N}$ let H_k be the stabilizer of Δ_k in Γ. If $i \neq j$, then either $\overline{\Delta}_i \cap \overline{\Delta}_j = \emptyset$ and $H_i \cap H_j = \emptyset$, or $\overline{\Delta}_i \cap \overline{\Delta}_j = \{p\}$ and $H_i \cap H_j$ is the rank-$(n-2)$ parabolic stabilizer of p in Γ.

2. Some preliminary lemmas

The following result is a slight generalization of Lemma 5.1 in [5], which is due to J.P. Otal. Notation is kept from the preceding section.

Lemma 2.1. *Let $j : S^{n-2} \to \Lambda(\Gamma)$ be a topological embedding. Then $\Lambda(\Gamma) \setminus j(S^{n-2})$ is path connected if and only if $j(S^{n-2}) = \partial\Delta_l$ for some $l \in \mathbb{N}$.*

Proof. If $\partial N = \emptyset$, then $\Lambda(\Gamma) \cong S^{n-1}$ and the claimed result is readily deduced from Jordan–Brower separation theorem, so we can assume $\partial N \neq \emptyset$.

Suppose that $j(S^{n-2}) = \partial\Delta_0$. Using the upper half-space model of hyperbolic space, we identify $\partial\mathbb{H}^n$ with $(R^{n-1} \times \{0\}) \cup \{\infty\}$ in such a way that Δ_0 corresponds to $\mathbb{H} = \{(x,0) \in \mathbb{R}^{n-1} \times \{0\}\colon x_{n-1} > 0\}$. Now let $p_1, p_2 \in \Lambda(\Gamma) \setminus \partial\Delta_0$ and let $\alpha : [0,1] \to (\mathbb{R}^{n-1} \times \{0\}) \setminus \overline{\mathbb{H}}$ be the straight Euclidean segment which joins p_1 to p_2. If $\{(a_i, b_i) \subset [0,1],\ i \geqslant 1\}$ is the set of the connected components of $\alpha^{-1}(\Omega(\Gamma))$, then, up to reordering the Δ_i's with $i \geqslant 1$, we have $\alpha([a_i, b_i]) \subset \overline{\Delta}_i$. Let r_i be the Euclidean radius of Δ_i. Since $\partial\Delta_i$ can touch $\partial\Delta_0$ at most in one point, for any $i \geqslant 1$ there exists a path $\beta_i : [a_i, b_i] \to \partial\Delta_i$ with $\beta_i(a_i) = \alpha(a_i)$, $\beta_i(b_i) = \alpha(b_i)$ and length $(\beta_i) \leqslant 2\pi r_i$. Now let α_i be the path inductively defined as follows: $\alpha_0 = \alpha$, $\alpha_{i+1} = \beta_{i+1}(t)$ if $t \in [a_{i+1}, b_{i+1}]$ and $\alpha_{i+1}(t) = \alpha_i(t)$ if $t \in [0, a_{i+1}] \cup [b_{i+1}, 1]$. The path α_i is obviously continuous for any $i \in \mathbb{N}$. Moreover, since $\lim_{i\to\infty} r_i = 0$, the sequence of paths $\{\alpha_i, i \in \mathbb{N}\}$ uniformly converges to the desired continuous path $\alpha_\infty : [0, 1] \to \Lambda(\Gamma)\setminus\partial\Delta_0$.

Suppose now that $\Lambda(\Gamma)\setminus j(S^{n-2})$ is path connected. The Jordan–Brower separation theorem implies that $\partial\mathbb{H}^n\setminus j(S^{n-2}) = A_1 \cup A_2$, where the A_i's are disjoint open subset of $\partial\mathbb{H}^n$ with $\partial A_i = j(S^{n-2})$ for $i = 1, 2$ (since we are not assuming that j is tame, at this stage we are not allowed to claim that the A_i's are topological balls). Our hypothesis now forces $A_k \cap \Lambda(\Gamma) = \emptyset$ for some $k \in \{1, 2\}$, so $A_k \subset \Delta_l$; for some $l \in \mathbb{N}$. Moreover, since $\partial A_k = j(S^{n-2}) \subset \Lambda(\Gamma)$, it is easily seen that $j(S^{n-2}) = \partial\Delta_l$, and we are done. $\square$

From now on let N_1 and N_2 be hyperbolic n-manifolds, let $\pi_i : \mathbb{H}^n \supset \widetilde{N}_i \to N_i$ be the universal covering of N_i and let Γ_i be a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$ such that $N_i \cong \widetilde{N}_i/\Gamma_i$. Let also $\varphi : \Gamma_1 \to \Gamma_2$ be a group isomorphism satisfying the condition of Theorem 1.2. If $f : N_1 \to N_2$ is a continuous map, it is easily seen that φ is induced by f if and only if f admits a continuous lift $\tilde{f} : \widetilde{N}_1 \to \widetilde{N}_2$ such that $\tilde{f} \circ \gamma = \varphi(\gamma) \circ \tilde{f}$ for every $\gamma \in \Gamma_1$.

Lemma 2.2. *The isomorphism φ satisfies the following conditions*:

(1) *$\varphi(\gamma)$ is a parabolic element of Γ_2 if and only if γ is a parabolic element of Γ_1*;
(2) *There exists a homeomorphism $\hat{\varphi} : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$ such that $\hat{\varphi}(\gamma(x)) = \varphi(\gamma)(\hat{\varphi}(x))$ for any $x \in \Lambda(\Gamma_1)$, $\gamma \in \Gamma_1$*;
(3) *$\varphi(H)$ is a peripheral subgroup of Γ_2 if and only if H is a peripheral subgroup Γ_1; if so we also have $\hat{\varphi}(\Lambda(H)) = \Lambda(\varphi(H))$.*

Proof. Since φ is supposed to satisfy the condition of Theorem 1.2, point (1) is an immediate consequence of Proposition 1.5.

Now a general result in the theory of discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ (see, e.g., [1, Theorem 4.41]) applies providing the equivariant homeomorphism $\hat{\varphi} : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$ described in point (2).

Let $H = \mathrm{stab}(\Delta)$ be a peripheral subgroup of Γ_1, where Δ is a component of $\Omega(\Gamma_1)$. By Lemma 2.1, $\Lambda(\Gamma_1)\setminus\Lambda(H) = \Lambda(\Gamma_1)\setminus\partial\Delta$ is path connected, so $\Lambda(\Gamma_2)\setminus\hat{\varphi}(\Lambda(H)) = \hat{\varphi}(\Lambda(\Gamma_1)\setminus\Lambda(H))$ is also path connected, and $\hat{\varphi}(\Lambda(H))$ is equal to $\Lambda(K)$ for some peripheral subgroup K of Γ_2. Let $K = \mathrm{stab}(\Delta')$, where Δ' is a component of $\Omega(\Gamma_2)$. Now let h be a loxodromic element of H with fixed points p_1, p_2 in $\Lambda(H)$. Since $\hat{\varphi}$ is φ-equivariant, we have that $\varphi(h)$ is a loxodromic element of Γ_2 with fixed points $\hat{\varphi}(p_1), \hat{\varphi}(p_2)$ which lie in $\Lambda(K)$. Since the boundaries of two different components of $\Omega(\Gamma_2)$ can intersect at most in one point, it easily follows that $\varphi(h) \in \mathrm{stab}(\Delta') = K$. Now H is generated by its loxodromic elements, so $\varphi(H)$ is contained in K. On the other hand, the same argument applied to φ^{-1} shows that $\varphi^{-1}(K)$ is contained in a peripheral subgroup of Γ_1, say H', with $H \subset H'$. Now Remark 1.6 implies that $H = H'$, so $\varphi(H) = K$ and point (3) is proved. $\square$

Corollary 2.3. $\partial N_1 = \emptyset$ *if and only if* $\partial N_2 = \emptyset$.

If $\partial N_1 = \partial N_2 = \emptyset$, Mostow–Prasad's rigidity theorem applies ensuring geometricity of φ. Then from now on we shall assume that both N_1 and N_2 have non-empty boundary.

3. The n-dimensional case, $n \geqslant 4$

The next proposition easily implies Theorem 1.2 under the assumption that the dimension of N_1 and N_2 is at least 4.

Proposition 3.1. *Let $n \geqslant 4$. Then there exists a conformal map $f : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ such that $f \circ \gamma = \varphi(\gamma) \circ f$ for any $\gamma \in \Gamma_1$.*

Proof. Let Δ^1 be a connected component of $\Omega(\Gamma_1)$, and H_1 be the stabilizer of Δ^1 in Γ_1. By point (3) of Lemma 2.2, the group $H_2 = \varphi(H_1)$ is a peripheral subgroup of Γ_2. Let now Δ^2 be the H_2-invariant component of $\Omega(\Gamma_2)$, i.e., the unique component of $\Omega(\Gamma_2)$ whose boundary is equal to $\Lambda(H_2)$. By construction, the homeomorphism $\hat{\varphi} : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$ described in Lemma 2.2 restricts to a homeomorphism $\hat{\varphi}|_{\partial\Delta^1} : \partial\Delta^1 \to \partial\Delta^2$ such that $\hat{\varphi}|_{\partial\Delta^1} \circ \gamma = \varphi(\gamma) \circ \hat{\varphi}|_{\partial\Delta^1}$ for every $\gamma \in H_1$. Let now S^1, S^2 be the hyperplanes of $\mathbb{H}^n$ bounded respectively by $\partial\Delta^1$ and $\partial\Delta^2$. Then S^k/H_k is isometric to a component of the geodesic boundary of N_k for $k = 1, 2$, so it is a finite-volume complete hyperbolic $(n-1)$-manifold without boundary. Since $n \geqslant 4$, Mostow–Prasad's rigidity theorem applies providing an isometry $g : S^1 \to S^2$ whose continuous extension to $\partial\Delta^1$ is equal to $\hat{\varphi}|_{\partial\Delta^1}$. Let now $p_k, k = 1, 2$ be the orthogonal projection of S^k onto Δ^k, i.e., the function which maps a point $q \in S^k$ to the point $p \in \Delta^k$ such that the geodesic ray $[q, p)$ is orthogonal to S^k. The map $g' : \Delta^1 \to \Delta^2$ defined by $g' = p_2 \circ g \circ p_1^{-1}$ is conformal, and its continuous extension to $\partial\Delta^1$ is equal to $\hat{\varphi}|_{\partial\Delta^1}$.

By repeating the construction described above for each component of $\Omega(\Gamma_1)$, we can construct a conformal map $t : \Omega(\Gamma_1) \to \Omega(\Gamma_2)$. This map is a homeomorphism, since it admits a continuous inverse which can be constructed from the isomorphism $\varphi^{-1} : \Gamma_2 \to \Gamma_1$. We want now to show that for any $\gamma \in \Gamma_1$, we have $t \circ \gamma = \varphi(\gamma) \circ t$. Let Δ be a component of $\Omega(\Gamma_1)$. By the very definition of t it follows that $t(\Delta)$ is the unique component of $\Omega(\Gamma_2)$ which is bounded by $\hat{\varphi}(\partial\Delta)$, so

$$\begin{aligned}\partial\bigl(\varphi(\gamma)\bigl(t(\Delta)\bigr)\bigr) &= \varphi(\gamma)\bigl(\partial\bigl(t(\Delta)\bigr)\bigr) = \varphi(\gamma)\bigl(\hat{\varphi}(\partial\Delta)\bigr) = \hat{\varphi}\bigl(\gamma(\partial\Delta)\bigr)\\ &= \hat{\varphi}\bigl(\partial\bigl(\gamma(\Delta)\bigr)\bigr) = \partial\bigl(t\bigl(\gamma(\Delta)\bigr)\bigr).\end{aligned}$$

This shows that both $t \circ \gamma$ and $\varphi(\gamma) \circ t$ map Δ onto the same component Δ' of $\Omega(\Gamma_2)$. Moreover, the continuous extensions of $t \circ \gamma$ and $\varphi(\gamma) \circ t$ to $\partial\Delta$ are respectively equal to $\hat{\varphi} \circ \gamma$ and $\varphi(\gamma) \circ \hat{\varphi}$, which are in turn equal to each other because of the φ-equivariance of $\hat{\varphi}$. Being conformal, the maps $t \circ \gamma$ and $\varphi(\gamma) \circ t$ must then be equal on Δ, and this proves the required φ-equivariance of t.

Now let $f : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ be defined by $f(x) = t(x)$ if $x \in \Omega(\Gamma_1)$, and $f(x) = \hat{\varphi}(x)$ if $x \in \Lambda(\Gamma_1)$. To conclude the proof we only have to observe that since f is φ-equivariant and conformal on $\Omega(\Gamma_1)$, a result of Tukia [12] ensures that f is a conformal map. $\square$

We can now conclude the proof of Theorem 1.2, under the assumption that the dimension of N_1 and N_2 is greater than 3. Let $\tilde{\psi}$ be the unique isometry of $\mathbb{H}^n$ whose continuous extension to $\partial\mathbb{H}^n$ is equal to f. The φ-equivariance of f readily implies that $\tilde{\psi}(\gamma(x)) = \varphi(\gamma)(\tilde{\psi}(x))$ for every $x \in \mathbb{H}^n$, $\gamma \in \Gamma_1$. If we identify N_i with the convex core of the manifold $\mathbb{H}^n/\Gamma_i$ for $i = 1, 2$, then $\tilde{\psi}$ induces an isometry $\psi : N_1 \to N_2$ with $\psi_* = \varphi$.

4. The 3-dimensional case

As briefly explained in the Introduction, the 3-dimensional case needs a different approach.

Lemma 4.1. *There exists a homeomorphism $g : N_1 \to N_2$ such that $\varphi = g_*$.*

Proof. Let $M_i = (\mathbb{H}^3 \cup \Omega(\Gamma_i))/\Gamma_i$ for $i = 1, 2$. By Lemma 2.2 and Remark 1.6, we can apply Theorem 1 of [8] to φ, obtaining a homeomorphism $g' : M_1 \to M_2$ inducing φ (note that our definition of *geometric* is stronger than the one in [8]). Now N_i is canonically embedded in M_i in such a way that $M_i \backslash N_i$ is an open collar of ∂M_i. This implies that g' can be isotoped to a $g'' : M_1 \to M_2$ such that $g''(N_1) = N_2$ and $g = g''|_{N_1}$ is the required homeomorphism. □

Remark 4.2. If N_1 and N_2 have compact geodesic boundary, then Lemma 4.1 can also be deduced by the following result of Johannson [4,10]: Any homotopy equivalence between compact orientable boundary-irreducible anannular Haken 3-manifolds can be homotoped to a homeomorphism.

We can now conclude the proof of Theorem 1.2 in the case when N_1 and N_2 are 3-dimensional manifolds. Let $g : N_1 \to N_2$ be the homeomorphism constructed in Lemma 4.1, let $D(N_i)$ be the double of N_i for $i = 1, 2$ and let $D(g) : D(N_1) \to D(N_2)$ be the homeomorphism obtained by doubling g. By Mostow–Prasad's rigidity theorem, $D(g)$ is homotopic to an isometry $h : D(N_1) \to D(N_2)$. Since $\partial N_2 = g(\partial N_1)$ and $h(\partial N_1)$ are embedded totally geodesic homotopic surfaces in N_2, we get that $h(\partial N_1) = \partial N_2$, so $h(N_1) = N_2$. Moreover, $h_* = g_*$ on $\pi_1(D(N_1))$, and the inclusion of $\pi_1(N_i)$ in $\pi_1(D(N_i))$ is injective for $i = 1, 2$, so $h_* = g_* = \varphi$ on Γ_1. In conclusion, we have shown that $h|_{N_1} : N_1 \to N_2$ is an isometry inducing φ, so φ is geometric.

4.1. Counterexamples in the non-compact boundary case

We now show that the conclusions of Corollaries 1.3 and 1.4 are no longer true if we consider hyperbolic 3-manifolds with non-compact geodesic boundary. More precisely, we will prove the following:

Proposition 4.3. *There exist hyperbolic 3-manifolds with non-compact geodesic boundary N_1, N_2 such that*:

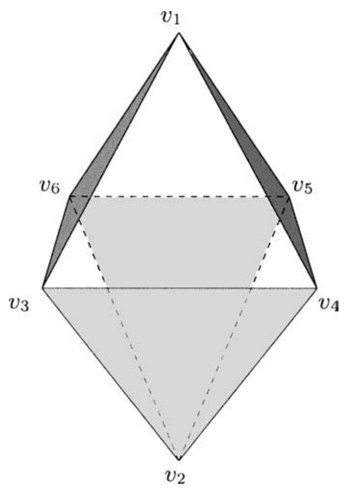

Fig. 1. The manifolds N_1, M_1 and $N_2 = M_2$ are obtained by gluing in pairs the non-shadowed faces of the regular ideal octahedron along suitable isometries.

(1) $\pi_1(N_1) \cong \pi_1(N_2)$ *but* ∂N_1 *is not homeomorphic to* ∂N_2 (*so,* a fortiori, N_1 *is not homeomorphic to* N_2);
(2) $\mathrm{Out}(\pi_1(N_i)) \ncong \mathrm{Isom}(N_i)$ *for* $i = 1, 2$.

Proof. We will give an explicit construction of N_1 and N_2. Let $O \subset \mathbb{H}^3$ be the regular ideal octahedron and let $v_1, \ldots, v_6$ be the vertices of O as shown in Fig. 1. We denote by F_{ijk} the face of O with vertices v_i, v_j, v_k. Let $g : F_{134} \to F_{156}$ be the unique isometry such that $g(v_1) = v_1$, $g(v_3) = v_6$ and $g(v_4) = v_5$, and $h_1, h_2 : F_{236} \to F_{254}$ be the unique isometries such that $h_1(v_2) = v_2$, $h_1(v_3) = v_4$, $h_1(v_6) = v_5$, $h_2(v_2) = v_5$, $h_2(v_3) = v_2$, $h_2(v_6) = v_4$. We now define N_1 to be the manifold obtained by gluing O along g and h_1, and N_2 to be the manifold obtained by gluing O along g and h_2. Since all the dihedral angles of O are right, it is easily seen that the metric on O induces a complete finite-volume hyperbolic structure on the N_i's such that the shadowed faces in Fig. 1 are glued along their edges to give a non-compact totally geodesic boundary.

Now the natural compactification of N_i is homeomorphic to the genus-2 handlebody for $i = 1, 2$, so $\pi_1(N_1) \cong \pi_1(N_2) \cong Z * Z$ (see also Remark 4.5 and Fig. 6).

Moreover, we claim that the boundary of N_1 is homeomorphic to the disjoint union of two 3-punctured spheres, while the boundary of N_2 is homeomorphic to the 4-punctured sphere. To see this, note that the shadowed faces of O glue up to give an ideal triangulation of ∂N_i, $i = 1, 2$. The combinatorial structure of this triangulation can be easily deduced from the rules g, h_i defining N_i as a quotient of O, and is explicitly described in Figs. 2 and 3. Using this description we can readily compute the Euler characteristic and the number of punctures of any component of ∂N_i, thus proving point (1).

In order to prove point (2), we only have to observe that the group of the outer isomorphisms of $\mathbb{Z} * \mathbb{Z}$ is of infinite order, while the group of isometries of any complete

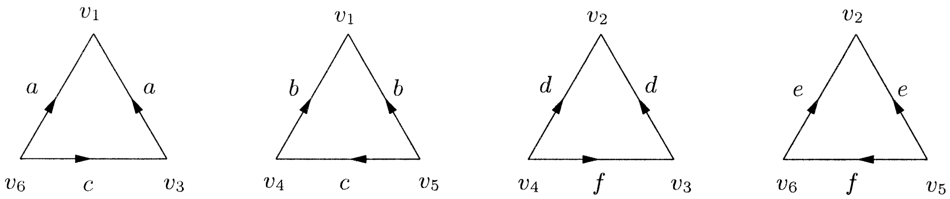

Fig. 2. The pairings defining ∂N_1: the two triangles on the left are glued to each other giving a 3-punctured sphere, and the same holds also for the two triangles on the right.

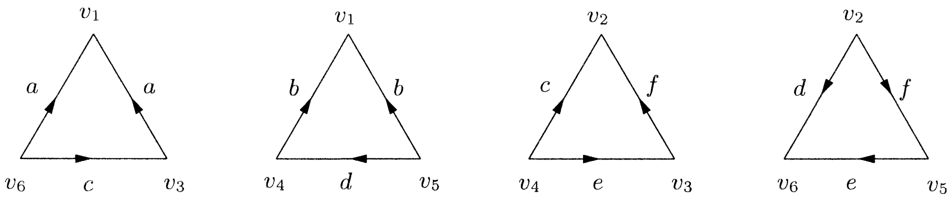

Fig. 3. The pairings defining ∂N_2: the resulting space is clearly connected, and the ideal vertices glue up giving exactly four punctures. Moreover, $\chi(\partial N_2) = \#\{\text{triangles}\} - \#\{\text{edges}\} = -2$, so ∂N_2 is homeomorphic to the 4-punctured sphere.

finite-volume hyperbolic n-manifold with geodesic boundary has a finite number of elements. $\square$

The construction just described can be slightly modified to provide manifolds with homeomorphic (but not isometric) boundaries, as the following proposition shows.

Proposition 4.4. *There exist hyperbolic* 3*-manifolds with non-compact geodesic boundary* M_1, M_2 *such that*:

(1) $\pi_1(M_1) \cong \pi_1(M_2)$ *and* ∂M_1 *is homeomorphic to* ∂M_2;
(2) ∂M_1 *is not isometric to* ∂M_2;
(3) M_1 *is not homeomorphic to* M_2;
(4) $\mathrm{Out}(\pi_1(M_i)) \ncong \mathrm{Isom}(M_i)$ *for* $i = 1, 2$.

Proof. We set $M_2 = N_2$ the manifold described in the preceding proposition, and we define M_1 to be the hyperbolic manifold with non-compact geodesic boundary obtained by gluing the faces of O along g' and h_3, where $g' : F_{134} \to F_{156}$, $h_3 : F_{236} \to F_{254}$ are the unique isometries such that $g'(v_1) = v_5$, $g'(v_3) = v_1$, $g'(v_4) = v_6$ and $h_3(v_2) = v_4, h_3(v_3) = v_5, h_3(v_6) = v_2$. As before, the natural compactification of the M_i's is the genus-2 handlebody, so $\pi(M_1) \cong \pi(M_2) \cong \mathbb{Z} * \mathbb{Z}$. Let now T_i be the ideal triangulation of ∂M_i given by the shadowed faces of O. The description of T_2 and T_1 in Figs. 3 and 4 implies that both ∂M_1 and ∂M_2 are homeomorphic to the 4-punctured sphere, which proves point (1).

In order to prove point (2) we need now to recall some elementary facts about the geometry of ideal triangles and to lead a more accurate analysis of T_1 and T_2. First of

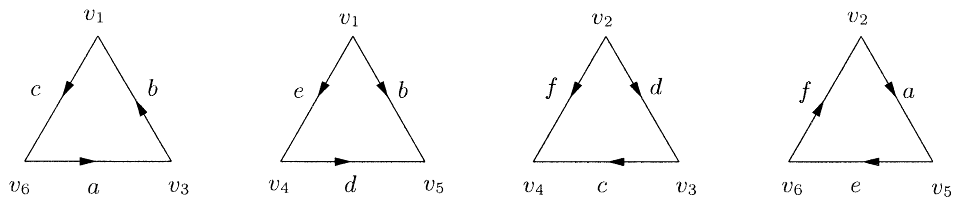

Fig. 4. The ideal triangulation T_1: the resulting surface ∂M_1 is connected with four punctures and has Euler characteristic equal to -2, so it is homeomorphic to the 4-punctured sphere.

all, let e_1, e_2 be distinct edges of an ideal triangle $\Delta \subset \mathbb{H}^2$, let $v \in \partial\mathbb{H}^2$ be their common endpoint, and let V_i be the hyperbolic half-plane containing Δ bounded by e_i for $i = 1, 2$. Then for any $r > 0$ we denote by $H_r(v, \Delta)$ the open horodisc centered at v such that $\mathrm{Area}(H_r(v, \Delta) \cap V_1 \cap V_2) = r$ (note that $H_r(v, \Delta) \cap V_1 \cap V_2 = H_r(v, \Delta) \cap \Delta$ if $r \leqslant 2$, i.e., if $H_r(v, \Delta)$ does not intersect the edge of Δ opposite to v). We refer to $H_1(v, \Delta)$ as to the *standard* horocycle neighbourhood of v in Δ, and recall that there exists a well-defined notion of *midpoint* of an edge in a hyperbolic ideal triangle: namely, if e is an edge of Δ with vertices r_1, r_2, then the midpoint of e is given by $e \cap \partial H_1(w_1, \Delta) = e \cap \partial H_1(w_2, \Delta)$. We also observe that if Δ, w_1, w_2 are as above, then $H_{r_1}(w_1, \Delta) \cap H_{r_2}(w_2, \Delta) = \emptyset$ if and only if $r_1 r_2 \leqslant 1$.

Coming back to the surfaces we are interested in, let us observe that the isometries realizing the gluings between the non-shadowed faces of O preserve the midpoints of the edges of the shadowed ones. This means that the standard horocycle neighbourhoods of the vertices of the triangles in T_i glue up to a horocycle neighbourhood of the punctures of ∂M_i for $i = 1, 2$. Since the number of vertices corresponding to any puncture in ∂M_i is exactly 3, this gives in turn four pairwise disjoint subsets of ∂M_1, each of which is a horocycle neighbourhood of a puncture having area equal to 3. Let now P_1, P_2, P_3, P_4 be the punctures of ∂M_2 and suppose ∂M_1 is isometric to ∂M_2. Under this assumption, for any $i = 1, \ldots, 4$ there exists a horocycle neighbourhood U_i of P_i such that $\mathrm{Area}(U_i) = 3$ and $U_i \cap U_j = \emptyset$ for any $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

Let Δ be the triangle of T_2 with vertices v_1, v_6 and v_3. An easy analysis of the combinatorics of T_2 shows that v_1 and v_6 correspond to distinct punctures of ∂M_2, say P_1 and P_2, respectively. Moreover, v_1 is the only vertex of triangles of T_2 incident to P_1, while the number of vertices incident to P_2 is equal to 5. Since $\mathrm{Area}(U_1) = \mathrm{Area}(U_2) = 3$, this implies that $U_1 \cap \Delta \supseteq H_3(v_1, \Delta) \cap \Delta$ and $U_2 \cap \Delta \supseteq H_{3/5}(v_6, \Delta)$, which gives in turn $U_1 \cap U_2 \neq \emptyset$, a contradiction. We have eventually shown that ∂M_1 is not isometric to ∂M_2, thus proving point (2).

To prove point (3) it is sufficient to observe that if M_1 and M_2 were homeomorphic, then by Theorem 1.2 they should be isometric, so ∂M_1 should be isometric to ∂M_2, against point (2).

The same argument given in the proof of Proposition 4.3 applies here yielding point (4). $\square$

Remark 4.5. From a topological and combinatorial point of view, an ideal octahedron with four marked faces as in Fig. 1 is equivalent to a truncated tetrahedron with the edges connecting truncation triangles removed, which is in turn equivalent to a “tetrapod” with

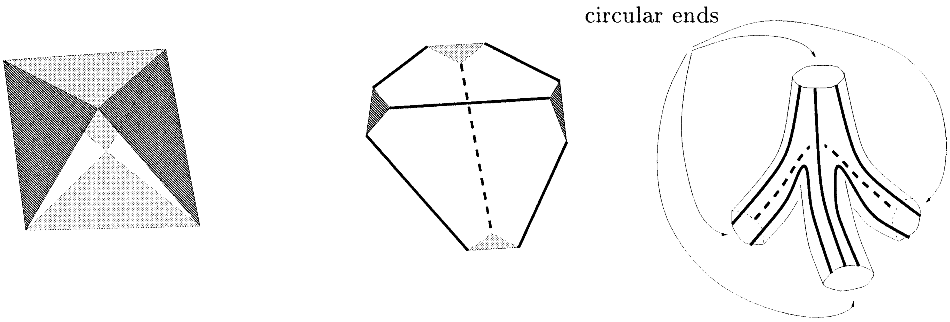

Fig. 5. An ideal octahedron, a truncated tetrahedron, and a tetrapod with arcs.

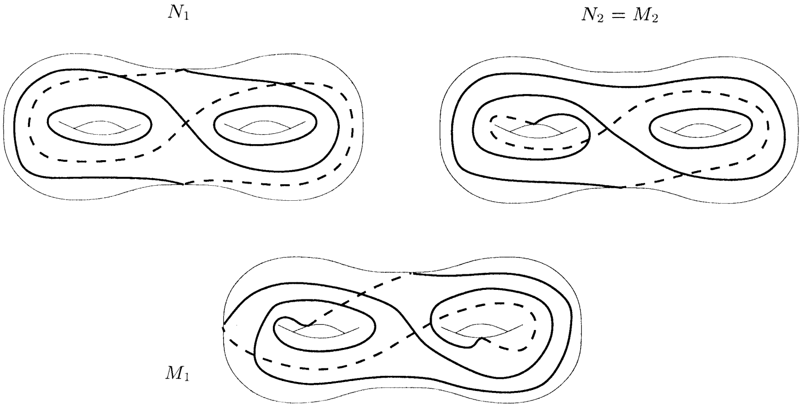

Fig. 6. The natural compactifications of N_1, M_1 and $N_2 = M_2$ are genus-2 handlebodies with boundary annuli. Here we represent annuli by drawing their core curves.

six arcs connecting circular ends removed, as shown in Fig. 5. Under this identification, the four shadowed ideal faces of O correspond to the four regions into which the lateral surface of the tetrapod is cut by the 6 arcs, while the non-shadowed ideal faces of O correspond to the four discs at the ends of the tetrapod. Therefore N_1, M_1 and $N_2 = M_2$ are obtained from the tetrapod by suitably gluing together in pairs the discs at its four ends. So these manifolds are homeomorphic to handlebodies with boundary loops removed. Using this correspondence we can easily draw pictures of the natural compactifications of N_1, M_1 and $N_2 = M_2$. These pictures are shown in Fig. 6. For a more detailed description of the natural compactification of hyperbolic 3-manifolds with non-compact geodesic boundary obtained by gluing regular ideal octahedra see [2].

4.2. A more general construction

We now briefly describe a different method of constructing homotopically-equivalent non-homeomorphic hyperbolic 3-manifolds with non-compact geodesic boundary. To this aim we first recall that Thurston's hyperbolization theorem for Haken manifolds [11] gives necessary and sufficient topological conditions on a 3-manifold to be hyperbolic with geodesic boundary:

Theorem 4.6. *Let $\overline{M}$ be a compact orientable* 3*-manifold with non-empty boundary, let $\mathcal{T}$ be the set of boundary tori of $\overline{M}$ and let $\mathcal{A}$ be a family of disjoint closed annuli in $\partial\overline{M}\setminus\mathcal{T}$. Then $M = \overline{M}\setminus(\mathcal{T}\cup\mathcal{A})$ is hyperbolic if and only if the pair $(\overline{M},\mathcal{A})$ satisfies the following conditions*:

- *the components of ∂M have negative Euler characteristic*;
- *$\overline{M}\setminus\mathcal{A}$ is boundary-irreducible and geometrically atoroidal*;
- *the only proper essential annuli contained in M are parallel in $\overline{M}$ to the annuli in $\mathcal{A}$.*

Using Theorem 4.6 we now prove the following:

Proposition 4.7. *Let N be a hyperbolic* 3*-manifold with non-empty geodesic boundary, and suppose that at least one component of ∂N is not a* 3*-punctured sphere. Then there exists a hyperbolic* 3*-manifold with geodesic boundary which is homotopically equivalent but not homeomorphic to N.*

Proof. By assumption ∂N contains an essential loop α. We then define N' as $N\setminus\alpha$ and note that N and N' have a common compactification $\overline{M}$ such that $N = \overline{M}\setminus(\mathcal{T}\cup\mathcal{A})$, $N' = \overline{M}\setminus(\mathcal{T}\cup\mathcal{A}')$, with $\mathcal{A}\subset\mathcal{A}'$ and $\#\mathcal{A}' = \#\mathcal{A}+1$. Moreover, since $(\overline{M},\mathcal{T},\mathcal{A})$ satisfies the assumptions of Theorem 4.6, so does $(\overline{M},\mathcal{T},\mathcal{A}')$, so N' is hyperbolic. Of course N' is homotopically equivalent to N, but $\partial N'$ is not homeomorphic to ∂N, so *a fortiori* N and N' are not homeomorphic to each other. □

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