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Lifetime prediction based on Gamma processes from accelerated degradation data

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Abstract Accelerated degradation test is a useful approach to predict the product lifetime at the normal use stress level, especially for highly reliable products. Two kinds of the lifetime prediction based on Gamma processes were studied. One was to predict the lifetime of the population from accelerated degradation data, and the other was to predict the lifetime of an individual by taking the accelerated degradation data as prior information. For an extensive application, the Gamma process with a time transformation and random effects was considered. A novel contribution is that a deducing method for determining the relationships between the shape and scale parameters of Gamma processes and accelerated stresses was presented. When predicting the lifetime of an individual, Bayesian inference methods were adopted to improve the prediction accuracy, in which the conjugate prior distribution and the non-conjugate prior distribution of random parameters were studied. The conjugate prior distribution only considers the random effect of the scale parameter while the non-conjugate prior distribution considers the random effects of both the scale and shape parameter. The application and usefulness of the proposed method was demonstrated by the accelerated degradation data of carbon-film resistors.

1. Introduction

With the development of engineering and science technology, more and more products with long lifetime and high reliability have emerged. Thus, accelerated degradation test (ADT), which can provide degradation data in a more timely fashion, has become an appealing approach in reliability assessment and lifetime prediction. An effective ADT must guarantee that the failure mechanism of products under different accelerated stresses remains consistent, so the lifetime characteristics at the normal use stress level can be extrapolated from accelerated degradation data. The applications of ADT can be referred by light bars,[1] integrated logic family,[2] light-emitting diodes[3,4] and carbon-film resistors,[5] etc.

However, the accuracy of lifetime prediction greatly depends on modeling accelerated degradation data (ADD) well or not. With the feature of reflecting the random influences of the internal and external environments, the

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stochastic process models are naturally applicable to modeling degradation data. Among them, the Wiener process has been widely studied by Whitmore and Schenkelberg,[6] Padgett and Tomlinson,[7] Wang,[8] Si et al.[9–11] and Peng and Tseng.[12] As far as the Gamma process, it also attracts great enthusiasm in the field of modeling degradation data after it was firstly used as a degradation model by Abdel-Hamee.[13] Park and Padgett[14] proposed an approach of approximating a Gamma distribution by an inverse Gaussian distribution. Lawless and Crowder[15] and van Noortwijk[16] studied a Gamma process with covariates and random effect. van Noortwijk[16] provided a comprehensive introduction for Gamma processes and surveyed their applications in maintenance. Pan and Balakrishnan[17] used Gamma processes to model the degradation data with multiple performance characteristics. Tsai et al.[18] analyzed and evaluated the effects of misspecification of Gamma and Wiener processes. Besides, there are lots of researches on the optimal design of ADT based on Gamma processes, see Liao and Tseng,[19] Tsai et al.[20]

When modeling ADD, an inevitable work is to specify which parameter of a degradation model should change with accelerated stresses varying and which parameter should remain unchanged. However, it is intractable to determine the relationships between parameters and accelerated stresses since there is no feasible deducting method. Instead, at present the relationships are specified by some assumptions which are made from testing data or engineering experience. For the Gamma degradation model, there are two different assumptions. The most widely applied assumption is that the shape parameter should change with accelerated stresses varying while the scale parameter should remain unchanged, see Park and Padgett,[14,21] Tseng et al.[22] The contrary assumption deemed that the shape parameter should remain unchanged with accelerated stresses varying while the scale parameter should change, see Lawless and Crowder,[15] Wang.[23] In the paper, according to acceleration factor constant hypothesis proposed by Zhou et al.,[24] the relationships between the parameters of the Gamma process and accelerated stresses were mathematically deduced.

Two kinds of the lifetime prediction based on Gamma processes were studied. One is to predict the lifetime of the population from accelerated degradation data, by which the reliability and quality of product can be assessed. The other is to predict the lifetime of an individual by taking the accelerated degradation data as prior information, which can provide decision support for prognostics and health management (PHM). It is well known that the conjugate prior distribution of a Gamma distribution only considers the random effect of the shape parameter. Thus, we also studied a non-conjugate prior distribution which allows for the random effects of both the shape parameter and the scale parameter.

The remainder of this paper is organized as follows. In Section 2, we studied the lifetime prediction of the population. Firstly, the lifetime prediction model based on a Gamma process was introduced, and then the relationships between the parameters of a Gamma process and accelerated stresses were deduced, lastly the method of estimating parameters was also illustrated in the section. In Section 3, we discussed the lifetime prediction of an individual based on Bayesian inference with a conjugate prior distribution and a non-conjugate prior distribution. In Section 4 we provided a case study to illustrate the application and usefulness of the proposed methods. Some conclusions were drawn in Section 5.

2. Lifetime prediction of the population

2.1. Lifetime prediction model

Gamma process is a stochastic process which is applicable to modeling the always positive and strictly increasing degradation data. In mathematics, Gamma process $\{Y(t); Y(0) = 0\}$ has independent, non-negative increments $\Delta Y(t) = Y(t+\Delta t) - Y(t)$ that follow a Gamma distribution as

$$\Delta Y(t) \sim \mathrm{Ga}(\alpha(\Lambda(t+\Delta t) - \Lambda(t)), \beta) \tag{1}$$

where $\beta(\beta > 0)$ is a scale parameter, $\alpha(\alpha > 0)$ is a shape parameter and $\Lambda(t)$ is a monotone increasing function of time t with $\Lambda(0) = 0$. According to the additivity of a Gamma distribution, it can be deduced that $Y(t)$ should follow the Gamma distribution $\mathrm{Ga}(\alpha\Lambda(t), \beta)$. The probability density function (PDF) of $Y(t)$ is expressed by

$$f(Y) = \frac{\beta^{\alpha\Lambda(t)}}{\Gamma(\alpha\Lambda(t))} Y^{\alpha\Lambda(t)-1} \exp(-Y\beta) \tag{2}$$

From Eq. (2), the mean of $Y(t)$ is obtained as $\alpha\Lambda(t)/\beta$ and the variance is obtained as $\alpha\Lambda(t)/\beta^2$. Suppose that $y(t)$ is a degradation process and the lifetime ξ is defined as the first passage time of $y(t)$ reaches the failure threshold l. Let y_0 denote the initial value of $y(t)$, then $y(t) - y_0$ follows the distribution $\mathrm{Ga}(\alpha\Lambda(t), \beta)$. Thus, the reliability function can be given by

$$\begin{aligned} P(\xi > t) &= P(y(t) < l) = P(y(t) - y_0 < l - y_0) \\ &= \int_0^{l-y_0} \frac{\beta^{\alpha\Lambda(t)}}{\Gamma(\alpha\Lambda(t))} y^{\alpha\Lambda(t)-1} \exp(-y\beta)\mathrm{d}y \\ &= \frac{1}{\Gamma(\alpha\Lambda(t))} \int_0^{l_\beta} x^{\alpha\Lambda(t)-1} \exp(-x)\mathrm{d}x \end{aligned} \tag{3}$$

where $l_\beta = (l - y_0)\beta$, $x = y\beta$. Substitute the incomplete Gamma function $\Gamma(a, z) = \int_z^\infty x^{a-1} \exp(-x)\mathrm{d}x$ into Eq. (3), then the cumulative distribution function (CDF) of ξ be obtained as

$$F(t) = \frac{\Gamma(\alpha\Lambda(t), l_\beta)}{\Gamma(\alpha\Lambda(t))} \tag{4}$$

From Eq. (4), the exact PDF of ξ for a Gamma degradation model can be extrapolated, but the exact PDF is too complex to practical applications. For mathematical convenience, Park and Padgett[14] provided an approach that a form of Birnbaum–Saunders (BS) distribution was used to approximate the CDF of ξ. With a time transformation $z = \Lambda(t)$, the CDF can be expressed as

$$F_{\mathrm{BS}}(z) \approx \Phi\left[\frac{1}{a}\left(\sqrt{\frac{z}{b}} - \sqrt{\frac{b}{z}}\right)\right] \tag{5}$$

where $a = 1/\sqrt{l_\beta}$ and $b = l_\beta/\alpha$. The PDF can be expressed as

$$f_{\mathrm{BS}}(z) = \frac{1}{2\sqrt{2}ab}\left[\left(\frac{b}{z}\right)^{1/2} + \left(\frac{b}{z}\right)^{3/2}\right] \exp\left[-\frac{(b-z)^2}{2a^2bz}\right] \tag{6}$$

As described by van Noortwijk,[16] empirical studies show that the expected deterioration of $y(t)$ is often proportional to a power law function of time as $E(y(t)) = \alpha\Lambda(t)/\beta =$

$\rho\Lambda(t) \propto t^c$, where $\rho = \alpha/\beta$ and c is a positive parameter. Thus, the time function can be specified as $\Lambda(t) = t^c$. From Eq. (6), the mean lifetime $\hat{\xi}_{BS}$ can be obtained as

$$\hat{\xi}_{BS} = \int_0^{+\infty} z f_{BS}(z) dz = \left[\frac{(l-y_0)\beta}{\alpha} + \frac{1}{2\alpha}\right]^{\frac{1}{c}} \tag{7}$$

2.2. Deducing the relationships between parameters and accelerated stresses

According to Pieruschka assumption, the degradation processes of products under different accelerated stresses should follow the same degradation model but the model parameters may change with accelerated stresses varying. However, Pieruschka assumption did not point out how the model parameters would change. In the paper, the acceleration factor constant hypothesis was used to deduce the changing rule for the parameters of the Gamma process.

There are several definitions of acceleration factor, which are equivalent in essence. One equivalent definition derives from the Nelson assumption.[25] Suppose that $F_k(t_k), F_h(t_h)$ represents the CDF of product under any two different stresses S_k, S_h. If $F_k(t_k) = F_h(t_h)$, then the acceleration factor $A_{k,h}$ of stress S_k relative to stress S_h can be defined as

$$A_{k,h} = t_h/t_k \tag{8}$$

Zhou et al.[24] pointed out that $A_{k,h}$ should be a constant which does not change with time t_k, t_h and depends only on S_k, S_h, otherwise $A_{k,h}$ cannot be applied to engineering application. Moreover, to guarantee that $A_{k,h}$ is a constant, the failure mechanism of products must remain consistent at S_k, S_h. It is well-known that the failure mechanism of products under an effective accelerated test should remain consistent. According to the above analysis, for any $t_k, t_h > 0$, the following equation should be always identical:

$$F_k(t_k) = F_h(A_{k,h}t_h) \tag{9}$$

Substitute Eq. (4) into Eq. (9), then

$$\frac{\Gamma(\alpha_k t_k^{c_k}, (l-y_0)\beta_k)}{\Gamma(\alpha_k t_k^{c_k})} = \frac{\Gamma(\alpha_h (A_{k,h}t_k)^{c_h}, (l-y_0)\beta_h)}{\Gamma(\alpha_h (A_{k,h}t_k)^{c_h})} \tag{10}$$

To ensure that Eq. (10) is always identical, the following relationship must be satisfied:

$$\begin{cases} (l-y_0)\beta_k = (l-y_0)\beta_h \\ \alpha_k t_k^{c_k} = \alpha_h (A_{k,h}t_k)^{c_h} \end{cases} \tag{11}$$

Hence, it can be deduced that

$$\begin{cases} \beta_k = \beta_h \\ c_k = c_h \\ A_{k,h} = (\alpha_k/\alpha_h)^{1/c_h} \end{cases} \tag{12}$$

From Eq. (12), it can be concluded that α should change with stresses varying while β and c should remain unchanged. Suppose that the accelerated stress is temperature T and Arrhenius model is correspondingly adopted as the reaction rate model, then the α, β, c under accelerated stresses can be expressed as

$$\begin{cases} \alpha(T) = \exp(\gamma_1 - \gamma_2/T) \\ \beta(T) = \gamma_3 \\ c(T) = \gamma_4 \end{cases} \tag{13}$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are unknown coefficients.

2.3. Parameter estimation

Suppose that a constant stress ADT was carried out. Let T_0 denote the normal use temperature level, T_k denote the kth accelerated temperature level, y_{ijk} denote the ith observed degradation data of the jth sample at T_k, t_{ijk} denote the observing time, $\Delta y_{ijk} = y_{ijk} - y_{(i-1)jk}$ denote the degradation increment, where $i = 1, 2, \cdots, n_1$; $j = 1, 2, \cdots, n_2$; $k = 1, 2, \cdots, n_3$. According to Eq. (1), the maximum likelihood function for each sample can be set up as

$$L(\alpha_{jk}, c_{jk}, \beta_{jk}) = \prod_{i=1}^{n_1} \frac{\beta_{jk}^{\alpha_{jk}\left(t_{ijk}^{c_{jk}} - t_{(i-1)jk}^{c_{jk}}\right)}}{\Gamma\left(\alpha_{jk}\left(t_{ijk}^{c_{jk}} - t_{(i-1)jk}^{c_{jk}}\right)\right)} \cdot \exp\left(-\Delta y_{ijk}\beta_{jk}\right) \Delta y_{ijk}^{\alpha_{jk}\left(t_{ijk}^{c_{jk}} - t_{(i-1)jk}^{c_{jk}}\right)-1} \tag{14}$$

Through Eq. (14), the maximum likelihood estimates $(\hat{\alpha}_{jk}, \hat{c}_{jk}, \hat{\beta}_{jk})$ of all samples can be obtained, then the random effects of α, β can be evaluated. To predict the product lifetime at T_0, the maximum likelihood function synthesizing all the degradation data is expressed as

$$L(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3} \frac{\gamma_3^{\exp(\gamma_1 - \gamma_2/T_k)\left(t_{ijk}^{\gamma_4} - t_{(i-1)jk}^{\gamma_4}\right)}}{\Gamma\left(\exp(\gamma_1 - \gamma_2/T_k)\left(t_{ijk}^{\gamma_4} - t_{(i-1)jk}^{\gamma_4}\right)\right)} \cdot \Delta y_{ijk}^{\exp(\gamma_1 - \gamma_2/T_k)\left(t_{ijk}^{\gamma_4} - t_{(i-1)jk}^{\gamma_4}\right)-1} \exp(-\Delta y_{ijk}\gamma_3) \tag{15}$$

Through Eq. (15), the maximum likelihood estimates $(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4)$ can be obtained. Substitute $(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4)$ into Eq. (13), $(\hat{\alpha}_0, \hat{\beta}_0, \hat{c}_0)$ at T_0 can be obtained, so the lifetime of products at T_0 can be evaluated.

3. Lifetime prediction of an individual based on Bayesian inference

For analytical and mathematical tractability, conjugate prior distributions are widely applied in Bayesian inference. However, for Gamma processes, the conjugate prior distribution only considers the random effect of β and assumes that β follows a Gamma distribution. The non-conjugate prior distribution can consider the random effects of both α and β, and need not specify the distribution type of α, β. The lifetime prediction methods based on the two kinds of prior distribution were studied in the paper, respectively.

3.1. Inference using conjugate prior distribution

For a Gamma distribution, the conjugate prior distribution is that β follows a Gamma distribution when α is given, denoted as $\beta|\alpha \sim \mathrm{Ga}(\lambda, \delta)$. Suppose that $\boldsymbol{x} = [x_0, x_1, \cdots, x_n]$ is the observed degradation data of an individual at T_0, $\boldsymbol{t} = [t_0, t_1, \cdots, t_n]$ is the corresponding observed time, $\Delta x_i = x_i - x_{i-1}$ denote the degradation increment, $\Delta z_i = t_i^c - t_{i-1}^c$ denote the time increment, $L(\Delta\boldsymbol{x}|\beta)$ denote a maximum likelihood function, $\pi(\beta)$ denote the prior distribution function and $\pi(\beta|\Delta\boldsymbol{x})$ denote a posterior distribution function. Then the $\pi(\beta|\Delta\boldsymbol{x})$ can be deduced according to Bayesian formula

$$\pi(\beta|\Delta\boldsymbol{x}) = \frac{L(\Delta\boldsymbol{x}|\beta)\pi(\beta)}{\int_0^\infty L(\Delta\boldsymbol{x}|\beta)\pi(\beta)\mathrm{d}\beta}$$
$$\propto \prod_{i=1}^{n} \frac{\beta^{\alpha\Delta z_i}\Delta x_i^{\alpha\Delta z_i - 1}}{\Gamma(\alpha\Delta z_i)} \exp(-\beta\Delta x_i)\frac{\delta^\lambda \beta^{\lambda-1}}{\Gamma(\lambda)}\exp(-\delta\beta)$$
$$\propto \beta^{\lambda-1+\alpha\sum\Delta z_i}\exp\left(-\beta\left(\delta+\sum\Delta x_i\right)\right)$$
$$\propto \beta^{\lambda+\alpha(t_n^c - t_0^c)-1}\exp(-\beta(\delta + x_n - x_0)) \tag{16}$$

Since the posterior distribution of β has the same form as that of the conjugate prior distribution, it can be deduced that $\beta \sim \mathrm{Ga}(\lambda + \alpha(t_n^c - t_0^c), \delta + x_n - x_0)$. The posterior mean of β can be written as

$$E(\beta|\Delta\boldsymbol{x}) = \frac{\lambda + \alpha\left(t_n^c - t_0^c\right)}{\delta + x_n - x_0} \tag{17}$$

Thus, the posterior mean of β can be updated as soon as any new observed degradation data is obtained, and the estimate of the lifetime of an individual can subsequently be evaluated.

3.2. Inference using non-conjugate prior distribution

When the random effects of both β and α are considered, there is no conjugate prior distribution available. Thus, non-conjugate prior distribution was alternatively adopted. Furthermore, α and β were supposed to be mutually independent in order to simplify the structure of the joint prior distribution function $\pi(\alpha, \beta)$, expressed as $\pi(\alpha, \beta) = \pi(\alpha)\pi(\beta)$. The distribution types of α and β can be determined by goodness-of-fit tests, and the hyper parameters can be estimated by maximum likelihood estimation.

Suppose that $L(\Delta\boldsymbol{x}|\alpha, \beta)$ is a maximum likelihood function and $\pi(\alpha, \beta|\Delta\boldsymbol{x})$ is a joint posterior distribution function. The following Bayesian formula can be established.

$$\pi(\alpha, \beta|\Delta\boldsymbol{x}) = \frac{L(\Delta\boldsymbol{x}|\alpha, \beta)\pi(\alpha, \beta)}{\int_0^\infty\int_0^\infty L(\Delta\boldsymbol{x}|\alpha, \beta)\pi(\alpha, \beta)\mathrm{d}\alpha\mathrm{d}\beta} \tag{18}$$

The marginal distribution functions $\pi(\alpha|\Delta\boldsymbol{x})$ and $\pi(\beta|\Delta\boldsymbol{x})$ can be obtained as

$$\begin{cases} \pi(\alpha|\Delta\boldsymbol{x}) = \int_0^\infty \pi(\alpha, \beta|\Delta\boldsymbol{x})\ \mathrm{d}\beta \\ \pi(\beta|\Delta\boldsymbol{x}) = \int_0^\infty \pi(\alpha, \beta|\Delta\boldsymbol{x})\mathrm{d}\alpha \end{cases} \tag{19}$$

The posterior means of α and β can be computed as

$$\begin{cases} E(\alpha|\Delta\boldsymbol{x}) = \int_0^\infty \alpha\pi(\alpha|\Delta\boldsymbol{x})\mathrm{d}\alpha \\ E(\beta|\Delta\boldsymbol{x}) = \int_0^\infty \beta\pi(\beta|\Delta\boldsymbol{x})\mathrm{d}\beta \end{cases} \tag{20}$$

Note that $\pi(\alpha|\Delta\boldsymbol{x})$ and $\pi(\beta|\Delta\boldsymbol{x})$ are commonly unknown forms. Thus, it is probably complex to obtain the values of $E(\alpha|\Delta\boldsymbol{x})$ and $E(\beta|\Delta\boldsymbol{x})$ by mathematical computation. Fortunately, WinBUGS software,[26,27] which can carry out Markov chain Monte Carlo (MCMC) simulations with Gibbs sampling, provides an efficient approach to evaluate the posterior means of random parameters.

4. A case study

The accelerated degradation data of Carbon-film resistors which was also shown in Table C.3 by Meeker and Escobar[5] was adopted for a case study. There were total 29 resistors in a constant stress ADT, where 9 samples were observed at 83 °C, 10 samples were observed at 133 °C and the rest 10 sample were observed at 173 °C. All the samples were simultaneously observed every time at $t_0 = 0$, $t_1 = 0.452$, $t_2 = 1.030$, $t_3 = 4.341$ and $t_4 = 8.084$ (10^3 h). It was assumed that the normal use temperature was 50 °C and the threshold value for percent increase in resistance was $l = 5$. The sample numbered 27 was omitted in our example, because its degradation data was not strictly increasing and the Gamma process was expected to model the degradation data.

4.1. Lifetime prediction of the population

To illustrate the degradation path of resistors, the degradation data was plotted in Fig. 1. It can be seen that the resistances of the samples at three different stress levels uniformly show a sudden augment at the beginning of the ADT, and then reach a stably increasing process. Although the stably increasing process is approximately linear, the whole degradation path is nonlinear. Thus, when modeling the degradation data with a Gamma process, we specified $\Lambda_1(t) = t^c$, $\Lambda_2(t) = t$, respectively. The maximum likelihood estimates and Akaike information criterion[28] (AIC) of each sample were listed in Table 1.

It can be concluded that a Gamma process with $\Lambda_1(t)$ is more suitable for modeling the degradation data of the resistors by comparing the AICs in Table 1. Through Eq. (15), $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4$ were obtained and listed in Table 2. For a compare purpose, a wiener process was also used to model the acceler-

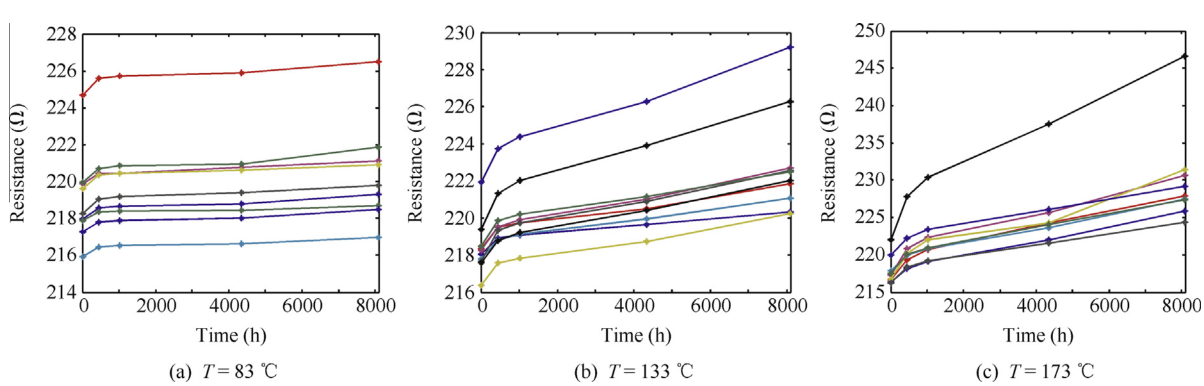


Fig. 1 Degradation curves.

Table 1 The maximum likelihood estimates and AIC of each sample.

Temperature (°C)	Item	Gamma processes with $\Lambda_1(t)$				Gamma processes with $\Lambda_2(t)$		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	AIC	$\hat{\alpha}$	$\hat{\beta}$	AIC
83	1	5.868	16.522	0.267	−2.617	0.544	7.092	1.290
	2	6.786	25.598	0.172	−6.986	0.357	7.585	−1.733
	3	7.923	15.815	0.230	−1.392	0.499	4.979	3.796
	4	6.456	20.732	0.207	−4.468	0.405	6.824	0.544
	5	7.555	25.578	0.315	−6.394	0.737	10.452	−2.345
	6	26.255	70.265	0.210	−8.445	0.509	7.095	1.513
	7	19.319	45.158	0.236	−5.818	0.565	6.526	2.664
	8	6.413	21.006	0.282	−3.821	0.590	8.673	0.263
	9	3.021	6.399	0.281	2.077	0.423	4.025	4.424
133	11	16.996	34.296	0.359	−3.071	0.960	7.394	3.570
	12	24.769	20.983	0.470	3.813	1.501	3.852	10.781
	13	10.005	14.839	0.413	2.002	1.130	5.709	6.112
	14	22.522	36.425	0.424	−1.336	1.227	6.612	5.848
	15	14.200	18.808	0.473	2.360	1.476	5.876	7.145
	16	9.969	14.185	0.447	2.602	1.294	5.842	6.327
	17	14.384	14.482	0.395	3.705	1.086	3.868	9.377
	18	12.288	11.000	0.517	6.418	1.731	4.254	10.175
	19	17.886	21.869	0.388	1.339	1.060	4.659	7.971
	20	19.338	26.244	0.485	1.097	1.589	6.326	6.897
	21	18.830	15.199	0.611	6.376	2.791	5.081	10.351
173	22	68.677	38.192	0.511	3.647	1.781	2.753	14.253
	23	24.411	6.644	0.530	13.492	1.948	1.416	19.622
	24	17.516	13.053	0.570	6.869	2.312	4.229	11.131
	25	14.536	6.461	0.480	11.404	1.459	1.921	16.397
	26	4.404	1.840	0.496	16.637	1.113	1.333	17.972
	28	34.241	25.488	0.490	3.782	1.647	3.561	11.783
	29	40.511	27.539	0.497	4.061	1.668	3.241	12.744
	30	40.190	25.050	0.496	4.384	1.720	3.076	12.876

ated degradation data. Through the acceleration factor constant hypothesis, the relationships between parameters of a Wiener process and temperature stress was deduced as $\mu = \exp(\gamma_1 - \gamma_2/T)$ and $\sigma = \exp(0.5(\gamma_3 - \gamma_2/T))$, where μ denote the drift parameter, σ denote the diffusion parameter, and $\gamma_1, \gamma_2, \gamma_3$ are coefficients. The deducing process was given in Appendix A.

The Gamma process was more suitable to modeling the accelerated degradation data than the wiener process, since

Table 2 The maximum likelihood estimates and AICs obtained by modeling all accelerated degradation data.

Model	Parameter				AIC
	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$	$\hat{\gamma}_4$	
Gamma	10.642	3642.020	6.025	0.466	9.782
Wiener	10.962	4584.951	9.375	0.510	45.948

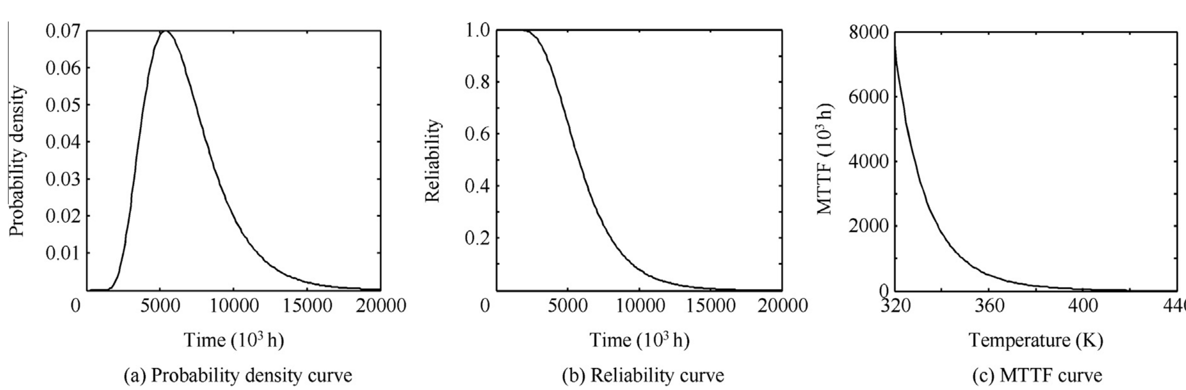


Fig. 2 Prediction curves of the resistors.

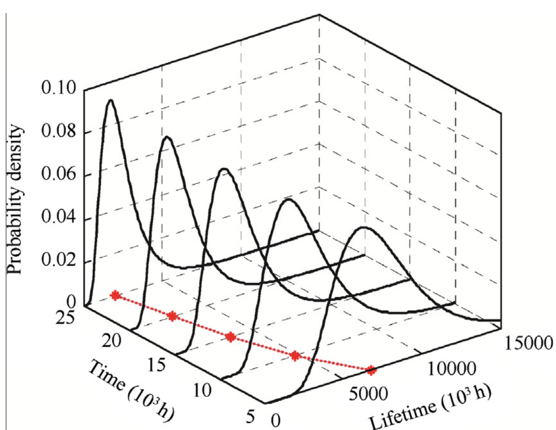
the AIC of the Gamma process was smaller. The estimated $\hat{\alpha}_0$ at 50 °C is 0.534, and the mean time to failure (MTTF) of resistors at 50 °C can be evaluated as $\hat{\xi}_{BS} = 5.991 \times 10^6$ h. The probability density curve, reliability curve at 50 °C and the MTTFs of resistors at different temperatures were illustrated in Fig. 2.

Table 3 Simulation field degradation data of an individual resistor.

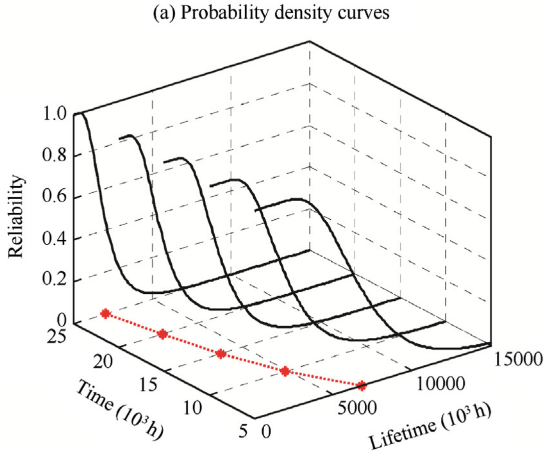
Observed degradation	Time (h)				
	5000	10000	15000	20000	25000
Resistance (Ω)	218.13	218.52	218.92	219.32	219.88
Increment (%)	0.465	0.645	0.829	1.013	1.271

Table 4 Posterior estimates of parameters using a conjugate prior distribution.

Estimator	Time (h)				
	5000	10000	15000	20000	25000
$\hat{\lambda}$	3.736	4.167	4.492	4.762	4.999
$\hat{\delta}$	0.307	0.344	0.417	0.459	0.528
$E(\beta_0\|\Delta\boldsymbol{x})$	6.411	5.463	4.744	4.212	3.599
$\hat{\xi}_{BS}(10^6/\text{h})$	6.771	4.831	3.590	2.796	2.013



(a) Probability density curves



(b) Reliability curves

Fig. 3 Prediction curves using a conjugate prior distribution.

4.2. Lifetime prediction of an individual

Assumed that there was field degradation data of an individual carbon-film resistor observed at 50 °C. The initial value of resistance observed at $t = 0$ h is specified as 217.12 Ω, and the rest simulated data was specified in Table 3.

4.2.1. Bayesian inference using a conjugate prior distribution

Through the Anderson-Darling hypothesis testing with a 95% confidence level, it was concluded that the $\hat{\beta}$ shown in Table 1 follows a Gamma distribution. Furthermore, the maximum likelihood estimates of hyper parameters were obtained and the prior distribution was obtained as $\beta_0 \sim \text{Ga}(2.605, 0.118)$, When $\hat{\alpha}_0 = 0.534$, $\hat{c}_0 = 0.466$ were adopted, and the posterior estimates were listed in Table 4. The PDF curves and the reliability curves were shown in Fig. 3.

4.2.2. Bayesian inference using a non-conjugate prior distribution

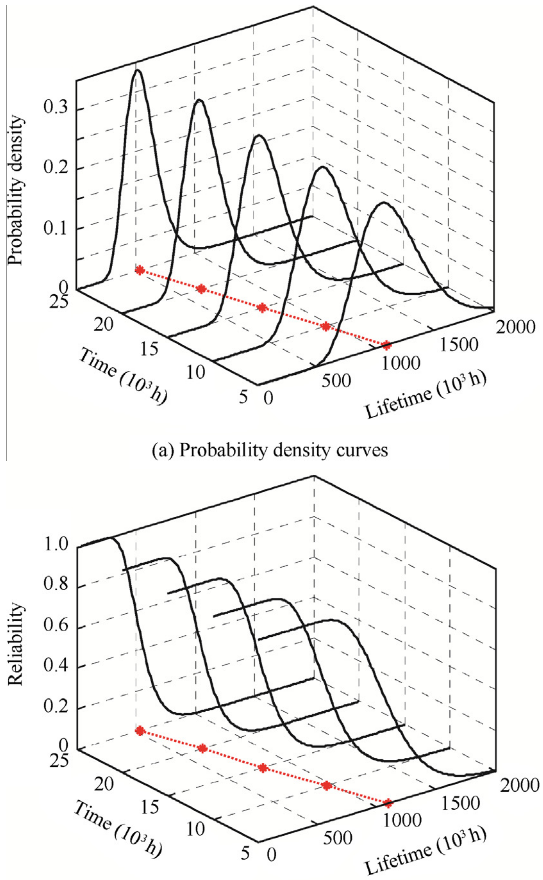
With $\hat{\gamma}_1$ and $\hat{\gamma}_2$, the acceleration factors of α were computed as $A_{1,0} = 2.841$, $A_{2,0} = 10.006$, $A_{3,0} = 22.357$. Thus, the $\hat{\alpha}$ at 83 °C, 133 °C, 173 °C was converted to the $\hat{\alpha}_0$ at 50 °C, as shown in Table 5. Because the values of α_0 and β_0 should be non-negative, we adopted Exponential distribution, Weibull distribution and Gamma distribution as the candidates for the best fitting distribution. The Anderson Darling (AD) statistic was used as the guideline for distribution type selection that the candidate with the smallest AD is the best fitting distribution type. It was concluded that the Gamma distribution was

Table 5 Converted $\hat{\alpha}_0$ from accelerated stresses.

83 °C		133 °C		173 °C	
Item	$\hat{\alpha}_0$	Item	$\hat{\alpha}_0$	Item	$\hat{\alpha}_0$
1	2.065	11	1.699	21	0.842
2	2.388	12	2.475	22	3.072
3	2.788	13	1.000	23	1.092
4	2.272	14	2.251	24	0.783
5	2.659	15	1.419	25	0.650
6	9.240	16	0.996	26	0.197
7	6.799	17	1.438		
8	2.257	18	1.228	28	1.532
9	1.063	19	1.788	29	1.812
		20	1.933	30	1.798

Table 6 Posterior estimates of parameters using a non-conjugate prior distribution.

Estimator	Time (h)				
	5000	10000	15000	20000	25000
$E(\alpha_0\|\Delta\boldsymbol{x})$	2.955	3.455	3.749	4.125	4.145
$E(\beta_0\|\Delta\boldsymbol{x})$	15.330	16.840	16.830	17.100	15.380
$\hat{\xi}_{BS}(10^5/\text{h})$	10.972	9.586	8.035	6.771	5.345



(a) Probability density curves

(b) Reliability curves

Fig. 4 Prediction curves using a non-conjugate prior distribution.

the best fitting distribution for both α_0 and β_0. The prior distributions of α_0 and β_0 were obtained as $\alpha_0 \sim \mathrm{Ga}(2.170, 1.020)$ and $\beta_0 \sim \mathrm{Ga}(2.605, 0.118)$. When $\hat{c}_0 = 0.466$ is specified, the posterior estimates were obtained, as shown in Table 6. The prediction curves were illustrated in Fig. 4.

5. Conclusions

(1) The acceleration factor constant hypothesis provides an appealing approach to deduce the relationships between parameters of a degradation model and accelerated stresses. Although the deduced conclusions still need abundant test data to verify, the hypothesis and deducing method are more convincing than the assumptions made from experience.
(2) The lifetime prediction of an individual based on Bayesian inference has been a research hotspot. Acceleration factor was used to convert the estimates of parameters at accelerated stresses to those at the normal use stress, which realized the objective of taking accelerated degradation data as prior information. The conversion method with acceleration factor can be extended to solve lots of problems about accelerated degradation data.
(3) For analytical and mathematical tractability, the conjugate prior distributions of the random parameters are widely applied in Bayesian inference. However, the non-conjugate prior distributions can be an alternative approach when the conjugate prior distributions are unavailable or the random effects of more parameters need to be considered. With WinBUGS software, Bayesian inference with non-conjugate prior distributions becomes easier and shows a potential engineering application value.

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Appendix A

According to the acceleration factor constant hypothesis, the following equation should always be identical:

$$F_k(t_k) = F_h(A_{k,h}t_k) \tag{A1}$$

For a wiener process, its first passage times follow an inverse Gaussian distribution when a threshold is specified. The CDF of the inverse Gaussian distribution is too complex to compute, so the PDF was used

$$f(t) = \frac{l}{\sqrt{2\pi\sigma^2 t^3}} \exp\left[-\frac{(l-\mu t)^2}{2\sigma^2 t}\right] \tag{A2}$$

From Eq. (A1), the following identical equation can be obtained

$$f_k(t_k) = A_{k,h} f_h(t_h) \tag{A3}$$

The deduction process is illustrated as

$$\begin{aligned} f_k(t_k) &= \frac{\mathrm{d}F_k(t_k)}{\mathrm{d}t_k} = A_{k,h}\frac{\mathrm{d}F_h(A_{k,h}t_k)}{\mathrm{d}(A_{k,h}t_k)} = A_{k,h}\frac{\mathrm{d}F_h(t_h)}{\mathrm{d}(t_h)} \\ &= A_{k,h} f_h(t_h) \end{aligned} \tag{A4}$$

Substitute Eq. (A2) into Eq. (A3), then

$$\begin{aligned} A_{k,h} &= \frac{f_k(t_k)}{f_h(t_h)} = \frac{f_k(t_k)}{f_h(A_{k,h}t_k)} = \frac{\sigma_h A_{k,h}^{3/2}}{\sigma_k} \\ &\cdot \exp\left[\left(\frac{l\mu_h}{\sigma_h^2} - \frac{l\mu_k}{\sigma_k^2}\right) + \frac{1}{t_k}\left(\frac{l^2}{2\sigma_h^2 A_{k,h}} - \frac{l^2}{2\sigma_k^2}\right) + t_k\left(\frac{\mu_h^2 A_{k,h}}{2\sigma_h^2} - \frac{\mu_k^2}{2\sigma_k^2}\right)\right] \end{aligned} \tag{A5}$$

To ensure $A_{k,h}$ is a constant which does not change with t_k, the following relationship must be satisfied

$$\begin{cases} \sigma_k^2 = \sigma_h^2 A_{k,h} \\ \sigma_h^2 \mu_k^2 = \sigma_k^2 \mu_h^2 A_{k,h} \end{cases} \tag{A6}$$

So the following relationship can be deduced

$$A_{k,h} = \mu_k/\mu_h = \sigma_k^2/\sigma_h^2 \tag{A7}$$

It can be concluded that both μ and σ should change with accelerated stresses varying and the ratios of μ_k, μ_h should be equal to that of σ_k^2, σ_h^2.

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