Available online at www.sciencedirect.com
SCIENCE DIRECT
Annals of Pure and Applied Logic 140 (2006) 75–103
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Power function on stationary classes

Moti Gitik[a,*], Carmi Merimovich[b]

[a] School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv 69978, Israel
[b] Department of Computer Science, The Academic College of Tel Aviv, 4 Antokolsky Street, Tel Aviv 64044, Israel

Available online 7 October 2005

Abstract

We show that under certain large cardinal requirements there is a generic extension in which the power function behaves differently on different stationary classes. We achieve this by doing an Easton support iteration of the Radin on extenders forcing.

MSC: primary 03E35; 03E55

Keywords: Cardinal arithmetic; Large cardinals; Extender based forcing; Stationary sets

1. Introduction

This work is part of the general project to understand all possible behaviors of the power set function according to the size of large cardinals in the core model. We deal here with the power function below a strongly inaccessible cardinal or just globally. Usually, there is a club subset with the power function having a uniform behavior along it; see [2,3,14]. It is natural to ask if a uniform behavior on a club is necessary. For a singular of uncountable cofinality there are limitations posed by the Silver Theorem. Also [11] provides additional limitations. The present work answers the above question negatively and provides a method of constructing models with different behavior of the power function on different stationary subsets of an inaccessible or on different stationary classes. In [6] other methods are used to deal with the same situation but below a singular of uncountable cofinality.

We demonstrate some possibilities by proving the following theorems.

Theorem 5.1. *Let $\xi < \kappa$ be regular cardinals in K (the core model) and $\xi \notin \omega - \{0\}$. Suppose that the set $\{\lambda < \kappa \mid o(\lambda) = \lambda^{++} + \xi\}$ is stationary. Then there is a cardinal preserving generic extension of K in which the sets*

$$\{\lambda < \kappa \mid 2^\lambda = \lambda^+ \text{ and } (\operatorname{cf}\lambda = \xi \text{ if } \xi \neq 0 \text{ or } \operatorname{cf}\lambda = \omega \text{ if } \xi = 0)\}$$

and

$$\{\lambda < \kappa \mid 2^\lambda = \lambda^{++} \text{ and } (\operatorname{cf}\lambda = \xi \text{ if } \xi \neq 0 \text{ or } \operatorname{cf}\lambda = \omega \text{ if } \xi = 0)\}$$

are stationary.

* Corresponding author.
E-mail addresses: gitik@post.tau.ac.il (M. Gitik), carmi@mta.ac.il (C. Merimovich).

0168-0072/$ - see front matter
doi:10.1016/j.apal.2005.09.004

A similar result is possible if κ is replaced by On:

Theorem 5.2. *Let ξ be a regular cardinal in K and $\xi \notin \omega - \{0\}$. Suppose that $\{\lambda \mid \mathrm{o}(\lambda) = \lambda^{++} + \xi\}$ is a stationary class. Then there is a cardinal preserving class generic extension of K in which the classes*

$$\{\lambda \mid 2^\lambda = \lambda^+ \textit{ and } (\mathrm{cf}\,\lambda = \xi \textit{ if } \xi \neq 0 \textit{ or } \mathrm{cf}\,\lambda = \omega \textit{ if } \xi = 0)\}$$

and

$$\{\lambda \mid 2^\lambda = \lambda^{++} \textit{ and } (\mathrm{cf}\,\lambda = \xi \textit{ if } \xi \neq 0 \textit{ or } \mathrm{cf}\,\lambda = \omega \textit{ if } \xi = 0)\}$$

are stationary.

By the results of [10], the above theorems are optimal for each $\xi \neq \omega_1$.

Theorem 5.3. *Let κ be a regular cardinal in K. Suppose that $\{\lambda < \kappa \mid \mathrm{o}(\lambda) = \lambda^{+3} + 1\}$ is stationary. Then there is a cardinal preserving generic extension of K in which the sets*

$$\{\lambda < \kappa \mid \mathrm{cf}\,\lambda = \omega,\ 2^\lambda = \lambda^+\},$$
$$\{\lambda < \kappa \mid \mathrm{cf}\,\lambda = \omega,\ 2^\lambda = \lambda^{++}\},$$

and

$$\{\lambda < \kappa \mid \mathrm{cf}\,\lambda = \omega,\ 2^\lambda = \lambda^{+3}\}$$

are stationary.

Theorem 5.4. *Suppose that $\{\lambda \mid \mathrm{o}(\lambda) = \lambda^{+3} + 1\}$ is a stationary class. Then there is a cardinal preserving class generic extension of K in which the classes*

$$\{\lambda \mid \mathrm{cf}\,\lambda = \omega,\ 2^\lambda = \lambda^+\},$$
$$\{\lambda \mid \mathrm{cf}\,\lambda = \omega,\ 2^\lambda = \lambda^{++}\},$$

and

$$\{\lambda \mid \mathrm{cf}\,\lambda = \omega,\ 2^\lambda = \lambda^{+3}\}$$

are stationary.

By [10] the assumptions are almost optimal.

Theorem 5.5. *Let κ be a regular cardinal in K. Suppose that for each $\xi < \kappa$ the set $\{\lambda < \kappa \mid \mathrm{o}(\lambda) = \lambda^{+3} + \xi\}$ is stationary. Then there is a cardinal preserving generic extension of K in which $\{\lambda < \kappa \mid 2^\lambda = \lambda^+$ or λ is regular$\}$ is nonstationary and both sets $\{\lambda < \kappa \mid 2^\lambda = \lambda^{++}\}$ and $\{\lambda < \kappa \mid 2^\lambda = \lambda^{+3}\}$ are stationary.*

Theorem 5.6. *Suppose that for each $\xi \in On$, $\{\xi < \lambda < \kappa \mid \mathrm{o}(\lambda) = \lambda^{+3} + \xi\}$ is a stationary class. Then there is a cardinal preserving class generic extension of K in which $\{\lambda \mid 2^\lambda = \lambda^+$ or λ is regular$\}$ is a nonstationary class and both sets $\{\lambda \mid 2^\lambda = \lambda^{++}\}$ and $\{\lambda \mid 2^\lambda = \lambda^{+3}\}$ are stationary classes.*

By [10] the assumptions are optimal.

The structure of this work is as follows: In Section 2 we review the needed results from the Easton iteration of Prikry type forcing notions theory. In Section 3 we review facts about extenders, and the Prikry on extenders forcing notion. In Section 4 we present the iteration of the Radin on extenders forcing notion. Section 5 presents the usage of the iterated forcing to control the power set function on stationary sets.

The notation we use is standard. We assume fluency with forcing ($p \leq q$ means p is stronger than q), iterated forcing, and large cardinals methods (namely, extenders, ultrapowers, and their elementary embeddings).

2. The Easton iteration

The Easton iteration of Prikry type forcing notions was introduced in [7], and appears in a simplified form in [5]. In this section we review results from [5] used in the current work, stripped down to the special cases we need. We refer the reader to [5] for the proofs.

Definition 2.1. The forcing $\langle P, \leq\rangle$ is called of Prikry type if there is auxiliary partial order $\langle P, \leq^*\rangle$ such that

(1) $\leq^* \subseteq \leq$.
(2) (Prikry Condition) For each $p \in P$ and σ a formula in the P-forcing language, there is $p^* \leq^* p$ such that $p^* \parallel_P \sigma$.

When we refer to the forcing notion '$\langle P, \leq, \leq^*\rangle$', we mean that P is of Prikry type in the above sense. Namely, we force with $\langle P, \leq\rangle$ and $\leq^*$ is the auxiliary order.

Note that, trivially, $\langle P, \leq, \leq\rangle$ is of Prikry type.

Definition 2.2. The Easton iteration of Prikry type forcing notions, $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$, is defined as follows: For each $\alpha < \kappa$, $p \in P_\alpha$ is of the form $p = \langle \dot{p}_\beta \mid \beta \in s\rangle$ where

(1) $s \subseteq \alpha$.
(2) (Easton support) For each $\beta \leq \alpha$ inaccessible $|s \cap \beta| < \beta$.
(3) $\forall \beta \in s\ p{\restriction}\beta = \langle \dot{p}_\gamma \mid \gamma \in s \cap \beta\rangle \in P_\beta$.
(4) $\forall \beta \in s \Vdash_{P_\beta} \ulcorner \langle \dot{Q}_\beta, \leq, \leq^*\rangle$ is of Prikry type$\urcorner$.
(5) $\forall \beta \in s\ p{\restriction}\beta \Vdash_{P_\beta} \ulcorner \dot{p}_\beta \in \dot{Q}_\beta \urcorner$.

We call s the support of p and write $\operatorname{supp} p$ for it.

Definition 2.3. For $p, q \in P_\kappa$ we say $p \leq^* q$ (p is a Prikry extension of q) if

(1) $\operatorname{supp} p \supseteq \operatorname{supp} q$.
(2) $\forall \alpha \in \operatorname{supp} q\ p{\restriction}\alpha \Vdash_{P_\alpha} \ulcorner \dot{p}_\alpha \leq^*_{\dot{Q}_\alpha} \dot{q}_\alpha \urcorner$.

Definition 2.4. For $p, q \in P_\kappa$ we say $p \leq q$ (p is an extension of q) if

(1) $\operatorname{supp} p \supseteq \operatorname{supp} q$.
(2) $\forall \alpha \in \operatorname{supp} q\ p{\restriction}\alpha \Vdash_{P_\alpha} \ulcorner \dot{p}_\alpha \leq_{\dot{Q}_\alpha} \dot{q}_\alpha \urcorner$.
(3) $|\operatorname{supp} q \setminus \{\alpha \in \operatorname{supp} q \mid p{\restriction}\alpha \Vdash_{P_\alpha} \ulcorner \dot{p}_\alpha \leq^*_{\dot{Q}_\alpha} \dot{q}_\alpha \urcorner\}| < \aleph_0$.

Lemma 2.5. *Assume $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa\rangle$ is an Easton iteration of Prikry type forcing notions. Assume $p \in P_\kappa$, and let σ be a statement in the P_κ-forcing language. Then there is $p^* \leq^*_{P_\kappa} p$ such that $p^* \parallel_{P_\kappa} \sigma$.*

That is, $\langle P_\kappa, \leq_{P_\kappa}, \leq^*_{P_\kappa}\rangle$ is of Prikry type.

The following definition is from the general theory of iterated forcing.

Definition 2.6. Assume $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa\rangle$ is an iteration, and $0 < \alpha < \kappa$.

(1) If $p \in P_\kappa$ then $\dot{p}_{\alpha,\kappa} = p{\restriction}[\alpha, \kappa)$. We consider $\dot{p}_{\alpha,\kappa}$ to be a P_α-name.
(2) $\dot{P}_{\alpha,\kappa}$ is the P_α-name satisfying

$$\Vdash_{P_\alpha} \ulcorner \dot{r} \in \dot{P}_{\alpha,\kappa} \urcorner \iff \forall p \in P_\kappa\ \exists q \leq_{P_\kappa} p\ q{\restriction}\alpha \Vdash_{P_\alpha} \ulcorner \dot{q}_{\alpha,\kappa} = \dot{r} \urcorner.$$

(3) The order $\leq_{\dot{P}_{\alpha,\kappa}}$ on $\dot{P}_{\alpha,\kappa}$ is defined by

$$p \leq_{P_\kappa} q \implies p{\restriction}\alpha \Vdash_{P_\alpha} \ulcorner \dot{p}_{\alpha,\kappa} \leq_{\dot{P}_{\alpha,\kappa}} \dot{q}_{\alpha,\kappa} \urcorner.$$

The Prikry ordering on P_κ induces an ordering on $\dot{P}_{\alpha,\kappa}$:

Definition 2.7. The order $\leq^*_{\dot{P}_{\alpha,\kappa}}$ on $\dot{P}_{\alpha,\kappa}$ is defined by

$$p \leq_{P_\kappa} q,\ \forall \beta \in \operatorname{supp} q \setminus \alpha\ p{\restriction}\beta \Vdash_{P_\beta} {}^\ulcorner p_\beta \leq^*_{\dot{Q}_\beta} q_\beta{}^\urcorner \implies$$

$$p{\restriction}\alpha \Vdash_{P_\alpha} {}^\ulcorner \dot{p}_{\alpha,\kappa} \leq^*_{\dot{P}_{\alpha,\kappa}} \dot{q}_{\alpha,\kappa}{}^\urcorner.$$

Claim 2.8. *Assume $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ is an Easton iteration of Prikry type forcings such that $\forall \alpha < \kappa$ $\Vdash_{P_\alpha} {}^\ulcorner\langle \dot{Q}_\alpha, \leq^*\rangle$ is α-closed${}^\urcorner$. Then all cardinals $\lambda \geq \kappa$ are preserved.*

Claim 2.9. *Let κ be Mahlo cardinal, and assume $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa\rangle$ is an Easton iteration of Prikry type forcing notions. Then P_κ satisfies the κ-cc.*

3. Extenders, the Prikry on extenders forcing notion

3.1. κ-Extenders

Definition 3.1. Let $j : V \to M$ be an elementary embedding. The generators of j are defined by induction as

$$\kappa_0 = \operatorname{crit}(j),$$
$$\kappa_\xi = \min\{\lambda \in \mathrm{On} \mid \forall \xi' < \xi\ \forall \mu \in \mathrm{On}\ \forall f : \mu \to \mathrm{On}\ j(f)(\kappa_{\xi'}) \neq \lambda\}.$$

If the induction terminates, then we have a set of generators for j:

$$\mathfrak{g}(j) = \{\kappa_\xi \mid \xi < \xi^*\}.$$

The measures in this work are not on $\operatorname{crit}(j)$ but on functions taking values inside $\operatorname{crit}(j)$. These objects are named OB in this work.

Definition 3.2. Assume $d \in [j(\kappa)]^{\leq\kappa}$ and $\kappa \in d$. Then $\nu \in \mathrm{OB}(d) \iff$

(1) $\nu : \operatorname{dom}\nu \to \kappa$.
(2) $\kappa \in \operatorname{dom}\nu \subseteq d$.
(3) $|\nu| \leq \nu(\kappa)$.
(4) $\forall \alpha, \beta \in \operatorname{dom}\nu\ \alpha < \beta \implies \nu(\alpha) < \nu(\beta)$.

In the following definition of extender we note that the interesting case is when $\bigcup \mathfrak{g}(j) \geq \operatorname{crit}(j)^{++}$.

Definition 3.3. Assume $j : V \to M \supset M^\kappa$ is an elementary embedding, $\operatorname{crit}(j) = \kappa$, and $\mathfrak{g}(j) \subset j(\kappa)$. The κ-extender E derived from j is the system

$$\langle\langle E(d) \mid \kappa \in d \in [j(\kappa)]^{\leq\kappa}\rangle, \langle \pi_{d_2,d_1} \mid d_1, d_2 \in [j(\kappa)]^{\leq\kappa},\ \kappa \in d_1 \subseteq d_2\rangle\rangle$$

where for each $\kappa \in d$, $d_1, d_2 \in [j(\kappa)]^{\leq\kappa}$, and $\kappa \in d_1 \subseteq d_2$,

(1) $A \in E(d) \iff$
  (1.1) $A \subseteq \mathrm{OB}(d)$.
  (1.2) $\{\langle j(\alpha), \alpha\rangle \mid \alpha \in d\} \in j(A)$.
(2) $\pi_{d_2,d_1} : \mathrm{OB}(d_2) \to \mathrm{OB}(d_1)$ is defined by $\pi_{d_2,d_1}(A) = \{\nu{\restriction}d_1 \mid \nu \in A\}$.

Given E, a κ-extender, we let $j_E : V \to M \simeq \mathrm{Ult}(V, E)$ be the corresponding elementary embedding.

We use the objects OB and not just κ in order to solve a technicality appearing in the Radin on extenders forcing. That is, if we have a long enough coherent sequence of extenders, and a large set in the sense of all of them, then we cannot know from which extender a specific point from this large set was taken. Hence we will not be able to use the projection of the right extender. Our solution is to use OB, where each 'point' is in fact a function. This function contains all the information we need from the extender, that is the projection and where to project.

Assume $d \in [j(\kappa)]^{\leq\kappa}$. As usual, a set $T \subseteq \mathrm{OB}(d)^{<\omega}$ ordered by end-extension and closed downwards is called a tree. We use the following notation for a tree T:

$$\forall n < \omega \ \mathrm{Lev}_n(T) = \{\langle \nu_0, \ldots, \nu_n\rangle \mid \langle \nu_0, \ldots, \nu_n\rangle \in T\},$$
$$T_{\langle \mu_0,\ldots,\mu_n\rangle} = \{\langle \nu_0, \ldots, \nu_k\rangle \mid k < \omega,\ \langle \mu_0, \ldots, \mu_n, \nu_0, \ldots, \nu_k\rangle \in T\},$$
$$\mathrm{Suc}_T(\nu_0, \ldots, \nu_n) = \{\nu \mid \langle \nu_0, \ldots, \nu_n, \nu\rangle \in T\}.$$

For our purposes we need special trees called $E(d)$-trees:

Definition 3.4. Assume $d \in [j(\kappa)]^{\leq\kappa}$. A tree T of height ω is an $E(d)$-tree if

$$\forall\langle \nu_0, \ldots, \nu_{k-1}\rangle \in T \ \ \mathrm{Suc}_T(\nu_0, \ldots, \nu_{k-1}) \in E(d)$$

and for each $\langle \nu\rangle \in T_{\langle \nu_0,\ldots,\nu_{k-1}\rangle}$

(1) $\mathrm{dom}\,\nu_{k-1} \subseteq \mathrm{dom}\,\nu$.
(2) $\forall\alpha \in \mathrm{dom}\,\nu_{k-1}\ \nu_{k-1}(\alpha) < \nu(\alpha)$.
(3) $\forall\langle\mu\rangle \in T_{\langle \nu_0,\ldots,\nu_{k-1}\rangle}\ \ \nu(\kappa) < \mu(\kappa) \implies \mathrm{dom}\,\nu \subseteq \mathrm{dom}\,\mu$.

Note that we use the convention $\mathrm{Suc}_T() = \mathrm{Lev}_0(T)$.

Definition 3.5. Assume T, T', are $E(d)$, $E(d')$-trees, respectively, and $d' \subseteq d$. Then

$$\pi_{d,d'}(T) = T\restriction d' = \{\langle \bar\nu_0\restriction d', \ldots, \bar\nu_{k-1}\restriction d'\rangle \mid \langle \bar\nu_0, \ldots, \bar\nu_{k-1}\rangle \in T\},$$
$$\pi^{-1}_{d,d'}(T') = \{\langle \bar\nu_0, \ldots, \bar\nu_{k-1}\rangle \in \mathrm{OB}(d)^k \mid \langle \bar\nu_0\restriction d', \ldots, \bar\nu_{k-1}\restriction d'\rangle \in T'\}.$$

3.2. The Prikry on extender forcing notion

We review the definition and basic facts about the Prikry on extender forcing notion [9]. The form of the forcing we give is a simplification of the presentation in [15].

Assume $j : V \to M \supseteq M^\kappa$, $\mathrm{crit}(j) = \kappa$, $\mathfrak{g}(j) \subseteq j(\kappa)$, and let E be the κ-extender derived from j.

We begin by defining the forcing notion $\langle \mathbb{P}^*_E, \leq^*\rangle$:

Definition 3.6. $f \in \mathbb{P}^*_E$ iff $f : d \to [\kappa]^{<\omega}$ is such that

(1) $d \in [j(\kappa)]^{\leq\kappa}$.
(2) $\kappa \in d$.

$\mathbb{P}^*_E$ is equipped with the partial order $f \leq^* g \iff f \supseteq g$. (Note that $\langle \mathbb{P}^*_E, \leq^*\rangle$ is the Cohen forcing adding $|j(\kappa)|$ subsets to κ^+.)

Definition 3.7. A condition p in $\mathbb{P}_E$ is of the form $\langle f, T\rangle$, where

(1) $f \in \mathbb{P}^*_E$.
(2) T is an $E(\mathrm{dom}\, f)$-tree.

We write $\mathrm{supp}\, p$, f^p, and T^p, for $\mathrm{dom}\, f$, f, and T, respectively.

Definition 3.8. Let $p, q \in \mathbb{P}_E$. We say that p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if

(1) $\mathrm{supp}\, p \supseteq \mathrm{supp}\, q$.
(2) $f^p\restriction \mathrm{supp}\, q = f^q$.
(3) $\pi_{\mathrm{supp}\, p,\mathrm{supp}\, q} T^p \subseteq T^q$.

Definition 3.9. Let $q \in \mathbb{P}_E$ and $\langle\nu\rangle \in T^q$. We define $q_{\langle\nu\rangle} \in \mathbb{P}_E$ to be p where

(1) $\mathrm{supp}\, p = \mathrm{supp}\, q$.
(2) $\forall\alpha \in \mathrm{supp}\, p\ \ f^p(\alpha) = \begin{cases} f^q(\alpha) ^\frown \langle\nu(\alpha)\rangle & \alpha \in \mathrm{dom}\,\nu,\ \nu(\alpha) > f^q_{|f^q(\alpha)|-1}(\alpha). \\ f^q(\alpha) & \text{Otherwise.} \end{cases}$

(3) $T^p = T^q_{\langle \nu \rangle}$.

When we write $q_{\langle \nu_0, \dots, \nu_k \rangle}$ we mean $(\cdots (q_{\langle \nu_0 \rangle})_{\langle \nu_1 \rangle} \cdots)_{\langle \nu_k \rangle}$.

Definition 3.10. Let $p, q \in \mathbb{P}_E$. We say that p is a 1-point extension of q ($p \leq^1 q$) if there is $\langle \nu \rangle \in T^q$ such that $p \leq^* q_{\langle \nu \rangle}$.

Definition 3.11. Let $p, q \in \mathbb{P}_E$ and $n < \omega$. We say that p is an n-point extension of q ($p \leq^n q$) if there are $p^n, \dots, p^0$ such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

Definition 3.12. Let $p, q \in \mathbb{P}_E$. We say that p is an extension of q ($p \leq q$) if there is $n < \omega$ such that $p \leq^n q$.

The properties of $\mathbb{P}_E$ we need are summarized in the following theorems. The reader is referred to [9] or [15] for the proofs.

Theorem 3.13. (1) $\langle \mathbb{P}_E, \leq, \leq^* \rangle$ *is of Prikry type.*
(2) $\langle \mathbb{P}_E, \leq^* \rangle$ *is* κ*-closed.*
(3) $\Vdash_{\mathbb{P}_E} \ulcorner (\kappa^+)_V$ *is a cardinal*$\urcorner$.
(4) $\mathbb{P}_E$ *satisfies the* κ^{++}*-cc.*

Theorem 3.14. *Let* G *be* $\mathbb{P}_E$*-generic. Then in* $V[G]$

(1) *No cardinals are collapsed.*
(2) $\operatorname{cf} \kappa = \omega$.
(3) *No bounded subsets are added to* κ.
(4) $2^\kappa = |j(\kappa)|$.

4. The Easton iteration of the Radin on extenders forcing

This section is modeled after Section 3 of [7]. The major change is that instead of the measures used there we use extenders. The main theorem iterates the Radin on extenders forcing notion [13] along a Mitchell style (i.e., using double indexing) coherent sequence of extenders.

The next definition adopts the general notion of coherency [16,17] to our context. Note the last requirement is a restriction of the sequence to *non-overlapping* extenders.

Definition 4.1. A function E is called a coherent sequence of (non-overlapping) extenders if

(1) $\operatorname{dom} E = \{\langle \kappa, \xi \rangle \mid \kappa < \mathrm{l}(E),\ \xi < \mathrm{o^E}(\kappa)\}$, where $\mathrm{l}(E)$ is an ordinal and $\mathrm{o^E} : \mathrm{l}(E) \to \mathrm{On}$.
(2) $\forall \langle \kappa, \xi \rangle \in \operatorname{dom} E$ $E_{\kappa,\xi}$ is a κ-extender, and $j_{\kappa,\xi} : V \to M_{\kappa,\xi} \simeq \mathrm{Ult}(V, E_{\kappa,\xi})$ is the corresponding natural embedding.
(3) $j_{\kappa,\xi}(\mathrm{o^E})(\kappa) = \xi$ and for each $\xi' < \xi$, $j_{\kappa,\xi}(E)_{\kappa,\xi'} = E_{\kappa,\xi'}$.
(4) $\forall \kappa_1 < \kappa_2 < \mathrm{l}(E)$

$$\mathrm{o^E}(\kappa_1) < \kappa_2,$$
$$\sup_{\xi < \mathrm{o^E}(\kappa_1)} j_{\kappa_1,\xi}(\kappa_1) < \kappa_2.$$

Since $\operatorname{dom} E$ consists of pairs of ordinals and we need to access the first coordinate from time to time, we use the notation $\operatorname{dom}^1 E$ for the projection of $\operatorname{dom} E$ to the first coordinate.

Theorem 4.2. *Let* E *be a coherent sequence of extenders such that*

$$\forall \langle \kappa, \xi \rangle \in \operatorname{dom} E\ \exists f_\xi : \kappa \to \kappa$$
$$\sup \mathfrak{g}(j_{\kappa,\xi}) < j_{\kappa,\xi}(f_\xi)(\kappa) < \min((\operatorname{dom}^1 j_{\kappa,\xi}(E)) \setminus (\kappa + 1)).$$

Then there is a cardinal preserving generic extension in which

$$\forall \kappa \in \operatorname{dom}^1 E\ 2^\kappa = \sup_{\xi < \mathrm{o^E}(\kappa)} |j_{\kappa,\xi}(\kappa)|.$$

Proof. The forcing notion we use is the Easton iteration of the Radin on extenders forcing notion. The proof is by induction on the length of the iteration $\kappa \leq \bigcup \operatorname{dom}^1 E$.

- $\kappa = 0$: As usual $P_0 = \mathbb{1}$ and there is nothing to prove.
- κ is a limit ordinal: If $\{\alpha < \kappa \mid\Vdash_{P_\alpha} \ulcorner \dot{Q}_\alpha \neq \mathbb{1} \urcorner\}$ is bounded in κ, then there is an $\alpha < \kappa$ such that $P_\kappa \simeq P_\alpha$ and there is nothing to prove. So we assume this is not the case.

Let G_κ be a P_κ-generic filter, and set $\forall\alpha < \kappa\ G_\alpha = G_\kappa \cap P_\alpha$ (hence for each $\alpha < \kappa$, G_α is a P_α-generic filter). By the induction hypothesis we have for each $\alpha < \kappa$,

$$V[G_\alpha] \vDash \ulcorner \forall\lambda \in \operatorname{dom}^1 E \cap \alpha\ \ 2^\lambda = \sup_{\xi < \mathrm{o}^{\mathrm{E}}(\lambda)} |j_{\lambda,\xi}(\lambda)| \urcorner,$$

and

$$V \text{ and } V[G_\alpha] \text{ have the same cardinals.}$$

Let $\lambda \in \operatorname{dom}^1 E \cap \kappa$. Pick $\alpha < \kappa$ such that $\lambda < \alpha$. Then $P_\kappa = P_\alpha * \dot{P}_{\alpha,\kappa}$. Since

$$\Vdash_{P_\alpha} \ulcorner \langle \dot{P}_{\alpha,\kappa}, \leq, \leq^* \rangle \text{ is of Prikry type} \urcorner,$$

and

$$\Vdash_{P_\alpha} \ulcorner \langle \dot{P}_{\alpha,\kappa}, \leq^* \rangle \text{ is } |\alpha|\text{-closed} \urcorner,$$

the forcing $\dot{P}_{\alpha,\kappa}$ does not collapse cardinals below α, nor does it change $(2^\lambda)_{V[G_\alpha]}$. Hence $V[G_\kappa] \vDash \ulcorner 2^\lambda = \sup_{\beta < \mathrm{o}^{\mathrm{E}}(\lambda)} |j_{\lambda,\beta}(\lambda)| \urcorner$.

Since for each $\alpha < \kappa$, $V[G_\kappa]$ and $V[G_\alpha]$ have the same cardinals below α, we get that no cardinal below κ is collapsed in $V[G_\kappa]$.

Cardinals above κ are not collapsed by the general Easton iteration theory. Hence V and $V[G_\kappa]$ have the same cardinals.

- $\kappa + 1$: If $\kappa \notin \operatorname{dom}^1 E$ then we set $\dot{Q}_\kappa = \mathbb{1}$ and $P_{\kappa+1} = P_\kappa * \dot{Q}_\kappa$; thus there is nothing to prove.

So, we are left with $\kappa \in \operatorname{dom}^1 E$: We would have liked to let $\dot{Q}_\kappa$ be the P_κ-name of the Radin on extenders forcing with the extenders $\langle E_{\kappa,\xi} \mid \xi < \mathrm{o}^{\mathrm{E}}(\kappa) \rangle$. However, after forcing with P_κ, the extenders $E_{\kappa,\xi}$ no longer measure all subsets of OB. We begin by finding a good enough replacement for the lost extenders. So, Let G_κ be a P_κ-generic filter.

Lemma 4.2.1. *Let $i : V \to N \simeq \mathrm{Ult}(V, E_{\kappa,0}(\{\kappa\}))$ be the natural embedding. Then there is a $\leq^*$-decreasing sequence $\langle \dot{p}^{N,0}_\zeta \mid \zeta < \kappa^+ \rangle \subset N$ such that if $\dot{D} \in N$ is a P^N_κ-name of a dense open subset of $\langle i(P_\kappa)/P^N_\kappa, \leq^* \rangle$, where*

$$\langle P^N_\xi \mid \xi \leq i(\kappa) \rangle = i(\langle P_\xi \mid \xi \leq \kappa \rangle),$$

then there is $\zeta < \kappa^+$ such that $\Vdash_{P^N_\kappa} \ulcorner \dot{p}_\zeta \in \dot{D} \urcorner$.

Proof. Observe that $\forall\xi \leq \kappa\ P_\xi = P^N_\xi$, and that $\dot{Q}^N_\kappa = \mathbb{1}$ since $\kappa \notin \operatorname{dom}^1 i(E)$. Let $\langle \dot{A}_\zeta \mid \zeta < \kappa^+ \rangle$ be an enumeration of all P^N_κ-names of maximal anti-chains of $\langle i(P_\kappa)/P^N_\kappa, \leq^* \rangle$. Since for each $\zeta^* < \kappa^+$ we have that

$$\langle \dot{A}_\zeta \mid \zeta < \zeta^* \rangle \in N,$$
$$\Vdash_{P^N_\kappa} \ulcorner \langle i(P_\kappa)/P^N_\kappa, \leq, \leq^* \rangle \text{ is of Prikry type} \urcorner,$$

and

$$\Vdash_{P^N_\kappa} \ulcorner \langle i(P_\kappa)/P^N_\kappa, \leq^* \rangle \text{ is } \kappa^+\text{-closed} \urcorner,$$

the sequence $\langle \dot{p}^{N,0}_\zeta \mid \zeta < \kappa^+ \rangle$ can be constructed by induction. $\square$

Definition 4.2.2. Using 4.2.1 we fix a sequence $\langle \dot{p}^{N,0}_{\zeta} \mid \zeta < \kappa^+ \rangle$ and call it a master sequence for $\mathrm{Ult}(V, E_{\kappa,0}(\{\kappa\}))$.

Lemma 4.2.3. *Let $j_{\kappa,0} : V \to M_{\kappa,0} \simeq \mathrm{Ult}(V, E_{\kappa,0})$ be the natural embedding. Then there is a $\leq^*$-decreasing sequence $\langle \dot{p}^0_{\zeta} \mid \zeta < \kappa^+ \rangle \subset M_{\kappa,0}$ such that if $\dot{D} \in M_{\kappa,0}$ is a $P^{M_{\kappa,0}}_{\kappa}$-name of a dense open subset of $\langle j_{\kappa,0}(P_\kappa)/P^{M_{\kappa,0}}_{\kappa}, \leq^* \rangle$, where*

$$\langle P^{M_{\kappa,0}}_{\xi} \mid \xi \leq j_{\kappa,0}(\kappa) \rangle = j_{\kappa,0}(\langle P_\xi \mid \xi \leq \kappa \rangle),$$

then there is $\zeta < \kappa^+$ such that $\Vdash_{P^{M_{\kappa,0}}_{\kappa}} \ulcorner \dot{p}^0_{\zeta} \in \dot{D} \urcorner$.

Proof. Observe that $\forall \xi \leq \kappa\ P_\xi = P^{M_{\kappa,0}}_{\xi}$, and $\dot{Q}^{M_{\kappa,0}}_{\kappa} = \mathbb{1}$ since $\kappa \notin \mathrm{dom}^1 j_{\kappa,0}(E)$. We factor $j_{\kappa,0}$ through the normal measure as follows:

$$\begin{array}{ccc} V & \xrightarrow{\quad j \quad} & M \simeq \mathrm{Ult}(V, E_{\kappa,0}) \\ {\scriptstyle i}\big\downarrow & \nearrow_{k([f]) = j_{\kappa,0}(f)(\kappa)} & \\ N \simeq \mathrm{Ult}(V, E_{\kappa,0}(\{\kappa\})) & & \end{array}$$

Let $\dot{D} \in M_{\kappa,0}$ be such that

$$\Vdash_{P^{M_{\kappa,0}}_{\kappa}} \ulcorner \dot{D} \text{ is a dense open subset of } \langle j_{\kappa,0}(P_\kappa)/P^{M_{\kappa,0}}_{\kappa}, \leq^* \rangle \urcorner.$$

Pick $\alpha \in \mathfrak{g}(E_{\kappa,0})$ and $f \in V$ such that $j_{\kappa,0}(f)(\alpha) = \dot{D}$. Let

$$\kappa' = \min((\mathrm{dom}^1 j_{\kappa,0}(E)) \setminus \kappa).$$

Then $\Vdash_{P^{M_{\kappa,0}}_{\kappa}} \ulcorner \langle j_{\kappa,0}(P_\kappa)/P^M_\kappa, \leq^* \rangle$ is κ'-closed$\urcorner$. Pick $g \in V$ such that $j_{\kappa,0}(g)(\kappa) = \kappa'$ (namely, for each inaccessible $\zeta < \kappa$ set $g(\zeta) = \min(\mathrm{dom}^1 E \setminus \zeta)$). Since $\sup \mathfrak{g}(E_{\kappa,0}) < j_{\kappa,0}(f_0)(\kappa) < k'$, we have in particular $\alpha < j_{\kappa,0}(f_0)(\kappa) < j_{\kappa,0}(g)(\kappa)$. Thus reflection to V yields

$$\begin{aligned} &\{\alpha\} \times \{\zeta < \kappa \mid \zeta < f_0(\pi_{\alpha,\kappa}(\zeta)), \\ &\qquad \Vdash_{P_{\pi_{\alpha,\kappa}(\zeta)}} \ulcorner f(\zeta) \text{ is a dense open subset of } \langle P_\kappa/P_{\pi_{\alpha,\kappa}(\zeta)}, \leq^* \rangle, \\ &\qquad\qquad \langle P_\kappa/P_{\pi_{\alpha,\kappa}(\zeta)}, \leq^* \rangle \text{ is } g(\pi_{\alpha,\kappa}(\zeta))\text{-closed}\urcorner\} \in E_{\kappa,0}(\{\alpha\}). \end{aligned}$$

Noting the existence of $X \in E_{\kappa,0}(\{\kappa\})$ such that

$$\forall \langle \kappa, \zeta_1 \rangle, \langle \kappa, \zeta_2 \rangle \in X\ [\zeta_1, f_0(\zeta_1)] \cap [\zeta_2, f_0(\zeta_2)] = \emptyset,$$
$$\bigcup_{\langle \kappa, \zeta \rangle \in X} (\zeta, f_0(\zeta)) \in E_{\kappa,0}(\alpha),$$

we can define a function $f^* : X \to V$ such that

$$\forall \langle \kappa, \zeta^* \rangle \in X\ \Vdash_{P_{\zeta^*}} \ulcorner f^*(\zeta^*) = \bigcap \{ f(\zeta) \mid \pi_{\alpha,\kappa}(\zeta) = \zeta^*,\ \zeta \in (\zeta^*, f_0(\zeta^*)) \} \urcorner.$$

Hence

$$\begin{aligned} &\{\kappa\} \times \{\zeta^* < \kappa \mid \Vdash_{P_{\zeta^*}} \ulcorner f^*(\zeta^*) \text{ is a dense open subset of } \langle P_\kappa/P_{\zeta^*}, \leq^* \rangle, \\ &\qquad (\zeta \in (\zeta^*, f_0(\zeta^*)),\ \zeta^* = \pi_{\alpha,\kappa}(\zeta)) \implies f^*(\zeta^*) \subseteq f(\zeta) \urcorner\} \in E_{\kappa,0}(\{\kappa\}). \end{aligned}$$

That is

$$\Vdash_{P^{M_{\kappa,0}}_{\kappa}} \ulcorner j_{\kappa,0}(f^*)(\kappa) \text{ is a dense open subset of } \langle j_{\kappa,0}(P_\kappa)/P^{M_{\kappa,0}}_{\kappa}, \leq^* \rangle \urcorner,$$

and

$$\Vdash_{P^{M_{\kappa,0}}_{\kappa}} \ulcorner j_{\kappa,0}(f^*)(\kappa) \subseteq j_{\kappa,0}(f)(\alpha) \urcorner.$$

Retreating to $N \simeq \mathrm{Ult}(V, E_{\kappa,0}(\{\kappa\}))$ we get

$$\Vdash_{P^N_\kappa} \ulcorner i(f^*)(\kappa) \text{ is a dense open subset of } \langle i(P_\kappa)/P^N_\kappa, \leq^*\rangle\urcorner.$$

Thus there is $\zeta < \kappa^+$ such that $\Vdash_{P^N_\kappa} \ulcorner \dot{p}^{N,0}_\zeta \in i(f^*)(\kappa)\urcorner$. Sending the last equation along k yields $\Vdash_{P_\kappa^{M_{\kappa,0}}} \ulcorner k(\dot{p}^{N,0}_\zeta) \in j_{\kappa,0}(f^*)(\kappa) \subseteq j_{\kappa,0}(f)(\alpha)\urcorner$. □

Definition 4.2.4. Using 4.2.3 we fix a sequence $\langle \dot{p}^0_\zeta \mid \zeta < \kappa\rangle$, and call it a master sequence for $M_{\kappa,0} \simeq \mathrm{Ult}(V, E_{\kappa,0})$.

The first extender, $E_{\kappa,0}$, is different from all other extenders $E_{\kappa,\xi}$ ($\xi > 0$) as we can lift it to an extender in $V[G_\kappa]$ as we show now.

The following definition makes sense since the master sequence $\langle \dot{p}^0_\zeta \mid \zeta < \kappa^+\rangle$ is $\leq^*$-decreasing, $j''_{\kappa,0} P_\kappa = P_\kappa$, and $j_{\kappa,0}(P_\kappa) \simeq P_\kappa * j_{\kappa,0}(P_\kappa)/P_\kappa^{M_{\kappa,0}}$. Note that:

(1) On the one hand, P_κ has added subsets to $j_{\kappa,0}(\kappa)$; hence there are ultrafilters which have no 'original' in the ground model.

(2) On the other hand, each new set in $[j_{\kappa,0}(\kappa)]^{\leq\kappa}$ is contained in an old set of $[j_{\kappa,0}(\kappa)]^{\leq\kappa}$. So we really do not need these orphan ultrafilters.

Definition 4.2.5. Assume $p \Vdash_{P_\kappa} \ulcorner \dot{d} \in [j_{\kappa,0}(\kappa)]^{\leq\kappa}\urcorner$. We define $\dot{E}_{\kappa,0}(\dot{d})$ as follows:

$$(p \Vdash_{P_\kappa} \ulcorner \dot{A} \subseteq \dot{\mathrm{OB}}(\dot{d})\urcorner \text{ and}$$
$$\exists \zeta < \kappa^+\ p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A})\urcorner) \implies$$
$$p \Vdash_{P_\kappa} \ulcorner \dot{A} \in \dot{E}_{\kappa,0}(\dot{d})\urcorner.$$

Claim 4.2.6. *Assume $p \Vdash_{P_\kappa} \ulcorner \dot{d} \in [j_{\kappa,0}(\kappa)]^{<\kappa}$ and $\dot{A} \subseteq \dot{OB}(\dot{d})\urcorner$. Then there are $p^* \leq^*_{P_\kappa} p$ and $\zeta < \kappa^+$ such that $p^* \frown \dot{p}^0_\zeta \parallel_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A})\urcorner$.*

Proof. Set $D = \{r \in j_{\kappa,0}(P_\kappa) \mid r \parallel_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A})\urcorner\}$. Since P_κ is of Prikry type, D is dense open in $\langle j_{\kappa,0}(P_\kappa), \leq^*\rangle$. Thus D/P_κ is a name of a dense open subset of $\langle j_{\kappa,0}(P_\kappa)/P_\kappa, \leq^*\rangle$. Hence there is $\zeta < \kappa^+$ such that $\Vdash_{P_\kappa} \ulcorner \dot{p}^0_\zeta \in \check{D}/P_\kappa\urcorner$. This means $\Vdash_{P_\kappa} \ulcorner \dot{p}^0_\zeta \parallel_{j_{\kappa,0}(P_\kappa)/P_\kappa} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A})\urcorner\urcorner$. Using the Prikry property we find $p^* \leq^*_{P_\kappa} p$ such that

$$p^* \frown \dot{p}^0_\zeta \parallel_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A})\urcorner. \quad \square$$

Corollary 4.2.7. *Assume $p \Vdash_{P_\kappa} \ulcorner \dot{d} \in [j_{\kappa,0}(\kappa]^{\leq\kappa})$ and $\dot{A} \subseteq \dot{OB}(\dot{d})\urcorner$. Then*

(1) $p \Vdash_{P_\kappa} \ulcorner \dot{A} \in \dot{E}_{\kappa,0}(\dot{d})\urcorner \iff$

$$\exists \zeta < \kappa^+\ p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A})\urcorner.$$

(2) $p \Vdash_{P_\kappa} \ulcorner \dot{A} \notin \dot{E}_{\kappa,0}(\dot{d})\urcorner \iff$

$$\exists \zeta < \kappa^+\ p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \notin j_{\kappa,0}(\dot{A})\urcorner.$$

Proof. The ($\Longleftarrow$) direction is immediate from the definition. We prove the ($\Longrightarrow$) direction.

(1) Assume $p \Vdash_{P_\kappa} \ulcorner \dot{A} \in \dot{E}_{\kappa,0}(\dot{d})\urcorner$: This means that there exists X, a maximal anti-chain below p, such that for each $q \in X$ there is $\zeta_q < \kappa^+$ such that $q \frown \dot{p}^0_{\zeta_q} \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A})\urcorner$. We set $\zeta = \bigcup_{q \in X} \zeta_q$. Since $|X| < \kappa$ we get $\zeta < \kappa^+$. Thus

$$p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A})\urcorner.$$

(2) Assume $p \Vdash_{P_\kappa} \ulcorner \dot{A} \notin \dot{E}_{\kappa,0}(\dot{d}) \urcorner$: Necessarily, if $q \leq_{P_\kappa} p$ and $\zeta < \kappa^+$, then $q \frown \dot{p}^0_\zeta \nVdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A}) \urcorner$. Invoking 4.2.6, we construct X, a maximal anti-chain below p, such that for each $q \in X$, there is $\zeta_q < \kappa^+$ such that $q \frown \dot{p}^0_{\zeta_q} \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \notin j_{\kappa,0}(\dot{A}) \urcorner$. We set $\zeta = \bigcup_{q \in X} \zeta_q$. Since $|X| < \kappa$, we have $\zeta < \kappa^+$. Thus

$$p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \notin j_{\kappa,0}(\dot{A}) \urcorner. \quad \square$$

Claim 4.2.8. *Assume* $\Vdash_{P_\kappa} \ulcorner \dot{d} \in [j_{\kappa,0}(\kappa)]^{\leq\kappa} \urcorner$. *Then*

$$\Vdash_{P_\kappa} \ulcorner \dot{E}_{\kappa,0}(\dot{d}) \textit{ is a } \kappa\textit{-closed ultrafilter on } \dot{\mathrm{O}}\mathrm{B}(\dot{d}) \urcorner.$$

Proof. We prove the four conditions showing $\dot{E}_{\kappa,0}(\dot{d})$ is a κ-complete ultrafilter.

(1) Assume $p \Vdash_{P_\kappa} \ulcorner \dot{A} \subseteq \dot{\mathrm{O}}\mathrm{B}(\dot{d})$ and $\dot{A} \notin \dot{E}_{\kappa,0}(\dot{d}) \urcorner$: Then there is $\zeta < \kappa^+$ such that $p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \notin j_{\kappa,0}(\dot{A}) \urcorner$. That is $p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in \dot{\mathrm{O}}\mathrm{B}(\dot{d}) \setminus j_{\kappa,0}(\dot{A}) \urcorner$. Hence

$$p \Vdash_{P_\kappa} \ulcorner \dot{\mathrm{O}}\mathrm{B}(\dot{d}) \setminus \dot{A} \in \dot{E}_{\kappa,0}(\dot{d}) \urcorner.$$

(2) Assume $\lambda < \kappa$ and $p \Vdash_{P_\kappa} \ulcorner \forall \mu < \lambda\ \dot{A}_\mu \in \dot{E}_{\kappa,0}(\dot{d}) \urcorner$: That is, for each $\mu < \lambda$ there is $\zeta_\mu < \kappa^+$ such that

$$p \frown \dot{p}^0_{\zeta_\mu} \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A}_\mu) \urcorner.$$

Let $\zeta = \bigcup_{\mu<\lambda} \zeta_\mu$. Then

$$p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in \bigcap_{\mu<\lambda} j_{\kappa,0}(\dot{A}_\mu) \urcorner.$$

Since $\mathrm{crit}(j) = \kappa > \lambda$ we get

$$p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\bigcap_{\mu<\lambda} \dot{A}_\mu) \urcorner.$$

That is $p \Vdash_{P_\kappa} \ulcorner \bigcap_{\mu<\lambda} \dot{A}_\mu \in \dot{E}_{\kappa,0}(\dot{d}) \urcorner$.

(3) Assume $p \Vdash_{P_\kappa} \ulcorner \dot{A} \subseteq \dot{B} \subseteq \dot{\mathrm{O}}\mathrm{B}(\dot{d})$ and $\dot{A} \in \dot{E}_{\kappa,0}(\dot{d}) \urcorner$: It is immediate that there is $\zeta < \kappa^+$ such that

$$p \frown \dot{p}^0_\zeta \Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner j_{\kappa,0}(\dot{A}) \subseteq j_{\kappa,0}(\dot{B}),\ \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,0}(\dot{A}) \urcorner. \quad \square$$

From 4.2.5–4.2.8 we deduce that we have constructed a lifting of $E_{\kappa,0}$.

Corollary 4.2.9. *The system*

$$\langle\langle \dot{E}_{\kappa,0}(d)[G_\kappa] \mid \kappa \in d \in [j(\kappa)]^{\leq\kappa}\rangle, \langle \pi_{d_2,d_1} \mid d_1, d_2 \in [j(\kappa)]^{\leq\kappa},\ \kappa \in d_1 \subseteq d_2\rangle\rangle$$

is a κ*-extender lifting* $E_{\kappa,0}$.

Proof. The only thing left to be proved is the lifting.

We work in V. First we note that if $d \in [j_{\kappa,0}(\kappa)]^{\leq\kappa}$ and $A \in E_{\kappa,0}(d)$, then $\{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in d\} \in j_{\kappa,0}(A)$. Trivially

$$\Vdash_{j_{\kappa,0}(P_\kappa)} \ulcorner \{\langle j_{\kappa,0}(\alpha), \alpha\rangle \mid \alpha \in \check{d}\} \in j_{\kappa,0}(\check{A}) \urcorner,$$

and hence $\Vdash_{P_\kappa} \ulcorner \check{A} \in \dot{E}_{\kappa,0}(\check{d}) \urcorner$.

The second thing to note is that if $\Vdash_{P_\kappa} \ulcorner \dot{d} \in [j_{\kappa,0}(\kappa)]^{\leq\kappa} \urcorner$ then due to the κ-c.c. of P_κ there is $d \in [j_{\kappa,0}(\kappa)]^{\leq\kappa}$ such that $\Vdash_{P_\kappa} \ulcorner \dot{d} \subseteq \check{d} \urcorner$. $\square$

When $\xi > 0$ we cannot lift the extender $E_{\kappa,\xi}$ to an extender in $V[G_\kappa]$. This is because we use the Prikry condition to decide when a set is large. When $\xi > 0$ we have that $\dot{Q}^{M_{\kappa,\xi}}_\kappa \neq \mathbb{1}$. In $\dot{Q}^{M_{\kappa,\xi}}_\kappa$ there might be two incompatible conditions (which are nonetheless Prikry extensions of the same condition!), one deciding that some set is large and the other that it is small.

What we do is construct an indexed set of *filters*. The properties of these filters will allow us to work almost as if we had ultrafilters. In fact this system of filters is the *name* of an extender which is found in a Cohen generic extension of $V[G_\kappa]$.

Lemma 4.2.10. *Assume* $0 < \xi < \mathrm{o}^{\mathrm{E}}(\kappa)$, *and let* $i : V \to N \simeq \mathrm{Ult}(V, E_{\kappa,\xi}(\{\kappa\}))$. *Then there is a* $\leq^*$*-decreasing sequence* $\langle \dot{p}^N_\zeta \mid \zeta < \kappa^+\rangle \subset N$ *such that if* $\dot{D} \in N$ *is a* $P^N_{\kappa+1}$*-name of a dense open subset of* $\langle i(P_\kappa)/P^N_{\kappa+1}, \leq^*\rangle$, *where*

$$\langle P^N_\xi \mid \xi \leq i(\kappa)\rangle = i(\langle P_\xi \mid \xi \leq \kappa\rangle),$$

then there is $\zeta < \kappa^+$ *such that* $\Vdash_{P^N_{\kappa+1}} \ulcorner \dot{p}_\zeta \in \dot{D}\urcorner$.

Proof. Observe that $\forall\alpha \leq \kappa\ P_\alpha = P^N_\alpha$. Let $\langle \dot{A}_\zeta \mid \zeta < \kappa^+\rangle$ be an enumeration of all $P^N_{\kappa+1}$-names of maximal anti-chains of $i(P_\kappa)/P^N_{\kappa+1}$. Since for each $\zeta^* < \kappa^+$ we have

$$\langle \dot{A}_\zeta \mid \zeta < \zeta^*\rangle \in N,$$
$$\Vdash_{P^N_{\kappa+1}} \ulcorner\langle i(P_\kappa)/P^N_{\kappa+1}, \leq, \leq^*\rangle \text{ is of Prikry type}\urcorner,$$

and

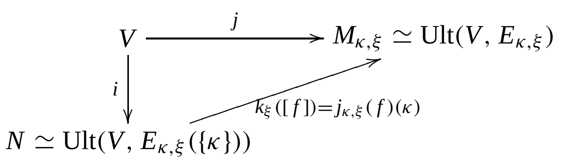
$$\Vdash_{P^N_{\kappa+1}} \ulcorner\langle i(P_\kappa)/P^N_{\kappa+1}, \leq^*\rangle \text{ is } \kappa^+\text{-closed}\urcorner,$$

the sequence $\langle \dot{p}^N_\zeta \mid \zeta < \kappa^+\rangle$ can be constructed by induction. □

Definition 4.2.11. For each $0 < \xi < \mathrm{o}^{\mathrm{E}}(\kappa)$, we use 4.2.10 to fix a sequence $\langle \dot{p}^{N,\xi}_\zeta \mid \zeta < \kappa^+\rangle$ and call it a master sequence for $\mathrm{Ult}(V, E_{\kappa,\xi}(\{\kappa\}))$.

Lemma 4.2.12. *Assume* $0 < \xi < \mathrm{o}^{\mathrm{E}}(\kappa)$. *Then there is a* $\leq^*$*-decreasing sequence* $\langle \dot{p}_\zeta \mid \zeta < \kappa^+\rangle \subset M_{\kappa,\xi}$ *such that if* $\dot{D} \in M_{\kappa,\xi}$ *is a* $P^{M_{\kappa,\xi}}_{\kappa+1}$*-name of a dense open subset of* $\langle j_{\kappa,\xi}(P_\kappa)/P^{M_{\kappa,\xi}}_{\kappa+1}, \leq^*\rangle$, *where* $\langle P^{M_{\kappa,\xi}}_\xi \mid \xi \leq j_{\kappa,\xi}(\kappa)\rangle = j_{\kappa,\xi}(\langle P_\xi \mid \xi \leq \kappa\rangle)$, *then there is* $\zeta < \kappa^+$ *such that* $\Vdash_{P^{M_{\kappa,\xi}}_{\kappa+1}} \ulcorner \dot{p}_\zeta \in \dot{D}\urcorner$.

Proof. Observe that $\forall\alpha \leq \kappa\ P_\alpha = P^{M_{\kappa,\xi}}_\alpha$. We factor $j_{\kappa,\xi}$ through the normal measure as follows:



Let $\dot{D}$ be such that $\Vdash_{P^{M_{\kappa,\xi}}_{\kappa+1}} \ulcorner \dot{D}$ is a dense open subset of $\langle j_{\kappa,\xi}(P_\kappa)/P^{M_{\kappa,\xi}}_{\kappa+1}, \leq^*\rangle\urcorner$.

Pick $\alpha \in \mathfrak{g}(E_{\kappa,\xi})$ and $f \in V$ such that $j_{\kappa,\xi}(f)(\alpha) = \dot{D}$. Let

$$\kappa' = \min((\mathrm{dom}^1 j_{\kappa,\xi}(E)) \setminus (\kappa + 1)).$$

Then $\Vdash_{P^{M_{\kappa,\xi}}_{\kappa+1}} \ulcorner\langle j_{\kappa,\xi}(P_\kappa)/P^{M_{\kappa,\xi}}_{\kappa+1}, \leq^*\rangle$ is κ'-closed$\urcorner$. Pick $g \in V$ such that $j_{\kappa,\xi}(g)(\kappa) = \kappa'$ (namely, for each inaccessible $\zeta < \kappa$ set $g(\zeta) = \min(\mathrm{dom}^1 E \setminus (\zeta + 1))$). Since $\sup \mathfrak{g}(E_{\kappa,\xi}) < j_{\kappa,\xi}(f_\xi)(\kappa) < k'$, we have in particular $\alpha < j_{\kappa,\xi}(f_\xi)(\kappa) < j_{\kappa,\xi}(g)(\kappa)$. Thus reflecting to V yields

$$\{\alpha\} \times \{\zeta < \kappa \mid \zeta < f_\xi(\pi_{\alpha,\kappa}(\zeta)),$$
$$\Vdash_{P_{\pi_{\alpha,\kappa}(\zeta)+1}} \ulcorner f(\zeta) \text{ is a dense open subset of } \langle P_\kappa/P_{\pi_{\alpha,\kappa}(\zeta)+1}, \leq^*\rangle,$$
$$\langle P_\kappa/P_{\pi_{\alpha,\kappa}(\zeta)+1}, \leq^*\rangle \text{ is } g(\pi_{\alpha,\kappa}(\zeta))\text{-closed}\urcorner\} \in E_{\kappa,\xi}(\{\alpha\}).$$

Noting the existence of $X \in E_{\kappa,\xi}(\{\kappa\})$ such that

$$\forall \langle \kappa, \zeta_1\rangle, \langle \kappa, \zeta_2\rangle \in X\ [\zeta_1,\ f_\xi(\zeta_1)] \cap [\zeta_2,\ f_\xi(\zeta_2)] = \emptyset,$$
$$\{\alpha\} \times \bigcup_{\langle \kappa,\zeta\rangle \in X} (\zeta,\ f_\xi(\zeta)) \in E_{\kappa,\xi}(\{\alpha\}),$$

we can define a function $f^* : X \to V$ such that

$$\forall \langle \kappa, \zeta^*\rangle \in X\ \Vdash_{P_{\zeta^*+1}} \ulcorner f^*(\zeta^*) = \bigcap \{ f(\zeta) \mid \pi_{\alpha,\kappa}(\zeta) = \zeta^*,\ \zeta \in (\zeta^*,\ f_\xi(\zeta^*))\} \urcorner.$$

Hence

$$\{\kappa\} \times \{\zeta^* < \kappa \mid \Vdash_{P_{\zeta^*+1}} \ulcorner f^*(\zeta^*) \text{ is a dense open subset of } \langle P_\kappa / P_{\zeta^*+1}, \leq^*\rangle,$$
$$(\zeta \in (\zeta^*,\ f_\xi(\alpha^*)),\ \zeta^* = \pi_{\alpha,\kappa}(\zeta)) \implies f^*(\zeta^*) \subseteq f(\zeta) \urcorner\} \in E_{\kappa,\xi}(\{\kappa\}).$$

That is

$$\Vdash_{P^{M_{\kappa,\xi}}_{\kappa+1}} \ulcorner j_{\kappa,\xi}(f^*)(\kappa) \text{ is a dense open subset of } \langle j_{\kappa,\xi}(P_\kappa)/P^{M_{\kappa,\xi}}_{\kappa+1}, \leq^*\rangle \urcorner,$$

and

$$\Vdash_{P^{M_{\kappa,\xi}}_{\kappa+1}} \ulcorner j_{\kappa,\xi}(f^*)(\kappa) \subseteq j_{\kappa,\xi}(f)(\alpha) \urcorner.$$

Retreating to $N \simeq \mathrm{Ult}(V, E_{\kappa,\xi}(\{\kappa\}))$ we get

$$\Vdash_{P^N_{\kappa+1}} \ulcorner i(f^*)(\kappa) \text{ is a dense open subset of } \langle i(P_\kappa)/P^N_{\kappa+1}, \leq^*\rangle \urcorner.$$

Thus there is $\zeta < \kappa^+$ such that $\Vdash_{P^N_{\kappa+1}} \ulcorner \dot{p}^{N,\xi}_\zeta \in i(f^*)(\kappa) \urcorner$. Sending the last equation along k_ξ yields $\Vdash_{P^M_{\kappa+1}} \ulcorner k_\xi(\dot{p}^{N,\xi}_\zeta) \in j_{\kappa,\xi}(f^*)(\kappa) \subseteq j_{\kappa,\xi}(f)(\alpha) \urcorner$. $\square$

Definition 4.2.13. For each $0 < \xi < \mathrm{o}^{\mathrm{E}}(\kappa)$ use 4.2.12 to fix a sequence $\langle \dot{p}^\xi_\zeta \mid \zeta < \kappa^+\rangle$ and call it a master sequence for $M_{\kappa,\xi} \simeq \mathrm{Ult}(V, E_{\kappa,\xi})$.

In order to lift the ultrafilters in $E_{\kappa,\xi}$ we define a forcing notion which will be used to index the lifting.

Definition 4.2.14. Let G_κ be P_κ-generic. In $V[G_\kappa]$ we define the forcing notion $\mathbb{P}^*_{\bar{E}}$: $f \in \mathbb{P}^*_{\bar{E}}$ iff

(1) $f : d \to [\kappa]^{<\omega}$. We use the convention $f(\alpha) = \langle f_n(\alpha) \mid n < |f(\alpha)|\rangle$.
(2) $d \in [\sup_{\xi < \mathrm{o}^{\mathrm{E}}(\kappa)} j_{\kappa,\xi}(\kappa)]^{\leq\kappa}$.
(3) $\kappa \in d$.
(4) $\forall n < |f(\kappa)|\ \mathrm{o}^{\mathrm{E}}(f_{n-1}(\kappa)) \geq \mathrm{o}^{\mathrm{E}}(f_n(\kappa))$.
(5) $\forall \alpha \in d\ \alpha \neq \kappa \implies \forall n < |f(\alpha)|\ \mathrm{o}^{\mathrm{E}}(f_n(\alpha)) = 0$.

$\mathbb{P}^*_{\bar{E}}$ is equipped with the partial order $\leq^*$: $f \leq^* g$ if $f \supseteq g$. We let $\dot{Q}^*_\kappa$ be the P_κ-name of $\mathbb{P}^*_{\bar{E}}$. We note the implicit existence of $\dot{Q}^{*M_{\kappa,\xi}}_\kappa$.

Note that $\langle \mathbb{P}^*_{\bar{E}}, \leq^*\rangle$ is the Cohen forcing for adding $|\sup_{\xi < \mathrm{o}^{\mathrm{E}}(\kappa)} j_{\kappa,\xi}(\kappa)|$ subsets to κ^+, and $\dot{Q}^{*M_{\kappa,\xi}}_\kappa[G_\kappa]$ is the Cohen forcing for adding $|\sup_{\xi' < \xi} j_{\kappa,\xi'}(\kappa)|$ subsets to κ^+. Hence if H^*_κ is $\mathbb{P}^*_{\bar{E}}$-generic (or $\dot{Q}^{*M_{\kappa,\xi}}_\kappa[G_\kappa]$-generic) over $V[G_\kappa]$, then $\mathcal{P}^{V[G_\kappa]}(\kappa) = \mathcal{P}^{V[G_\kappa][H^*_\kappa]}(\kappa)$.

Note that the above definition relates to our main forcing notion 4.2.26 in the same way as 3.6 relates to 3.7. That is a tree of large sets will be put alongside f. The complication here is that now we have filters instead of ultrafilters. Thus in 3.7 the largeness of the sets was dependent on dom f. Now the largeness depends on f (and not only its domain).

The requirement $\mathrm{o}^{\mathrm{E}}(f_{n-1}(\kappa)) \geq \mathrm{o}^{\mathrm{E}}(f_n(\kappa))$ stems from 4.2.26. The $f_{n-1}(\kappa)$'s codes a previously added Radin sequence and $\mathrm{o}^{\mathrm{E}}(f_{n-1}(\kappa))$ codes the order type of this sequence. If $\mathrm{o}^{\mathrm{E}}(f_{n-1}(\kappa)) < \mathrm{o}^{\mathrm{E}}(f_n(\kappa))$ then the sequence coded by $f_{n-1}(\kappa)$ is a prefix of the sequence coded by $f_n(\kappa)$, hence giving superfluous information.

We construct the filters which are the lifting of the ultrafilters in $E_{\kappa,\xi}$ ($\xi > 0$). The following definition makes sense since $j''_{\kappa,\xi} P_\kappa = P_\kappa$.

Definition 4.2.15. Assume ξ, p, $\dot{f}$, $\dot{A}$, and $\dot{d}$ are such that $0 < \xi < \mathrm{o}^{\mathrm{E}}(\kappa)$ and

$$p \Vdash_{P_\kappa} \ulcorner \dot{f} \in \dot{Q}_\kappa^{*M_{\kappa,\xi}},\ \dot{d} \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ \text{and } \dot{A} \subseteq \dot{\mathrm{O}}\mathrm{B}(\mathrm{dom}\,\dot{d})\urcorner.$$

We define a P_κ-name, $\dot{E}_{\kappa,\xi}(\dot{d}, \dot{f})$, as follows:

$$\exists \zeta < \kappa^+ \exists \dot{q} \in \dot{Q}_\kappa^{M_{\kappa,\xi}} \ (p \Vdash_{P_\kappa} \ulcorner \dot{f} = f^{\dot{q}}\urcorner \text{ and}$$
$$p \frown \dot{q} \frown \dot{p}^\xi_\zeta \Vdash_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \langle\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\rangle \in j_{\kappa,\xi}(\dot{A})\urcorner)$$
$$\Longrightarrow\ p \Vdash_{P_\kappa} \ulcorner \dot{A} \in \dot{E}_{\kappa,\xi}(\dot{d}, \dot{f})\urcorner.$$

Lemma 4.2.16. *Assume ξ, p, $\dot{f}$, $\dot{A}$, and $\dot{d}$ are such that $0 < \xi < \mathrm{o}^{\mathrm{E}}(\kappa)$, and*

$$p \Vdash_{P_\kappa} \ulcorner \dot{f} \in \dot{Q}_\kappa^{*M_{\kappa,\xi}},\ \dot{d} \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ \text{and } \dot{A} \subseteq OB(\mathrm{dom}\,\dot{d})\urcorner.$$

Then

$$p \Vdash_{P_\kappa} \ulcorner \dot{A} \in \dot{E}_{\kappa,\xi}(\dot{d}, \dot{f})\urcorner \iff$$
$$\exists \zeta < \kappa^+ \exists \dot{q} \in \dot{Q}_\kappa^{M_{\kappa,\xi}} \ (p \Vdash_{P_\kappa} \ulcorner \dot{f} = f^{\dot{q}}\urcorner,$$
$$p \frown \dot{q} \frown \dot{p}^\xi_\zeta \Vdash_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \langle\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\rangle \in j_{\kappa,\xi}(\dot{A})\urcorner).$$

Proof. The ($\Longleftarrow$) direction is immediate from the definition. We prove the ($\Longrightarrow$) direction. So, assume ξ, p, $\dot{f}$, $\dot{A}$, and $\dot{d}$ are such that $0 < \xi < \mathrm{o}^{\mathrm{E}}(\kappa)$, and

$$p \Vdash_{P_\kappa} \ulcorner \dot{f} \in \dot{Q}_\kappa^{*M_{\kappa,\xi}},\ \dot{d} \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ \text{and } \dot{A} \in \dot{E}_{\kappa,\xi}(\dot{d}, \dot{f})\urcorner.$$

This means that there exists X, a maximal anti-chain below p, such that for each $r \in X$ there are $\zeta_r < \kappa^+$, $\dot{q}_r$, such that

$$r \Vdash_{P_\kappa} \ulcorner f^{\dot{q}_r} = \dot{f}\urcorner$$

and

$$r \frown \dot{q}_r \frown \dot{p}^\xi_{\zeta_r} \Vdash_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \langle\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\rangle \in j_{\kappa,\xi}(\dot{A})\urcorner.$$

Hence we can construct a P_κ-name, $\dot{q}$, such that $\forall r \in X\ r \Vdash_{P_\kappa} \ulcorner \dot{q} = \dot{q}_r\urcorner$. We set $\zeta = \bigcup_{r\in X} \zeta_r$. Since $|X| < \kappa$ we get $\zeta < \kappa^+$. Thus

$$p \frown \dot{q} \frown \dot{p}^\xi_\zeta \Vdash_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \langle\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\rangle \in j_{\kappa,\xi}(\dot{A})\urcorner. \quad \square$$

Claim 4.2.17. *Assume ξ, p, $\dot{f}$, and $\dot{d}$ are such that $0 < \xi < \mathrm{o}^{\mathrm{E}}(\kappa)$ and*

$$p \Vdash_{P_\kappa} \ulcorner \dot{f} \in \dot{Q}_\kappa^{*M_{\kappa,\xi}} \text{ and } \dot{d} \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa}\urcorner.$$

Then

$$p \Vdash_{P_\kappa} \ulcorner \dot{E}_{\kappa,\xi}(\dot{d}, \dot{f}) \text{ is a } \kappa\text{-closed filter on } \dot{O}B(\dot{d})\urcorner.$$

Proof. The properties meaning a κ-closed filter are:

(1) Assume $\lambda < \kappa$ and $p \Vdash_{P_\kappa} \ulcorner \forall \mu < \lambda\ \dot{A}_\mu \in \dot{E}_{\kappa,\xi}(\dot{d}, \dot{f})\urcorner$: By 4.2.16, we can construct a $\leq^*$-decreasing sequence $\langle p_\mu \frown \dot{q}_\mu \frown \dot{p}^\xi_{\zeta_\mu} \mid \mu < \lambda\rangle$ satisfying

$$p_0 \leq^*_{P_\kappa} p,$$
$$p_\mu \Vdash_{P_\kappa} \ulcorner f^{\dot{q}}_\mu = \dot{f}\urcorner,$$

and

$$p_\mu \frown \dot{q}_\mu \frown \dot{p}^\xi_{\zeta_\mu} \Vdash_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \{\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,\xi}(\dot{A}_\mu)\urcorner.$$

Let $\zeta = \bigcup_{\mu<\lambda} \zeta_\mu$. Since $\lambda < \kappa$ we get $\zeta < \kappa^+$. Since $\langle P^{M_{\kappa,\xi}}_{\kappa+1}, \leq^*\rangle$ is κ-closed there are p^* and $\dot{q}^*$ such that $p^* \Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner f^{\dot{q}^*} = \dot{f}\urcorner$ and $\forall\mu < \lambda\ \langle p^*, \dot{q}^*\rangle \leq^*_{P^{M_{\kappa,\xi}}_{\kappa+1}} \langle p_\mu, \dot{q}_\mu\rangle$. Thus

$$p^* \frown \dot{q}^* \frown \dot{p}^\xi_\zeta \Vdash_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \{\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in \bigcap_{\mu<\lambda} j_{\kappa,\xi}(\dot{A}_\mu)\urcorner.$$

Since crit $j_{\kappa,\xi} = \kappa > \lambda$,

$$p^* \frown \dot{q}^* \frown \dot{p}^\xi_\zeta \Vdash_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \{\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,\xi}(\bigcap_{\mu<\lambda} \dot{A}_\mu)\urcorner.$$

Hence $p^* \Vdash_{P_\kappa} \ulcorner \bigcap_{\mu<\lambda} \dot{A}_\rho \in \dot{E}_{\kappa,\xi}(\dot{d}, \dot{f})\urcorner$.

(2) Assume $p \Vdash_{P_\kappa} \ulcorner \dot{d} \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa}$, $\dot{A} \subseteq \dot{B} \subseteq \dot{\text{OB}}(\dot{d})$, and $\dot{A} \in \dot{E}_{\kappa,\xi}(\dot{d}, \dot{f})\urcorner$: It is immediate that there are $\dot{q}$ and $\zeta < \kappa^+$ such that $p \Vdash_{P_\kappa} \ulcorner f^{\dot{q}} = \dot{f}\urcorner$ and

$$p \frown \dot{q} \frown \dot{p}^\xi_\zeta \Vdash_{j_{\kappa,\xi}(P_\kappa)} \ulcorner j_{\kappa,\xi}(\dot{A}) \subseteq j_{\kappa,\xi}(\dot{B}) \text{ and } \{\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\} \in j_{\kappa,\xi}(\dot{A})\urcorner. \quad \square$$

We can view the corollary of the following lemma as a form of 'ultrafilterness'.

Lemma 4.2.18. *Assume ξ, p, $\dot{q}$, $\dot{A}$, and $\dot{d}$ are such that $0 < \xi < \text{o}^{\text{E}}(\kappa)$, and*

$$p \Vdash_{P_\kappa} \ulcorner \dot{q} \in \dot{Q}^{M_{\kappa,\xi}}_\kappa,\ \dot{d} \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ \textit{and } \dot{A} \subseteq \dot{OB}(\dot{d})\urcorner.$$

Then there are $p^* \frown \dot{q}^* \leq^*_{P^{M_{\kappa,\xi}}_{\kappa+1}} p \frown \dot{q}$ and $\zeta < \kappa^+$ such that

$$p^* \frown \dot{q}^* \frown \dot{p}^\xi_\zeta \parallel_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \langle\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\rangle \in j_{\kappa,\xi}(\dot{A})\urcorner.$$

Proof. Assume $p \Vdash_{P_\kappa} \ulcorner \dot{d} \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa}$ and $\dot{A} \subseteq \text{OB}(\dot{d})\urcorner$. Set

$$D = \{r \in j_{\kappa,\xi}(P_\kappa) \mid r \parallel_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \langle\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\rangle \in j_{\kappa,\xi}(\dot{A})\urcorner\}.$$

Since P_κ is of Prikry type, D is a dense open subset of $\langle j_{\kappa,\xi}(P_\kappa), \leq^*\rangle$. Thus $D/P^{M_{\kappa,\xi}}_{\kappa+1}$ is a name of a dense open subset of $\langle j_{\kappa,\xi}(P_\kappa)/P^{M_{\kappa,\xi}}_{\kappa+1}, \leq^*\rangle$. Hence there is $\zeta < \kappa^+$ such that $\Vdash_{P^{M_{\kappa,\xi}}_{\kappa+1}} \ulcorner \dot{p}^\xi_\zeta \in D/P^{M_{\kappa,\xi}}_{\kappa+1}\urcorner$. That is

$$\Vdash_{P^{M_{\kappa,\xi}}_{\kappa+1}} \ulcorner \dot{p}^\xi_\zeta \parallel_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \langle\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\rangle \in j_{\kappa,\xi}(\dot{A})\urcorner\urcorner.$$

Using the Prikry property we find $p^* \frown \dot{q}^* \leq^*_{P^{M_{\kappa,\xi}}_{\kappa+1}} p \frown \dot{q}$ such that

$$p^* \frown \dot{q}^* \frown \dot{p}^\xi_\zeta \parallel_{j_{\kappa,\xi}(P_\kappa)} \ulcorner \langle\langle j_{\kappa,\xi}(\alpha), \alpha\rangle \mid \alpha \in \dot{d}\rangle \in j_{\kappa,\xi}(\dot{A})\urcorner. \quad \square$$

Corollary 4.2.19. *Assume ξ, p, $\dot{f}$, $\dot{A}$, and $\dot{d}$ are such that $0 < \xi < \text{o}^{\text{E}}(\kappa)$ and*

$$p \Vdash_{P_\kappa} \ulcorner \dot{f} \in \dot{Q}^{*M_{\kappa,\xi}}_\kappa,\ \dot{d} \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ \textit{and } \dot{A} \subseteq \dot{OB}(\dot{d})\urcorner.$$

Then there is f^* satisfying $p \Vdash_{P_\kappa} \ulcorner \dot{f}^* \leq^*_{\dot{Q}^{*M_{\kappa,\xi}}_\kappa} \dot{f}\urcorner$ such that either

$$p \Vdash_{P_\kappa} \ulcorner \dot{A} \in \dot{E}_{\kappa,\xi}(\dot{d}, \dot{f}^*)\urcorner$$

or

$$p \Vdash_{P_\kappa} \ulcorner (\dot{OB}(\dot{d}) \setminus \dot{A}) \in \dot{E}_{\kappa,\xi}(\dot{d}, \dot{f}^*)\urcorner.$$

A corollary of 4.2.15–4.2.19 is that the system

$$\langle\langle \dot{E}_{\kappa,\xi}(d,\ f)[G_\kappa] \mid d \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ f \in \dot{Q}_\kappa^{M_{\kappa,\xi}}[G_\kappa]\rangle,$$
$$\langle \pi_{d_2,d_1} \mid d_1, d_2 \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ \kappa \in d_1 \subseteq d_2\rangle\rangle$$

codes an extender in a $P_\kappa * \dot{Q}^*_\kappa$-generic extension. Written explicitly in $V[G_\kappa][H^*]$, where H^* is $\mathbb{P}^*_{\bar{E}}$-generic over $V[G_\kappa]$, this extender is

$$F_{\kappa,\xi} = \langle\langle F_{\kappa,\xi}(d) \mid d \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa}\rangle, \langle \pi_{d_2,d_1} \mid d_1, d_2 \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ \kappa \in d_1 \subseteq d_2\rangle\rangle$$

where the ultrafilters $F_{\kappa,\xi}(d)$ are defined by

$$F_{\kappa,\xi}(d) = \bigcup\{\dot{E}_{\kappa,\xi}(d,\ f) \mid d \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ f \in H^* \cap \dot{Q}_\kappa^{M_{\kappa,\xi}[G_\kappa]}\}.$$

Corollary 4.2.20. *Assume* $0 < \xi < \mathrm{o}^{\mathrm{E}}(\kappa)$ *and* H^* *is a* $\mathbb{P}^*_{\bar{E}}$*-generic filter over* $V[G_\kappa]$*. Then*

$$\langle\langle \dot{F}_{\kappa,\xi}(d)[G_\kappa] \mid d \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa}\rangle,\ \langle \pi_{d_2,d_1} \mid d_1, d_2 \in [j_{\kappa,\xi}(\kappa)]^{\leq\kappa},\ \kappa \in d_1 \subseteq d_2\rangle\rangle$$

is a κ*-extender lifting* $E_{\kappa,\xi}$*.*

A couple of remarks regarding the last corollary are in order:

(1) Of course we could have taken H^* to be a $\dot{Q}_\kappa^{*M_{\kappa,\xi}}[G_\kappa]$-generic filter over $V[G_\kappa]$.
(2) The demand $\xi > 0$ is not really needed. After all forcing with $\mathbb{P}^*_{\bar{E}}$ adds no subsets to κ; hence the lifted extender $\dot{E}_{\kappa,0}[G_\kappa]$ remains an extender in $V[G_\kappa][H^*]$. Thus we can set $F_{\kappa,0} = E_{\kappa,0}$.

The following is the substitute for the intersection of measures used in the Radin on extenders forcing.

Definition 4.2.21. Assume $\xi < \mathrm{o}^{\mathrm{E}}(\kappa)$ and $f \in \mathbb{P}^*_{\bar{E}}$. Then

$$E_{\kappa,\xi}(f) = E_{\kappa,\xi}(\mathrm{dom}\, f \restriction j_{\kappa,\xi}(\kappa),\ f \restriction \sup_{\xi'<\xi} j_{\kappa,\xi'}(\kappa)),$$
$$E_\kappa(f) = \bigcap_{\xi<\mathrm{o}^{\mathrm{E}}(\kappa)} E_{\kappa,\xi}(f).$$

Note that we have used $E_{\kappa,0}(d,\ f\restriction\emptyset)$ in the above definition. Obviously we just mean $E_{\kappa,0}(d)$. In addition, instead of writing $\pi_{\mathrm{dom}\, f,\mathrm{dom}\, g}$ we will write $\pi_{f,g}$.

After all these liftings, we are ready to define the forcing notion at stage κ of the iteration, $\dot{Q}_\kappa$. The definition is by induction on $\mathrm{o}^{\mathrm{E}}(\kappa)$.

$\mathrm{o}^{\mathrm{E}}(\kappa) = 1$: Then $\dot{Q}_\kappa[G_\kappa]$ is the Prikry on extenders forcing, reviewed in 3.2, with the (lifted) extender $E_{\kappa,0}$.

$\mathrm{o}^{\mathrm{E}}(\kappa) > 1$: Then $\dot{Q}_\kappa[G_\kappa]$ is the Radin on extenders forcing, defined as follows, in $V[G_\kappa]$. (Recall that $\dot{Q}^*_\kappa[G_\kappa] = \mathbb{P}^*_{\bar{E}}$.)

Definition 4.2.22. Assume $f \in \mathbb{P}^*_{\bar{E}}$ and $\bar{\nu} \in \mathrm{OB}(d)$. The function

$$f_{\langle\bar{\nu}\rangle} : \mathrm{dom}\, f \to [\kappa]^{<\omega}$$

is defined as

$$\forall\alpha \in \mathrm{dom}\, f$$

$$f_{\langle\bar{\nu}\rangle}(\alpha) = \begin{cases} & \alpha \in \mathrm{dom}\,\bar{\nu}, \\ f(\alpha)\restriction k \frown \langle\bar{\nu}(\alpha)\rangle & \bar{\nu}(\alpha) > f_{|f(\alpha)|-1}(\alpha), \\ & k = \max\{n+1 \mid \mathrm{o}^{\mathrm{E}}(\bar{\nu}(\kappa)) \leq \mathrm{o}^{\mathrm{E}}(f_n(\alpha))\}. \\ f(\alpha) & \text{Otherwise.} \end{cases}$$

Note that $f_{\langle\bar{\nu}\rangle} \in \mathbb{P}^*_{\bar{E}}$.

By writing $f_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ we mean $(\cdots(f_{\langle\bar{\nu}_0\rangle})_{\langle\bar{\nu}_1\rangle}\cdots)_{\langle\bar{\nu}_{k-1}\rangle}$.

Note in the above definition that when $\alpha \neq \kappa$ we have $\mathrm{o}^{\mathrm{E}}(f_n(\alpha)) = 0$.

In the following couple of definitions it is implicitly assumed that if T is a tree then $\mathrm{Suc}_T()$ is $\mathrm{Lev}_0(T)$.

Definition 4.2.23. Assume $f \in \mathbb{P}^*_{\bar{E}}$. A tree T of height ω is called an $E_\kappa(f)$-tree if

$$\forall \langle \bar{\nu}_0, \ldots, \bar{\nu}_{k-1} \rangle \in T \ \ \mathrm{Suc}_T(\bar{\nu}_0, \ldots, \bar{\nu}_{k-1}) \in E_\kappa(f_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{k-1} \rangle})$$

and for each $\langle \bar{\nu} \rangle \in T_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{k-1} \rangle}$

(1) $\mathrm{dom}\, \bar{\nu}_{k-1} \subseteq \mathrm{dom}\, \bar{\nu}$.
(2) $\forall \alpha \in \mathrm{dom}\, \bar{\nu}_{k-1} \ \bar{\nu}_{k-1}(\alpha) < \bar{\nu}(\alpha)$.
(3) $\forall \alpha \in \mathrm{dom}\, \bar{\nu}_{k-1} \ \alpha \neq \kappa \implies \mathrm{o}^{\mathrm{E}}(\nu(\alpha)) = 0$.
(4) $\forall \langle \bar{\mu} \rangle \in T_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{k-1} \rangle} \ \bar{\nu}(\kappa) < \bar{\mu}(\kappa) \implies \mathrm{dom}\, \bar{\nu} \subseteq \mathrm{dom}\, \bar{\mu}$.

Definition 4.2.24. Assume $f \in \mathbb{P}^*_{\bar{E}}$. A tree T of height $\mathrm{ht}(T) < \omega$ is called an $E_\kappa(f)$-fat if

$$\forall \langle \bar{\nu}_0, \ldots, \bar{\nu}_{k-1} \rangle \in T \ \exists \xi < \mathrm{o}^{\mathrm{E}}(\kappa) \ \ \mathrm{Suc}_T(\bar{\nu}_0, \ldots, \bar{\nu}_{k-1}) \in E_{\kappa,\xi}(f_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{k-1} \rangle})$$

and for each $\langle \bar{\nu} \rangle \in T_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{k-1} \rangle}$

(1) $\mathrm{dom}\, \bar{\nu}_{k-1} \subseteq \mathrm{dom}\, \bar{\nu}$.
(2) $\forall \alpha \in \mathrm{dom}\, \bar{\nu}_{k-1} \ \bar{\nu}_{k-1}(\alpha) < \bar{\nu}(\alpha)$.
(3) $\forall \alpha \in \mathrm{dom}\, \bar{\nu}_{k-1} \ \alpha \neq \kappa \implies \mathrm{o}^{\mathrm{E}}(\nu(\alpha)) = 0$.
(4) $\forall \langle \bar{\mu} \rangle \in T_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{k-1} \rangle} \ \bar{\nu}(\kappa) < \bar{\mu}(\kappa) \implies \mathrm{dom}\, \bar{\nu} \subseteq \mathrm{dom}\, \bar{\mu}$.

The definition of $\pi_{d',d'}$ and $\pi^{-1}_{d,d'}$ for $E_\kappa(f)$-trees (or $E_\kappa(f)$-fat trees) is taken verbatim from 3.5.

It is useful to observe that different levels in $E_\kappa(f)$-fat trees are big in the sense of product filters (and hence the different levels in $E(f)$-trees are big in the sense of *all* relevant product filters). Recall

Definition 4.2.25. Assume F_0 is a filter on A_0, and for each $\nu_0 \in A_0$, $F_1(\nu_0)$ is a filter on $A_1(\nu_0)$. Then the product filter $F_0 \times F_1$ on $A_0 \times A_1$ is defined by

$$X \in F_0 \times F_1 \iff \{\nu_0 \in A_0 \mid \{\nu_1 \in A_1(\nu_0) \mid \langle \nu_0, \nu_1 \rangle \in X\} \in F_1(\nu_0)\} \in F_0.$$

By induction we define $\prod_{i=0}^n F_i$ by

$$X \in \prod_{i=0}^{n} F_i \iff \Big\{ \langle \nu_0, \ldots, \nu_{n-1} \rangle \in \prod_{i=0}^{n-1} A_i \mid \{\nu_n \in A_n(\nu_0, \ldots, \nu_{n-1}) \mid \langle \nu_0, \ldots, \nu_{n-1}, \nu_n \rangle \in X\} \in F_n(\nu_0, \ldots, \nu_{n-1}) \Big\} \in \prod_{i=0}^{n-1} F_i.$$

Definition 4.2.26. A condition $p \in \mathbb{P}_{\bar{E}}$ is of the form

$$\langle f, T \rangle,$$

where

(1) $f \in \mathbb{P}^*_{\bar{E}}$.
(2) T is an $E_\kappa(f)$-tree.

We write $\mathrm{supp}\, p$, f^p, and T^p, for $\mathrm{dom}\, f$, f, and T, respectively.

Definition 4.2.27. Let $p, q \in \mathbb{P}_{\bar{E}}$. We say that p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if

(1) $f^p \restriction \mathrm{supp}\, q = f^q$.
(2) $\pi_{\mathrm{supp}\, p, \mathrm{supp}\, q} T^p \subseteq T^q$.

Definition 4.2.28. Let $q \in \mathbb{P}_{\bar{E}}$ and $\langle \bar{\nu} \rangle \in T^q$. We define $q_{\langle \bar{\nu} \rangle} \in \mathbb{P}_{\bar{E}}$ to be p where

(1) $\operatorname{supp} p = \operatorname{supp} q$.
(2) $f^p = f^q_{\langle \bar{\nu} \rangle}$.
(3) $T^p = T^q_{\langle \bar{\nu} \rangle}$.

When we write $q_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{k-1} \rangle}$ we mean $(\cdots (q_{\langle \bar{\nu}_0 \rangle})_{\langle \bar{\nu}_1 \rangle} \cdots)_{\langle \bar{\nu}_{k-1} \rangle}$.

Definition 4.2.29. Let $p, q \in \mathbb{P}_{\bar{E}}$. We say that p is a 1-point extension of q ($p \leq^1 q$) if there is $\langle \bar{\nu} \rangle \in T^q$ such that $p \leq^* q_{\langle \nu \rangle}$.

Definition 4.2.30. Let $p, q \in \mathbb{P}_{\bar{E}}$, $n < \omega$. We say that p is an n-point extension of q ($p \leq^n q$) if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

Definition 4.2.31. Let $p, q \in \mathbb{P}_{\bar{E}}$. We say that p is an extension of q ($p \leq q$) if there is $n < \omega$ such that $p \leq^n q$.

Proposition 4.2.32. *Assume $q \in \mathbb{P}_{\bar{E}}$ and $\alpha \in \sup_{\xi <_{o^{\mathrm{E}}(\kappa)}} j_{\kappa,\xi}(\kappa)$. Then there is $p \leq^*_{\mathbb{P}_{\bar{E}}} q$ with $\alpha \in \operatorname{supp} p$.*

Proof. If $\alpha \in \operatorname{supp} q$ then there is nothing to do; we set $p = q$.

Otherwise we set $p = \langle f^q \cup \langle \alpha, \langle\rangle \rangle, \pi^{-1}_{\operatorname{supp} q \cup \{\alpha\}, \operatorname{supp} q} T^q \rangle$. Then $p \leq^*_{\mathbb{P}_{\bar{E}}} q$, and $\alpha \in \operatorname{supp} p$. Note that strictly speaking $\pi^{-1}_{\operatorname{supp} q \cup \{\alpha\}, \operatorname{supp} q} T^q$ might not be legal as a tree for f^p. However the illegal points have measure zero in all the relevant filters, so we just remove them. $\square$

Proposition 4.2.33. *$\mathbb{P}_{\bar{E}}$ satisfies the κ^{++}-cc.*

Proof. Assume $X \subseteq \mathbb{P}_{\bar{E}}$ and $|X| = \kappa^{++}$. Since for each $p \in X$ we have $|\operatorname{supp} p| \leq \kappa$, we can assume that $\{\operatorname{supp} p \mid p \in X\}$ forms a Δ-system. That is, there is d such that $\forall p, q \in X$ $\operatorname{supp} p \cap \operatorname{supp} q = d$. Since $|d| \leq \kappa$ we have $|\{f \mid f : d \to [\kappa]^{<\omega}\}| \leq \kappa^+$, so we can assume that $\forall p, q \in X\ \forall \beta \in d\ f^p(\beta) = f^q(\beta)$.

Let us fix two conditions, $p, q \in X$, and let $f = f^p \cup f^q$. Then $f : \operatorname{supp} p \cup \operatorname{supp} q \to [\kappa]^{<\omega}$. We set $T = \pi^{-1}_{\operatorname{supp} p \cup \operatorname{supp} q, \operatorname{supp} p}(T^p) \cap \pi^{-1}_{\operatorname{supp} p \cup \operatorname{supp} q, \operatorname{supp} q}(T^q)$. Then $\langle f, T \rangle \leq_{\mathbb{P}_{\bar{E}}} p, q$. $\square$

Claim 4.2.34. *$\langle \mathbb{P}_{\bar{E}}, \leq^* \rangle$ is κ-closed.*

Proof. Assume $\lambda < \kappa$, and $\langle p_\xi \mid \xi < \lambda \rangle$ is a $\leq^*$-decreasing sequence in $\mathbb{P}_{\bar{E}}$. Then $\langle f^{p_\xi} \mid \xi < \lambda \rangle$ is $\leq^*$-decreasing sequence in $\mathbb{P}^*_{\bar{E}}$. Since $\langle \mathbb{P}^*_{\bar{E}}, \leq^* \rangle$ is κ^+-closed, there is $f \in \mathbb{P}^*_{\bar{E}}$ such that $\forall \xi < \lambda\ f \leq^*_{\mathbb{P}^*_{\bar{E}}} f^{p_\xi}$. Set $T = \bigcap_{\xi < \lambda} \pi^{-1}_{\operatorname{dom} f, \operatorname{supp}_\xi}(T^{p_\xi})$. Then $\forall \xi < \lambda\ \langle f, T \rangle \leq^*_{\mathbb{P}_{\bar{E}}} p_\xi$. $\square$

The notions $\langle N, P \rangle$-generic and properness are due to S. Shelah, originally used for countable N. We adapt these notions for our use, i.e. for N of size κ. H. Woodin initiated the use of properness in the context of Radin forcing.

Definition 4.2.35. Assume P is a forcing notion and χ is large enough so that $\chi > 2^{2^{|\mathcal{P}|}}$, $N \prec H_\chi$, and $P \in N$. We say that $p \in P$ is $\langle N, P \rangle$-generic if for each $D \in N$ a dense subset of P, $p \Vdash \ulcorner \check{D} \cap \check{N} \cap \underset{\sim}{G} \neq \emptyset \urcorner$, where $\underset{\sim}{G}$ is the canonical name of the P-generic filter.

Definition 4.2.36. Assume P is a forcing notion, and χ is large enough so that $\chi > 2^{2^{|\mathcal{P}|}}$. We say that P is κ-proper if for each $N \prec H_\chi$ and $p \in P \cap N$ such that $|N| = \kappa$, $N \supset N^{<\kappa}$, and $P \in N$ there is $q \leq p$ such that q is $\langle N, P \rangle$-generic.

The following lemma is an immediate corollary of the κ^+-closedness of the Cohen forcing $\mathbb{P}^*_{\bar{E}}$, and it amounts to the κ-properness of $\mathbb{P}^*_{\bar{E}}$

Lemma 4.2.37. *Let χ be large enough so that $\chi > 2^{2^{|\mathbb{P}^*_{\bar{E}}|}}$. Assume $N \prec H_\chi$ and $f \in \mathbb{P}^*_{\bar{E}} \cap N$ are such that $|N| = \kappa$, $N \supset N^{<\kappa}$ and $\mathbb{P}^*_{\bar{E}} \in N$. Then there is $f^* \leq_{\mathbb{P}^*_{\bar{E}}} f$ such that:*

(1) *f^* is $\langle N, \mathbb{P}^*_{\bar{E}}\rangle$-generic.*
(2) *If H^* is $\mathbb{P}^*_{\bar{E}}$-generic with $f^* \in H^*$, then for each dense open subset of $\mathbb{P}^*_{\bar{E}}$, D, appearing in N, there is $g \in D \cap H^* \cap N$ such that $f^* \leq^*_{\mathbb{P}^*_{\bar{E}}} g \leq^*_{\mathbb{P}^*_{\bar{E}}} f$.*
(3) *For each $\xi \in N \cap \mathrm{o}^{\mathrm{E}}(\kappa)$, $E_{\kappa,\xi}(f^*)$ is an N-extender. (Note that this allows the construction of* $\mathrm{Ult}(N, E_{\kappa,\xi}(f^*))$.*)*

Claim 4.2.38. *Assume that $p \in \mathbb{P}_{\bar{E}}$ and D is a dense open subset of $\mathbb{P}_{\bar{E}}$. Then there is $p^* \leq^*_{\mathbb{P}_{\bar{E}}} p$ such that if $q \leq^*_{\mathbb{P}_{\bar{E}}} p^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ and $q \in D$, then $p^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} \in D$.*

Proof. Let χ be large enough so that $\mathcal{P}(\mathbb{P}_{\bar{E}}) \in H_\chi$. Let $N \prec H_\chi$ be such that $|N| = \kappa$, $N \supset N^{<\kappa}$, and $\mathbb{P}_{\bar{E}}, D, p \in N$.

Since $\mathbb{P}_{\bar{E}} \in N$, also $\mathbb{P}^*_{\bar{E}} \in N$. By 4.2.37 there is $f^* \leq^*_{\mathbb{P}^*_{\bar{E}}} f$ which is $\langle N, \mathbb{P}^*_{\bar{E}}\rangle$-generic. Let $T = \pi^{-1}_{f^*,f}(T^p)$. For each $\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle \in T$ we define

$$D^{\in}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} = \{q \leq^*_{\mathbb{P}_{\bar{E}}} p_{\langle\bar{\nu}_0\restriction \operatorname{supp} p,\ldots,\bar{\nu}_{k-1}\restriction \operatorname{supp} p\rangle} \mid q \in D\},$$

and

$$D^{\perp}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} = \{q \leq^*_{\mathbb{P}_{\bar{E}}} p_{\langle\bar{\nu}_0\restriction \operatorname{supp} p,\ldots,\bar{\nu}_{k-1}\restriction \operatorname{supp} p\rangle} \mid \forall r \in D^{\in}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}\ q \perp_{\mathbb{P}_{\bar{E}}} r\}.$$

Since D is open, $D^{\in}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ is $\leq^*_{\mathbb{P}_{\bar{E}}}$-open below $p_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$. By its definition $D^{\perp}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ is $\leq^*_{\mathbb{P}_{\bar{E}}}$-open below $p_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$. Hence

$$D_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} = D^{\in}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} \cup D^{\perp}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$$

is $\leq^*_{\mathbb{P}_{\bar{E}}}$-open and $\leq^*_{\mathbb{P}_{\bar{E}}}$-dense below $p_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$. Let us set

$$D^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} = \{f \leq^*_{\mathbb{P}^*_{\bar{E}}} f^p \mid \exists T\ \langle f_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}, T\rangle \in D_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}\}.$$

The $\leq^*_{\mathbb{P}^*_{\bar{E}}}$-openness of $D^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ follows immediately from the $\leq^*_{\mathbb{P}_{\bar{E}}}$-openness of $D_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$. We show that $D^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ is a dense subset of $\mathbb{P}^*_{\bar{E}}$ below f^p. So, let $g' \leq^*_{\mathbb{P}^*_{\bar{E}}} f^p$.

Pick $g \leq^*_{\mathbb{P}^*_{\bar{E}}} g'$ such that $\operatorname{dom} g \supseteq \bigcup_{i<k} \operatorname{dom} \bar{\nu}_i$, and set

$$T' = \pi^{-1}_{g,f^p}(T^p_{\langle\bar{\nu}_0\restriction \operatorname{supp} p,\ldots,\bar{\nu}_{k-1}\restriction \operatorname{supp} p\rangle}).$$

Then $\langle g_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}, T'\rangle \leq^*_{\mathbb{P}_{\bar{E}}} p_{\langle\bar{\nu}_0\restriction \operatorname{supp} p,\ldots,\bar{\nu}_{k-1}\restriction \operatorname{supp} p\rangle}$. Since $D_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ is $\leq^*_{\mathbb{P}_{\bar{E}}}$-dense, there is $q \in D_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ such that $q \leq^*_{\mathbb{P}_{\bar{E}}} \langle g_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}, T'\rangle$. We set

$$f = g \cup (f^q \restriction (\operatorname{supp} q \setminus \operatorname{dom} g)).$$

Since $f_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} = f^q$, we get $f \in D^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$. Thus $D^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ is $\leq^*_{\mathbb{P}^*_{\bar{E}}}$-dense open below f^p.

The sets $D^{\in}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$, $D^{\perp}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$, $D_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ and $D^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ appear in N. Since f^* is in the intersection of all $\leq^*$-dense open subsets of $\mathbb{P}^*_{\bar{E}}$ appearing in N, we have that $f^* \in \bigcap\{D^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} \mid \langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle \in T\}$.

Hence for each $\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle \in T$ there is $T^{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} \subseteq T_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$ such that

$$\langle f^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}, T^{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}\rangle \in D_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}.$$

Let T^* be the tree T shrunken level by level (i.e., $T^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} = T_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} \cap T^{(\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle)}$. Thus for each $\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle \in T^*$

$$\langle f^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}, T^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}\rangle \in D_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}.$$

Let $p^* = \langle f^*, T^*\rangle$. We show that p^* is as required: Let $q \leq_{\mathbb{P}_{\bar{E}}} p^*$ and $q \in D$.

Then there is $\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle \in T^*$ such that $q \leq^*_{\mathbb{P}_{\bar{E}}} p^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$. By the construction of p^*, $p^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} \in D_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$. By the definition of $D^{\in}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$, $q \in D^{\in}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$. Necessarily $p^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} \in D^{\in}_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle}$. That is $p^*_{\langle\bar{\nu}_0,\ldots,\bar{\nu}_{k-1}\rangle} \in D$. $\square$

Claim 4.2.39. *Assume $p \in \mathbb{P}_{\bar{E}}$ and $D \subseteq \mathbb{P}_{\bar{E}}$ is dense open. Then there are $p^* \leq^*_{\mathbb{P}_{\bar{E}}} p$ and an $E_\kappa(f^{p^*})$-fat tree, S^*, such that*

$$\forall \langle \bar{\nu}_0, \dots, \bar{\nu}_{\mathrm{ht}(S^*)-1}\rangle \in S^* \ p^*_{\langle \bar{\nu}_0, \dots, \bar{\nu}_{\mathrm{ht}(S^*)-1}\rangle} \in D.$$

Proof. Let us assume, by contradiction, that there is no such p^*. We construct a $\leq^*_{\mathbb{P}_{\bar{E}}}$-decreasing sequence $\langle p^n \mid n < \omega\rangle$ such that

$$\forall \langle \bar{\nu}_0, \dots, \bar{\nu}_{n-1}\rangle \in T^{p^n} \ p^n_{\langle \bar{\nu}_0, \dots, \bar{\nu}_{n-1}\rangle} \notin D.$$

We construct $p^0 \leq^*_{\mathbb{P}_{\bar{E}}} p^*$ using 4.2.38. Let us assume p^n was constructed, and construct p^{n+1}. We set

$$X = \{\langle \bar{\nu}_0, \dots, \bar{\nu}_n\rangle \in T^{p^n} \mid p^n_{\langle \bar{\nu}_0, \dots, \bar{\nu}_{n-1}, \bar{\nu}_n\rangle} \in D\}.$$

Note that if $q \leq^*_{\mathbb{P}_{\bar{E}}} p^n$ is such that

$$\{\langle \bar{\nu}_0, \dots, \bar{\nu}_n\rangle \in T^q \mid \langle \bar{\nu}_0 \restriction \operatorname{supp} p^n, \dots, \bar{\nu}_n \restriction \operatorname{supp} p^n\rangle \in X\}$$

is an $E_\kappa(f^q)$-fat tree, then by the openness of D,

$$\{\langle \bar{\nu}_0, \dots, \bar{\nu}_n\rangle \in T^q \mid q_{\langle \bar{\nu}_0, \dots, \bar{\nu}_n\rangle} \in D\}$$

is an $E_\kappa(f^q)$-fat tree, contradicting our assumption. Hence there is $p^{n+1} \leq^*_{\mathbb{P}_{\bar{E}}} p^n$ such that

$$\forall \langle \bar{\nu}_0, \dots, \bar{\nu}_n\rangle \in T^{p^{n+1}} \ \langle \bar{\nu}_0 \restriction \operatorname{supp} p^n, \dots, \bar{\nu}_n \restriction \operatorname{supp} p^n\rangle \notin X.$$

In particular (since p^0 was constructed using 4.2.38)

$$\forall \langle \bar{\nu}_0, \dots, \bar{\nu}_n\rangle \in T^{p^{n+1}} \ p^{n+1}_{\langle \bar{\nu}_0, \dots, \bar{\nu}_n\rangle} \notin D.$$

Having constructed $\langle p^n \mid n < \omega\rangle$, we pick $p^* \in \mathbb{P}_{\bar{E}}$ such that $\forall n < \omega \ p^* \leq^*_{\mathbb{P}_{\bar{E}}} p^n$. Note that since p^0 was constructed using 4.2.38 then

$$\forall \langle \bar{\nu}_0, \dots, \bar{\nu}_{n-1}\rangle \in T^{p^*} \ p^*_{\langle \bar{\nu}_0, \dots, \bar{\nu}_{n-1}\rangle} \notin D.$$

Let us pick $q \leq_{\mathbb{P}_{\bar{E}}} p^*$ such that $q \in D$. Then there is $\langle \bar{\nu}_0, \dots, \bar{\nu}_{n-1}\rangle \in T^{p^*}$ such that $q \leq^*_{\mathbb{P}_{\bar{E}}} p^*_{\langle \bar{\nu}_0, \dots, \bar{\nu}_{n-1}\rangle}$. Then $q \leq^*_{\mathbb{P}_{\bar{E}}} p^0_{\langle \bar{\nu}_0 \restriction \operatorname{supp} p^0, \dots, \bar{\nu}_{n-1} \restriction \operatorname{supp} p^0\rangle}$; hence

$$p^0_{\langle \bar{\nu}_0 \restriction \operatorname{supp} p^0, \dots, \bar{\nu}_{n-1} \restriction \operatorname{supp} p^0\rangle} \in D.$$

By the openness of D, $p^*_{\langle \bar{\nu}_0, \dots, \bar{\nu}_{n-1}\rangle} \in D$. Contradiction. □

Lemma 4.2.40. *Assume $p \in \mathbb{P}_{\bar{E}}$, $\xi < \mathrm{o}^{\mathrm{E}}(\kappa)$, σ is a formula in the $\mathbb{P}_{\bar{E}}$-forcing language, and $\{\langle \bar{\nu}\rangle \in T^p \mid p_{\langle \bar{\nu}\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma\} \in E_{\kappa,\xi}(f^p)$. Then there is $p^* \leq^*_{\mathbb{P}_{\bar{E}}} p$ such that $p^* \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$.*

Proof. Let $D = \{q \in \mathbb{P}_{\bar{E}} \mid q \parallel_{\mathbb{P}_{\bar{E}}} \sigma\}$. Since D is dense open, we use 4.2.38 to construct $p^0 \leq^*_{\mathbb{P}_{\bar{E}}} p$ such that if $q \leq^*_{\mathbb{P}_{\bar{E}}} p^0_{\langle \bar{\nu}\rangle}$ and $q \parallel_{\mathbb{P}_{\bar{E}}} \sigma$, then $p^0_{\langle \bar{\nu}\rangle} \parallel_{\mathbb{P}_{\bar{E}}} \sigma$.

We construct by induction a $\leq^*_{\mathbb{P}_{\bar{E}}}$-decreasing sequence $\langle p^n \mid n < \omega\rangle$ such that $\forall n < \omega \ f^{p^n} = f^{p^0}$, and for each $\langle \bar{\mu}_0, \dots, \bar{\mu}_{n-1}\rangle \in T^{p^n}$ either

$$p^n_{\langle \bar{\mu}_0, \dots, \bar{\mu}_{n-1}\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma,$$

or

$$\{\langle \bar{\nu}_1\rangle \in T^{p^n}_{\langle \bar{\mu}_0, \dots, \bar{\mu}_{n-1}\rangle} \mid p^n_{\langle \bar{\mu}_0, \dots, \bar{\mu}_{n-1}, \bar{\nu}_1\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma\} \in E_{\kappa,\xi}(f^p_{\langle \bar{\mu}_0, \dots, \bar{\mu}_{m-1}\rangle}).$$

p^0 trivially satisfies the requirements. Let us construct p^{n+1} assuming p^n was constructed. What we do is construct for each $\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle \in T^{p^n}$ a set

$$X(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \in E_\kappa(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle}),$$

and then shrink T^{p^n} to these sets. If $p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle} \Vdash \sigma$ then there is nothing to do and we just set $X(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) = \mathrm{Suc}_{T^{p^n}}(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1})$. On the other hand, if $p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle} \nVdash \sigma$, then $X(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1})$ is the union of the three sets $X_0(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1})$, $X_1(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1})$, and $X_2(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1})$, such that

$$X_0(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \in \bigcap_{\xi_0<\xi} E_{\kappa,\xi_0}(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle}),$$
$$X_1(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \in E_{\kappa,\xi}(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle}),$$
$$X_2(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \in \bigcap_{\xi<\xi_2<\mathrm{o}^{\mathrm{E}}(\kappa)} E_{\kappa,\xi_2}(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle}).$$

Let us begin the construction. For each $\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle \in T^{p^n}$ such that

$$p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle} \nVdash \sigma$$

do the following three steps:

(1) Set

$$X_1(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) = \{\langle\bar{\nu}_1\rangle \in T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle} \mid p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma\}.$$

By the induction hypothesis $X_1(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \in E_{\kappa,\xi}(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle})$.

(2) For each $\langle\bar{\nu}_1\rangle \in X_1(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1})$ we set

$$X_2(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1) = \{\langle\bar{\nu}_2\rangle \in T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1\rangle} \mid f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_2\rangle} = f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1,\bar{\nu}_2\rangle}\}.$$

Then $X_2(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1) \in \bigcap_{\xi<\xi_2<\mathrm{o}^{\mathrm{E}}(\kappa)} E_{\kappa,\xi_2}(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1\rangle})$. We set

$$X_2'(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) = \bigcup\{X_2(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1) \mid \langle\bar{\nu}_1\rangle \in X_1(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1})\}.$$

Then $X_2'(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \in \bigcap_{\xi<\xi_2<\mathrm{o}^{\mathrm{E}}(\kappa)} E_{\kappa,\xi_2}(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle})$. We set

$$X_2(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) = X_2'(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \cap T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle}.$$

Then $X_2(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \in \bigcap_{\xi<\xi_2<\mathrm{o}^{\mathrm{E}}(\kappa)} E_{\kappa,\xi_2}(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle})$.

(3) For each $\langle\bar{\nu}_0\rangle \in T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle}$ we set

$$X_1'(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_0) = \{\langle\bar{\nu}_1\rangle \in T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_0\rangle} \mid f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1\rangle} = f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_0,\bar{\nu}_1\rangle}\}.$$

Then $X_1'(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_0) \in E_{\kappa,\xi}(f^{p}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_0\rangle})$. Since

$$X_1(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \in E_{\kappa,\xi}(f^{p}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle})$$

then also

$$\{\langle\bar{\nu}_0\rangle \in T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle} \mid X_1(\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}) \in E_{\kappa,\xi}(f^{p}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_0\rangle})\} \in \bigcap_{\xi_0<\xi} E_{\kappa,\xi_0}(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle}).$$

Hence by setting

$$X_1(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}, \bar{\nu}_0) = X'_1(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}, \bar{\nu}_0) \cap X_1(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}),$$

and

$$X_0(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}) = \{\langle\bar{\nu}_0\rangle \in T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle} \mid X_1(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}, \bar{\nu}_0) \in E_{\kappa,\xi}(f^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_0\rangle})\},$$

we get $X_0(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}) \in \bigcap_{\xi_0<\xi} E_{\kappa,\xi_0}(f^{p}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle})$.

We complete the set construction by setting

$$X(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}) = X_0(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}) \cup X_1(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}) \cup X_2(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}).$$

p^{n+1} is constructed from p^n by shrinking T^{p^n} as follows:

$$T^{p^{n+1}} \cap [\mathrm{OB}(\mathrm{dom}\, f^{p^n})]^n = T^{p^n} \cap [\mathrm{OB}(\mathrm{dom}\, f^{p^n})]^n,$$
$$\mathrm{Suc}_{T^{p^{n+1}}}(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}) = X(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}),$$

and

$$\forall\langle\bar{\nu}\rangle \in X(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1})\ T^{p^{n+1}}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}\rangle} = T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}\rangle}.$$

Let us show that p^{n+1} is as required. Thus let $\langle\bar{\mu}_0, \ldots, \bar{\mu}_n\rangle \in T^{p^{n+1}}$. If

$$p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$$

then trivially $p^{n+1}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$, and we are done. Thus let us assume that

$$p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1}\rangle} \nVdash_{\mathbb{P}_{\bar{E}}} \sigma.$$

We split the handling according to the whereabouts of $\bar{\mu}_n$:

(1) $\langle\bar{\mu}_n\rangle \in X_1(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1})$: The definition of $X_1(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1})$ implies immediately $p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$; hence $p^{n+1}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$.

(2) $\langle\bar{\mu}_n\rangle \in X_2(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1})$: Then there is $\langle\bar{\nu}_1\rangle \in X_1(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1})$ such that $\langle\bar{\mu}_n\rangle \in X_2(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}, \bar{\nu}_1)$. Since $\langle\bar{\nu}_1\rangle \in X_1(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1})$, we have $p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$. Thus, since $p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1,\bar{\mu}_n\rangle} \leq_{\mathbb{P}_{\bar{E}}} p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1\rangle}$, we have $p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1,\bar{\mu}_n\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$. Since $\langle\bar{\mu}_n\rangle \in X_2(\bar{\mu}_0, \ldots, \bar{\mu}_{n-1}, \bar{\nu}_1)$, we have $f^{p}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1,\bar{\mu}_n\rangle} = f^{p}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle}$. Since

$$\langle f^{p}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle}, T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle} \cap T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1,\bar{\mu}_n\rangle}\rangle \leq^*_{\mathbb{P}_{\bar{E}}} p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1,\bar{\mu}_n\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$$

we have

$$\langle f^{p}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle}, T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle} \cap T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1,\bar{\mu}_n\rangle}\rangle \Vdash_{\mathbb{P}_{\bar{E}}} \sigma.$$

Since

$$p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle} \geq^*_{\mathbb{P}_{\bar{E}}} \langle f^{p}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle}, T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle} \cap T^{p^n}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\nu}_1,\bar{\mu}_n\rangle}\rangle \Vdash_{\mathbb{P}_{\bar{E}}} \sigma,$$

and p^0 was constructed using 4.2.38 we conclude that $p^n_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$ and thus $p^{n+1}_{\langle\bar{\mu}_0,\ldots,\bar{\mu}_{n-1},\bar{\mu}_n\rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \sigma$.

(3) $\langle\bar\mu_n\rangle \in X_0(\bar\mu_0,\dots,\bar\mu_{n-1})$: The crucial points are that

$$X_1(\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n) \in E_{\kappa,\xi}(f^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n\rangle}),$$

and if $\langle\bar\nu_1\rangle \in X_1(\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n)$ then

$$\langle\bar\nu_1\rangle \in X_1(\bar\mu_0,\dots,\bar\mu_{n-1}),$$
$$f^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\nu_1\rangle} = f^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n,\bar\nu_1\rangle}.$$

So let us assume $\langle\bar\nu_1\rangle \in X_1(\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n)$. Then

$$\langle f^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n,\bar\nu_1\rangle}, T^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n,\bar\nu_1\rangle} \cap T^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\nu_1\rangle}\rangle \leq^*_{\mathbb{P}_{\bar E}} p^n_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\nu_1\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma.$$

Hence

$$\langle f^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n,\bar\nu_1\rangle}, T^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n,\bar\nu_1\rangle} \cap T^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\nu_1\rangle}\rangle \Vdash_{\mathbb{P}_{\bar E}} \sigma$$

Since p^0 was constructed using 4.2.38 we get $p^n_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n,\bar\nu_1\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma$; thus $p^{n+1}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n,\bar\nu_1\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma$. Since this last relation is true for each $\langle\bar\nu_1\rangle \in X_1(\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n)$ we got

$$\{\langle\bar\nu\rangle \in T^{p^{n+1}}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\nu\rangle)} \mid p^{n+1}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n,\bar\nu\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma\} \in E_{\kappa,\xi}(f^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\mu_n\rangle}).$$

With $\langle p^n \mid n < \omega\rangle$ constructed we pick $p^* \in \mathbb{P}_{\bar E}$ such that $\forall n < \omega\ p^* \leq^*_{\mathbb{P}_{\bar E}} p^n$. Obviously for each $\langle\bar\mu_0,\dots,\bar\mu_{n-1}\rangle \in T^{p^*}$ either

$$p^*_{\langle\bar\mu_0,\dots,\bar\mu_{n-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma,$$

or

$$\{\langle\bar\nu\rangle \in T^{p^*}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1}\rangle} \mid p^*_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\nu\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma\} \in E_{\kappa,\xi}(f^{p^n}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1}\rangle}).$$

We claim $p^* \Vdash_{\mathbb{P}_{\bar E}} \sigma$. To show this let us take $q \leq_{\mathbb{P}_{\bar E}} p^*$ such that $q \parallel_{\mathbb{P}_{\bar E}} \sigma$. Then there is $\langle\bar\mu_0,\dots,\bar\mu_{n-1}\rangle \in T^{p^*}$ such that $q \leq^*_{\mathbb{P}_{\bar E}} p^*_{\langle\bar\mu_0,\dots,\bar\mu_{n-1}\rangle}$. Then either $p^*_{\langle\bar\mu_0,\dots,\bar\mu_{n-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma$ and then $q \Vdash_{\mathbb{P}_{\bar E}} \sigma$, or there is $\langle\bar\nu'\rangle \in T^q$ such that

$$\langle\bar\nu' \restriction \operatorname{supp} p^*\rangle \in T^{p^*}_{\langle\bar\mu_0,\dots,\bar\mu_{n-1}\rangle},$$
$$p^*_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\nu'\restriction \operatorname{supp} p^*\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma,$$

and then $q \parallel_{\mathbb{P}_{\bar E}} p^*_{\langle\bar\mu_0,\dots,\bar\mu_{n-1},\bar\nu\rangle}$; thus $q \Vdash_{\mathbb{P}_{\bar E}} \sigma$. □

Claim 4.2.41. *Assume $p \in \mathbb{P}_{\bar E}$, $S \subseteq T^p$ is an $E_\kappa(f^p)$-fat tree, and σ is a formula in the $\mathbb{P}_{\bar E}$-forcing language such that $\forall\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-1}\rangle \in S\ p_{\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma$. Then there is $p^* \leq^*_{\mathbb{P}_{\bar E}} p$ such that $p^* \Vdash_{\mathbb{P}_{\bar E}} \sigma$.*

Proof. We invoke 4.2.40 for each of $p_{\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-2}\rangle}$ where $\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-2}\rangle \in S$. Since the condition constructed by 4.2.40 is just a shrinking of $T^p_{\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-2}\rangle}$ we can construct $p_1 \leq^*_{\mathbb{P}_{\bar E}} p$ such that $\forall\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-2}\rangle \in S \cap T^{p_1}\ p_{1\,\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-2}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma$. Repeating this process for $\mathrm{ht}(S)-1$ steps we construct a $\leq^*_{\mathbb{P}_{\bar E}}$-decreasing sequence $\langle p_{n+1} \mid n < \mathrm{ht}(S)\rangle$ such that

$$\forall\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-n-1}\rangle \in S \cap T^{p_n}\ p_{n\,\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-n-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma.$$

That is $p_{\mathrm{ht}(S)} \Vdash_{\mathbb{P}_{\bar E}} \sigma$. □

Corollary 4.2.42. *Assume $p \in \mathbb{P}_{\bar E}$ and S is an $E_\kappa(f^p)$-fat tree. Then there is $p^* \leq^*_{\mathbb{P}_{\bar E}} p$ such that $\{p_{\langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-1}\rangle} \mid \langle\bar\nu_0,\dots,\bar\nu_{\mathrm{ht}(S)-1}\rangle \in S\}$ is pre-dense below p^*.*

Proof. We set $A = \{p_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S)-1}\rangle} \mid \langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S)-1}\rangle \in S\}$. Trivially

$$\forall\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S)-1}\rangle \in S\ p_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S)-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \ulcorner \check A \cap \underset{\sim}{H} \neq \emptyset \urcorner,$$

where $\underset{\sim}{H}$ is the canonical name for a $\mathbb{P}_{\bar E}$-generic object. Hence, by 4.2.41, there is $p^* \leq^*_{\mathbb{P}_{\bar E}} p$ such that $p^* \Vdash_{\mathbb{P}_{\bar E}} \ulcorner \check A \cap \underset{\sim}{H}\urcorner \neq \emptyset$. □

Claim 4.2.43. *$\langle\mathbb{P}_{\bar E}, \leq, \leq^*\rangle$ is of Prikry type.*

Proof. Let $p \in \mathbb{P}_{\bar E}$ and σ be a statement in the $\mathbb{P}_{\bar E}$-forcing language. We set $D = \{q \in \mathbb{P}_{\bar E} \mid q \parallel_{\mathbb{P}_{\bar E}} \sigma\}$. Since D is dense open in $\mathbb{P}_{\bar E}$, by 4.2.39, there are $p' \leq^*_{\mathbb{P}_{\bar E}} p$, and an $E_\kappa(f^{p'})$-fat tree, S', such that $\forall\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S')-1}\rangle \in S'$ $p'_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S')-1}\rangle} \in D$. We set

$$S_0 = \{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S')-1}\rangle \in S' \mid p'_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S')-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \neg\sigma\},$$

and

$$S_1 = \{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S')-1}\rangle \in S' \mid p'_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S)-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma\}.$$

By 4.2.37 there is $p'' \leq^*_{\mathbb{P}_{\bar E}} p'$ and $S'' \subseteq T^{p''}$, an $E_\kappa(f^{p''})$-fat tree, such that either $S''\restriction \operatorname{supp} p' \subseteq S_0$ or $S''\restriction \operatorname{supp} p' \subseteq S_1$. That is either

$$\{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S'')-1}\rangle \in T^{p''} \mid p''_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S'')-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \neg\sigma\} \text{ is an } E_\kappa(f^{p''})\text{-fat tree},$$

or

$$\{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S'')-1}\rangle \in T^{p''} \mid p''_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S'')-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma\} \text{ is an } E_\kappa(f^{p''})\text{-fat tree}.$$

Hence there is $p^* \leq^*_{\mathbb{P}_{\bar E}} p''$ such that either

$$\{p''_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S'')-1}\rangle} \mid p''_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S'')-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \neg\sigma\} \text{ is pre-dense below } p^*,$$

or

$$\{p''_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S'')-1}\rangle} \mid p''_{\langle\bar\nu_0,\ldots,\bar\nu_{\mathrm{ht}(S'')-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \sigma\} \text{ is pre-dense below } p^*.$$

Hence either

$$p^* \Vdash_{\mathbb{P}_{\bar E}} \neg\sigma,$$

or

$$p^* \Vdash_{\mathbb{P}_{\bar E}} \sigma. \quad \square$$

Claim 4.2.44. *$\langle\mathbb{P}_{\bar E}, \leq\rangle$ is κ-proper.*

Proof. Let χ be large enough so that $\mathcal{P}^2(\mathbb{P}_{\bar E}) \in H_\chi$. Let $N \prec H_\chi$ and $p \in \mathbb{P}_{\bar E} \cap N$ be such that $|N| = \kappa$, $N \supset N^{<\kappa}$, and $\mathbb{P}_{\bar E} \in N$. We will construct $p^* \leq^*_{\mathbb{P}_{\bar E}} p$ such that p^* is $\langle N, \mathbb{P}_{\bar E}\rangle$-generic.

Use 4.2.37 to construct $f^* \leq^*_{\mathbb{P}^*_{\bar E}} f$, an $\langle N, \mathbb{P}^*_{\bar E}\rangle$-generic condition. Let $T = \pi^{-1}_{f^*,f}(T^p)$. Let $\prec$ be a well-ordering of T of order type κ. (Thus $\forall\langle\bar\nu_0,\ldots,\bar\nu_{n-1}\rangle \in T \prec \restriction\langle\bar\nu_0,\ldots,\bar\nu_{n-1}\rangle \in N$.) Let $D : T \xrightarrow{\text{onto}} \{D \in N \mid D \subseteq \mathbb{P}_{\bar E}$ is dense open$\}$.

For each $\langle \bar\nu_0, \dots, \bar\nu_{n-1}\rangle \in T$ we set

$$D^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle} = \Bigg\{ f \leq^*_{\mathbb{P}^*_{\bar E}} f^p \mid \operatorname{dom} f \supseteq \bigcup_{i<n} \operatorname{dom} \bar\nu_i,$$
$$\exists q \leq^*_{\mathbb{P}_{\bar E}} p_{\langle \bar\nu_0 \restriction \operatorname{supp} p,\dots,\bar\nu_{n-1}\restriction \operatorname{supp} p\rangle}\ f^q = f_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle}$$
$$\forall \langle \bar\nu'_0,\dots,\bar\nu'_{k-1}\rangle \prec \langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle\ \exists q \leq^*_{\mathbb{P}_{\bar E}} q' \leq^*_{\mathbb{P}_{\bar E}} p_{\langle \bar\nu_0 \restriction \operatorname{supp} p,\dots,\bar\nu_{n-1}\restriction \operatorname{supp} p\rangle}$$
$$\exists S \subseteq T \text{ an } E_\kappa(f^{q'})\text{-fat tree } \{q'_{\langle \bar\mu_0,\dots,\bar\mu_{\operatorname{ht}(S)-1}\rangle} \in D(\bar\nu'_0,\dots,\bar\nu'_{k-1}) \mid$$
$$\langle \bar\mu_0,\dots,\bar\mu_{\operatorname{ht}(S)-1}\rangle \in S\} \text{ is pre-dense below } q \Bigg\}.$$

By repeated invocations of 4.2.39 and 4.2.34, we get that $D^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle}$ is a dense open subset of $\mathbb{P}^*_{\bar E}$ below f^p. Thus $\forall \langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle \in T\ f^* \in D^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle}$.

So for each $\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle \in T$ we pick $T(\bar\nu_0,\dots,\bar\nu_{n-1})$ such that

(1) $\langle f^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle}, T(\bar\nu_0,\dots,\bar\nu_{n-1})\rangle \leq^*_{\mathbb{P}_{\bar E}} p_{\langle \bar\nu_0 \restriction \operatorname{supp} p,\dots,\bar\nu_{n-1}\restriction \operatorname{supp} p\rangle}$.
(2) $\forall \langle \bar\nu'_0,\dots,\bar\nu'_{k-1}\rangle \prec \langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle$ there is $q' \in \mathbb{P}_{\bar E} \cap N$ such that

$$\langle f^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle}, T(\bar\nu_0,\dots,\bar\nu_{n-1})\rangle \leq^*_{\mathbb{P}_{\bar E}} q' \leq^*_{\mathbb{P}_{\bar E}} p_{\langle \bar\nu_0 \restriction \operatorname{supp} p,\dots,\bar\nu_{n-1}\restriction \operatorname{supp} p\rangle}$$
$$\exists S \subseteq T^{q'} \text{ an } E_\kappa(f^{q'})\text{-fat tree } \{q'_{\langle \bar\mu_0,\dots,\bar\mu_{\operatorname{ht}(S)-1}\rangle} \in D(\bar\nu'_0,\dots,\bar\nu'_{k-1}) \mid$$
$$\langle \bar\mu_0,\dots,\bar\mu_{\operatorname{ht}(S)-1}\rangle \in S\} \text{ is pre-dense below } \langle f^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle}, T(\bar\nu_0,\dots,\bar\nu_{n-1})\rangle\big).$$

We construct the tree T^* from T by shrinking so as to get

$$\forall \langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle \in T^*\ T^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle} \subseteq T(\bar\nu_0,\dots,\bar\nu_{n-1}).$$

Then we set $p^* = \langle f^*, T^*\rangle$. What we got is

(1) $p^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle} \leq^*_{\mathbb{P}_{\bar E}} p_{\langle \bar\nu_0 \restriction \operatorname{supp} p,\dots,\bar\nu_{n-1}\restriction \operatorname{supp} p\rangle}$.
(2) $\forall \langle \bar\nu'_0,\dots,\bar\nu'_{k-1}\rangle \prec \langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle$ there is $q' \in \mathbb{P}_{\bar E} \cap N$ such that

$$p^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle} \leq^*_{\mathbb{P}_{\bar E}} q' \leq^*_{\mathbb{P}_{\bar E}} p_{\langle \bar\nu_0 \restriction \operatorname{supp} p,\dots,\bar\nu_{n-1}\restriction \operatorname{supp} p\rangle}$$
$$\exists S \subseteq T^{q'} \text{ an } E_\kappa(f^{q'})\text{-fat tree } \{q'_{\langle \bar\mu_0,\dots,\bar\mu_{\operatorname{ht}(S)-1}\rangle} \in D(\bar\nu'_0,\dots,\bar\nu'_{k-1}) \mid$$
$$\langle \bar\mu_0,\dots,\bar\mu_{\operatorname{ht}(S)-1}\rangle \in S\} \text{ is pre-dense below } p^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle}.$$

Let us show that p^* is as required. So, let $D \in N$ be a dense open subset of $\mathbb{P}_{\bar E}$.

Let $\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle \in T^{p^*}$ be such that there is $\langle \bar\nu'_0,\dots,\bar\nu'_{k-1}\rangle \prec \langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle$ satisfying $D = D(\bar\nu'_0,\dots,\bar\nu'_{k-1})$. Then there are $q' \in \mathbb{P}_{\bar E} \cap N$ and $S \in N$ such that

$$p^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle} \leq^*_{\mathbb{P}_{\bar E}} q',$$
$$S \subseteq T^{q'} \text{ an } E_\kappa(f^{q'})\text{-fat tree},$$

and

$$A = \{q'_{\langle \bar\mu_0,\dots,\bar\mu_{\operatorname{ht}(S)-1}\rangle} \in D(\bar\nu'_0,\dots,\bar\nu'_{k-1}) \mid \langle \bar\mu_0,\dots,\bar\mu_{\operatorname{ht}(S)-1}\rangle \in S\}$$
$$\text{is pre-dense below } p^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle}.$$

Hence $p^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \ulcorner \check A \cap \underset{\sim}{H} \neq \emptyset \urcorner$. Since $A \subset N \cap D(\bar\nu'_0,\dots,\bar\nu'_{k-1})$ we really have $p^*_{\langle \bar\nu_0,\dots,\bar\nu_{n-1}\rangle} \Vdash_{\mathbb{P}_{\bar E}} \ulcorner \check D \cap \underset{\sim}{H} \cap \check N \neq \emptyset \urcorner$. □

Corollary 4.2.45. *$\mathbb{P}_{\bar E}$ does not collapse κ^+.*

Claim 4.2.46. *Assume* $\operatorname{cf} \mathrm{o}^{\mathrm{E}}(\kappa) = \lambda < \kappa$. *Then* $\Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \operatorname{cf} \kappa = \lambda \urcorner$.

Proof. Assume $\mu < \lambda$ and $p \Vdash_{\mathbb{P}_E} \ulcorner \dot{f} : \check{\mu} \to \check{\kappa} \urcorner$. For each $\xi < \mu$ set

$$D_\xi = \{q \leq_{\mathbb{P}_{\bar{E}}} p \mid \exists \zeta < \kappa \; q \Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \dot{f}(\check{\xi}) = \check{\zeta} \urcorner\}.$$

Note that D_ξ is dense open below p. Hence, using 4.2.39, 4.2.42, and the κ-closedness of $\langle \mathbb{P}_{\bar{E}}, \leq^* \rangle$ we construct a $\leq^*_{\mathbb{P}_{\bar{E}}}$-decreasing sequence $\langle p_\xi \mid \xi \leq \mu \rangle$ together with $\langle S^\xi, f_\xi \mid \xi < \mu \rangle$ so that for each $\xi < \mu$

$$S^\xi \text{ is an } E_\kappa(f^{p_\xi})\text{-fat tree},$$
$$f_\xi : S^\xi \to \kappa,$$
$$\forall \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1} \rangle \in S^\xi \;\; p_{\xi \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1} \rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \dot{f}(\check{\xi}) = f_\xi(\bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1}) \urcorner,$$

and

$$\{p_{\xi \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1} \rangle} \mid \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1} \rangle \in S^\xi\} \text{ is pre-dense below } p_{\xi+1}.$$

For each $\xi < \mu$ and $\langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle \in S^\xi$ let $\tau(\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1}) < \mathrm{o}^{\mathrm{E}}(\kappa)$ be a witness of the $E_\kappa(f^p)$-fatness of S^ξ. That is

$$\forall \xi < \mu \; \forall \langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle \in S^\xi$$
$$\operatorname{Suc}_{S^\xi}(\bar{\nu}_0, \ldots, \bar{\nu}_{n-1}) \in E_{\tau(\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1})}(f^{p_\xi}_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle}).$$

Pick an increasing sequence $\{\lambda_\zeta \mid \zeta < \lambda\}$ so that $\mathrm{o}^{\mathrm{E}}(\kappa) = \bigcup_{\zeta < \lambda} \lambda_\zeta$. Then for each $\xi < \mu$ and $\langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle \in S^\xi$, there is $\zeta(\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1})$ such that $\tau(\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1}) < \lambda_{\zeta(\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1})}$. Since $\zeta_{\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1}} < \lambda < \kappa$ we can shrink S^ξ so that

$$\forall \langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle, \langle \bar{\mu}_0, \ldots, \bar{\mu}_{n-1} \rangle \in S^\xi \;\; \zeta_{\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1}} = \zeta_{\xi, \bar{\mu}_0, \ldots, \bar{\mu}_{n-1}}.$$

Then we set $\zeta^* = \sup\{\zeta_{\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1}} \mid \xi < \mu, \langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle \in S^\xi\}$, and get $\zeta^* < \lambda$. Hence there is τ^* such that $\lambda_{\zeta^*} < \tau^* < \mathrm{o}^{\mathrm{E}}(\kappa)$ and for each $\xi < \mu$ and $\langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle \in S^\xi$, we have $\tau(\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1}) < \tau^*$. Let

$$A = \{\langle \bar{\nu} \rangle \in T^{p_\mu} \mid \forall \xi < \mu \; \forall \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S)-1} \rangle \in S^\xi$$
$$p_{\mu \langle \bar{\nu} \rangle} \parallel_{\mathbb{P}_{\bar{E}}} p_{\xi \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1} \rangle} \implies \langle \bar{\nu} \restriction \operatorname{supp} p_\xi \rangle \in T^{p_\xi}_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1} \rangle}\}.$$

We note that $A \in E_{\tau^*}(f^{p_\mu})$. We set for each $\langle \bar{\nu} \rangle \in A$,

$$f^*(\bar{\nu}) = \sup\{ f_\xi(\bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1}) \mid$$
$$\xi < \mu, \; \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1} \rangle \in S^\xi, \; \langle \bar{\nu} \restriction \operatorname{supp} p_\xi \rangle \in T^{p_\xi}_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{\operatorname{ht}(S^\xi)-1} \rangle}\}.$$

The supremum is taken over less than κ elements hence for each $\langle \bar{\nu} \rangle \in A$ $f^*(\bar{\nu}) < \kappa$. Thus we get $p_{\mu \langle \bar{\nu} \rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \forall \xi < \mu \; \dot{f}(\xi) < f^*(\bar{\nu}) < \kappa \urcorner$.

We construct p^* from p_μ by shrinking T^{p_μ} so as to have $\{p_{\mu \langle \bar{\nu} \rangle} \mid \langle \bar{\nu} \rangle \in A\}$ is pre-dense below p^*. Since $p^* \Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \exists \langle \bar{\nu} \rangle \in A \; p_{\mu \langle \bar{\nu} \rangle} \in \underset{\sim}{H} \urcorner$, we are done. $\square$

Claim 4.2.47. *Assume* $\operatorname{cf} \mathrm{o}^{\mathrm{E}}(\kappa) > \kappa$. *Then* $\Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \operatorname{cf} \kappa = \kappa \urcorner$.

Proof. Assume $\lambda < \kappa$ and $p \Vdash_{\mathbb{P}_E} \ulcorner f : \check{\lambda} \to \check{\kappa} \urcorner$. For each $\xi < \lambda$ set

$$D_\xi = \{q \leq_{\mathbb{P}_{\bar{E}}} p \mid \exists \zeta < \kappa \; q \Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \dot{f}(\check{\xi}) = \check{\zeta} \urcorner\}.$$

Note that D_ξ is dense open below p. Hence, using 4.2.39, 4.2.42, and the κ-closedness of $\langle \mathbb{P}_{\bar{E}}, \leq^* \rangle$ we construct a $\leq^*_{\mathbb{P}_{\bar{E}}}$-decreasing sequence $\langle p_\xi \mid \xi \leq \lambda \rangle$ together with $\langle S^\xi, f_\xi \mid \xi < \lambda \rangle$ so that for each $\xi < \lambda$

$$S^\xi \text{ is an } E_\kappa(f^{p_\xi})\text{-fat tree,}$$

$$f_\xi : S^\xi \to \kappa,$$

$$\forall \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1} \rangle \in S^\xi \ p_{\xi\, \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1} \rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \dot{f}(\check{\xi}) = f_\xi(\bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1}) \urcorner,$$

and

$$\{ p_{\xi\, \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1} \rangle} \mid \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1} \rangle \in S^\xi \} \text{ is pre-dense below } p_{\xi+1}.$$

For each $\xi < \lambda$ and $\langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle \in S^\xi$ we let $\tau(\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1}) < \mathrm{o}^{\mathrm{E}}(\kappa)$ be the ordinals witnessing the $E_\kappa(f^{p_\xi})$-fatness of S^ξ. That is

$$\forall \xi < \lambda \ \forall \langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle \in S^\xi$$
$$\mathrm{Suc}_{S^\xi}(\bar{\nu}_0, \ldots, \bar{\nu}_{n-1}) \in E_{\tau(\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1})}(f^{p_\xi}_{\langle\rangle\, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1}}).$$

Since $\mathrm{cf}\, \mathrm{o}^{\mathrm{E}}(\kappa) > \kappa$, there is $\tau^* < \mathrm{o}^{\mathrm{E}}(\kappa)$ such that for each $\xi < \lambda$ and $\langle \bar{\nu}_0, \ldots, \bar{\nu}_{n-1} \rangle \in S^\xi$ we have $\tau(\xi, \bar{\nu}_0, \ldots, \bar{\nu}_{n-1}) < \tau^*$. Let

$$A = \{ \langle \bar{\nu} \rangle \in T^{p_\lambda} \mid \forall \xi < \lambda \ \forall \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1} \rangle \in S^\xi$$
$$p_{\lambda\, \langle \bar{\nu} \rangle} \parallel_{\mathbb{P}_{\bar{E}}} p_{\xi \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1} \rangle} \implies \langle \bar{\nu} \restriction \mathrm{supp}\, p_\xi \rangle \in T^{p_\xi}_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1} \rangle} \}.$$

We note that $A \in E_{\tau^*}(f^{p_\lambda})$. For each $\langle \bar{\nu} \rangle \in A$ we set

$$f(\bar{\nu}) = \sup \{ f_\xi(\bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1}) \mid$$
$$\xi < \lambda, \ \langle \bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1} \rangle \in S^\xi, \ \langle \bar{\nu} \rangle \in T^{p^*}_{\langle \bar{\nu}_0, \ldots, \bar{\nu}_{\mathrm{ht}(S^\xi)-1} \rangle} \}.$$

Since the supremum is taken over less than κ ordinals we have that for each $\langle \bar{\nu} \rangle \in A$, $f(\bar{\nu}) < \kappa$. In particular

$$p_{\lambda \langle \bar{\nu} \rangle} \Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \forall \xi < \lambda \ \dot{f}(\xi) < f(\bar{\nu}) < \kappa \urcorner.$$

We construct p^* from p_λ by shrinking T^{p_λ} so as to get that $\{ p_{\lambda \langle \bar{\nu} \rangle} \mid \langle \bar{\nu} \rangle \in A \}$ is pre-dense below p^*. Thus $p^* \Vdash_{\mathbb{P}_{\bar{E}}} \ulcorner \exists \langle \bar{\nu} \rangle \in A \ p_{\lambda \langle \bar{\nu} \rangle} \in \underset{\sim}{H} \urcorner$, and we are done. □

All in all we got

Corollary 4.2.48. *Let H_κ be $\mathbb{P}_{\bar{E}}$-generic over $V[G_\kappa]$. Then in $V[G_\kappa][H_\kappa]$:*

(1) *All $V[G_\kappa]$ cardinals remain cardinals.*

(2) $\mathrm{cf}\, \kappa = \begin{cases} \mathrm{cf}\, \mathrm{o}^{\mathrm{E}}(\kappa) & \mathrm{cf}\, \mathrm{o}^{\mathrm{E}}(\kappa) < \kappa. \\ \omega & \mathrm{o}^{\mathrm{E}}(\kappa) \text{ is successor, or } \mathrm{cf}_{V[G_\kappa]}\, \mathrm{o}^{\mathrm{E}}(\kappa) = \kappa. \\ \kappa & \mathrm{cf}\, \mathrm{o}^{\mathrm{E}}(\kappa) > \kappa. \end{cases}$

(3) *$V[G_\kappa]$ and $V[G_\kappa][H_\kappa]$ have the same bounded subsets of κ.*

(4) $2^\kappa = |\bigcup_{\xi < \mathrm{o}^{\mathrm{E}}(\kappa)} j_{\kappa,\xi}(\kappa)|$.

This step of the induction terminates by setting $P_{\kappa+1} = P_\kappa * \dot{Q}_\kappa$. □

5. Applications

In the following examples we use the iteration P_κ of the previous section with different coherent sequences E.

Theorem 5.1. *Let $\xi < \kappa$ be regular cardinals in K (the core model) and $\xi \notin \omega - \{0\}$. Suppose that the set $\{ \lambda < \kappa \mid \mathrm{o}(\lambda) = \lambda^{++} + \xi \}$ is stationary. Then there is a cardinal preserving generic extension of K in which the sets*

$$\{ \lambda < \kappa \mid 2^\lambda = \lambda^+ \text{ and } (\mathrm{cf}\, \lambda = \xi \text{ if } \xi \neq 0 \text{ or } \mathrm{cf}\, \lambda = \omega \text{ if } \xi = 0) \}$$

and

$$\{\lambda < \kappa \mid 2^\lambda = \lambda^{++} \text{ and } (\operatorname{cf}\lambda = \xi \text{ if } \xi \neq 0 \text{ or } \operatorname{cf}\lambda = \omega \text{ if } \xi = 0)\}$$

are stationary.

Proof. For $\xi = 0$ we iterate the forcing of [8] thus getting that $\{\lambda < \kappa \mid o(\lambda) = \lambda^{++} + 1\}$ is stationary. Constructing a coherent sequence E such that $\{\lambda < \kappa \mid o^E(\lambda) = \lambda^{++} + 1\}$ is immediate. Now force with P_κ of the previous section using this E. Since κ is Mahlo, P_κ preserves stationary subsets; hence

$$\{\lambda < \kappa \mid 2^\lambda = \lambda^{++},\ \operatorname{cf}\lambda = \omega\},$$

and

$$\{\lambda < \kappa \mid 2^\lambda = \lambda^{+},\ \operatorname{cf}\lambda = \omega\},$$

are stationary.

For $\xi > \omega$ we construct directly a coherent sequence E satisfying $\{\lambda < \kappa \mid o^E(\lambda) = \lambda^{++} + \xi\}$ is stationary, and then we proceed as above. □

A similar result is possible if κ is replaced by On:

Theorem 5.2. *Let ξ be a regular cardinal in K and $\xi \notin \omega - \{0\}$. Suppose that $\{\lambda \mid o(\lambda) = \lambda^{++} + \xi\}$ is a stationary class. Then there is a cardinal preserving class generic extension of K in which the classes*

$$\{\lambda \mid 2^\lambda = \lambda^{+} \text{ and } (\operatorname{cf}\lambda = \xi \text{ if } \xi \neq 0 \text{ or } \operatorname{cf}\lambda = \omega \text{ if } \xi = 0)\}$$

and

$$\{\lambda \mid 2^\lambda = \lambda^{++} \text{ and } (\operatorname{cf}\lambda = \xi \text{ if } \xi \neq 0 \text{ or } \operatorname{cf}\lambda = \omega \text{ if } \xi = 0)\}$$

are stationary.

Proof. Use class forcing and On instead of P_κ and κ in the previous proof; see [12] or [4]. □

By the results of [10], the above theorems are optimal for each $\xi \neq \omega_1$.

Theorem 5.3. *Let κ be a regular cardinal in K. Suppose that $\{\lambda < \kappa \mid o(\lambda) = \lambda^{+3} + 1\}$ is stationary. Then there is a cardinal preserving generic extension of K in which the sets*

$$\{\lambda < \kappa \mid \operatorname{cf}\lambda = \omega,\ 2^\lambda = \lambda^{+}\},$$

$$\{\lambda < \kappa \mid \operatorname{cf}\lambda = \omega,\ 2^\lambda = \lambda^{++}\},$$

and

$$\{\lambda < \kappa \mid \operatorname{cf}\lambda = \omega,\ 2^\lambda = \lambda^{+3}\}$$

are stationary.

Proof. Let $S = \{\lambda < \kappa \mid o(\lambda) = \lambda^{+3} + 1\}$. It is easy to define E such that

$$\{\lambda < \kappa \mid o^E(\lambda) = \lambda^{+2} + 1\}$$

and

$$\{\lambda < \kappa \mid o^E(\lambda) = \lambda^{+3} + 1\}$$

are stationaries: Just split S into disjoint stationaries S_2 and S_3. Then for $\lambda \in S_2$ restrict the extenders to size λ^{++}.

Now force with P_κ for this E. In the generic extension we have $\forall\lambda \in S$, $\operatorname{cf}\lambda = \omega$ and

$$2^\lambda = \begin{cases} \lambda^{++} & \lambda \in S_2, \\ \lambda^{+3} & \lambda \in S_3. \end{cases}$$

Since κ is Mahlo in V, stationary subsets of κ are preserved; thus in the extension

$$\{\lambda < \kappa \mid \operatorname{cf}\lambda = \omega,\ 2^\lambda = \lambda^{++}\}$$

and

$$\{\lambda < \kappa \mid \operatorname{cf}\lambda = \omega,\ 2^\lambda = \lambda^{+3}\}$$

are stationaries.

Since $\forall\lambda \in \kappa \setminus S, 2^\lambda = \lambda^+$ in the generic extension, and in V the set $\{\lambda < \kappa \mid \operatorname{cf}\lambda = \omega\} = \{\lambda < \kappa \setminus S \mid \operatorname{cf}\lambda = \omega\}$ is stationary, in the generic extension we get that

$$\{\lambda < \kappa \mid \operatorname{cf}\lambda = \omega,\ 2^\lambda = \lambda^{+}\}$$

is stationary. □

Theorem 5.4. *Suppose that* $\{\lambda \mid o(\lambda) = \lambda^{+3} + 1\}$ *is a stationary class. Then there is a cardinal preserving class generic extension of* K *in which the classes*

$$\{\lambda \mid \operatorname{cf}\lambda = \omega,\ 2^\lambda = \lambda^{+}\},$$
$$\{\lambda \mid \operatorname{cf}\lambda = \omega,\ 2^\lambda = \lambda^{++}\},$$

and

$$\{\lambda \mid \operatorname{cf}\lambda = \omega,\ 2^\lambda = \lambda^{+3}\}$$

are stationary.

Theorem 5.5. *Let* κ *be a regular cardinal in* K. *Suppose that for each* $\xi < \kappa$ *the set* $\{\xi < \lambda < \kappa \mid o(\lambda) = \lambda^{+3} + \xi\}$ *is stationary. Then there is a cardinal preserving generic extension of* K *in which* $\{\lambda < \kappa \mid 2^\lambda = \lambda^+$ *or* λ *is regular*$\}$ *is nonstationary and both sets* $\{\lambda < \kappa \mid 2^\lambda = \lambda^{++}\}$ *and* $\{\lambda < \kappa \mid 2^\lambda = \lambda^{+3}\}$ *are stationary.*

Proof. We construct a coherent sequence E such that for each $\xi < \kappa$ the sets $\{\xi < \lambda < \kappa \mid o^E(\lambda) = \lambda^{++} + \xi\}$ and $\{\xi < \lambda < \kappa \mid o^E(\lambda) = \lambda^{+3} + \xi\}$ are stationary. Then we force with P_κ of the previous section using this E. In the generic extension we get that for each regular $\xi < \kappa$,

$$\{\xi < \lambda < \kappa \mid 2^\lambda = \lambda^{++},\ \operatorname{cf}\lambda = \xi\}$$

and

$$\{\xi < \lambda < \kappa \mid 2^\lambda = \lambda^{+3},\ \operatorname{cf}\lambda = \xi\}$$

are stationary. In this model, as in 5.3, we have that $\{\lambda < \kappa \mid 2^\lambda = \lambda^+\}$ is stationary.

We note that the set $\{\lambda < \kappa \mid 2^\kappa \in \{\kappa^{++}, \kappa^{+3}\}\}$ is fat in the following sense:

Definition 5.5.1. A stationary set $S \subseteq \kappa$ is called fat if for each $\xi < \kappa$ and each club $C \subseteq \kappa$ there is a closed subset of order type ξ in $S \cap C$.

By [1], we can shoot a club through a fat stationary without adding bounded subsets. Thus after shooting the club the power function below κ does not change and in addition we have $\{\lambda < \kappa \mid 2^\lambda = \lambda^+ \text{ or } \lambda \text{ is regular}\}$ is nonstationary. □

Theorem 5.6. *Suppose that for each* $\xi \in On$, $\{\xi < \lambda < \kappa \mid o(\lambda) = \lambda^{+3} + \xi\}$ *is a stationary class. Then there is a cardinal preserving class generic extension of* K *in which* $\{\lambda \mid 2^\lambda = \lambda^+$ *or* λ *is regular*$\}$ *is a nonstationary class and both sets* $\{\lambda \mid 2^\lambda = \lambda^{++}\}$ *and* $\{\lambda \mid 2^\lambda = \lambda^{+3}\}$ *are stationary classes.*

With the forcing notion of this paper we were not able to eliminate the GCH behavior altogether.

Acknowledgement

The writing of this paper was partially supported by the Israel Science Foundation.

References

[1] Uri Avraham, Saharon Shelah, Forcing closed unbounded sets, Journal of Symbolic Logic 48 (1983) 643–648.
[2] James Cummings, A model in which GCH holds at successors but fails at limits, Transactions of the American Mathematical Society 329 (1) (1992) 1–39.
[3] Matthew Foreman, Hugh Woodin, GCH can fail everywhere, Annals of Mathematics 133 (2) (1991) 1–35.
[4] Sy D. Friedman, Fine Structure and Class Forcing, in: de Gruyter Series in Logic and Its Applications, vol. 3, de Gruyter.
[5] Moti Gitik, Prikry type forcings, in: Matt Foreman, Aki Kanamori, Menachem Magidor (Eds.), Handbook of Set Theory (in press).
[6] Moti Gitik, Two stationary sets with different gaps of the power function.
[7] Moti Gitik, Changing cofinalities and the non-stationary ideal, Israel Journal of Mathematics 56 (1986) 280–314.
[8] Moti Gitik, The negation of the singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$, Annals of Pure and Applied Logic 43 (1989) 209–234.
[9] Moti Gitik, Menachem Magidor, The singular cardinal hypothesis revisited, in: Haim Judah, Winfried Just, Hugh Woodin (Eds.), Set Theory of the Continuum, Springer, 1992, pp. 243–278.
[10] Moti Gitik, William J. Mitchell, Indiscernible sequences for extenders, and the singular cardinal hypothesis, Annals of Pure and Applied Logic 82 (3) (1996) 273–316.
[11] Moti Gitik, Ralf Schindler, Saharon Shelah, Pcf theory and Woodin cardinals, arXiv:math.LO/0001121.
[12] Thomas Jech, Set Theory, 3rd edition, in: Springer Monographs in Mathematics, Springer, 2003.
[13] Carmi Merimovich, Extender based Radin forcing, Transactions of the American Mathematical Society 355 (2003) 1729–1772. arXiv:math.LO/0001121.
[14] Carmi Merimovich, A power function with a fixed finite gap everywhere, Journal of Symbolic Logic (in press). arXiv:math.LO/0005179.
[15] Carmi Merimovich, Prikry on Extenders, Revisited (submitted for publication). arXiv:math.LO/0001121.
[16] William J. Mitchell, Sets constructible from sequences of ultrafilters, Journal of Symbolic Logic 39 (1974) 57–66.
[17] William J. Mitchell, John R. Steel, Fine Structure and Iteration Trees, in: Lecture Notes in Logic, vol. 3, Springer, Berlin, 1994.