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Note

Kernels in quasi-transitive digraphs

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Abstract

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively.

A kernel N of D is an independent set of vertices such that for every $w \in V(D) - N$ there exists an arc from w to N. A digraph is called *quasi-transitive* when $(u, v) \in A(D)$ and $(v, w) \in A(D)$ implies $(u, w) \in A(D)$ or $(w, u) \in A(D)$. This concept was introduced by Ghouilá–Houri [Caractérisation des graphes non orientés dont on peut orienter les arrêtes de maniere à obtenir le graphe d' un relation d'ordre, C.R. Acad. Sci. Paris 254 (1962) 1370–1371] and has been studied by several authors. In this paper the following result is proved: Let D be a digraph. Suppose $D = D_1 \cup D_2$ where D_i is a quasi-transitive digraph which contains no asymmetrical infinite outward path (in D_i) for $i \in \{1, 2\}$; and that every directed cycle of length 3 contained in D has at least two symmetrical arcs, then D has a kernel. All the conditions for the theorem are tight.

Keywords: Kernel; Kernel-perfect digraph; Quasi-transitive digraph

1. Introduction

For general concepts we refer the reader to [4]. In the paper we write digraph to mean 1-digraph in the sense of Berge [4]. In this paper D will denote a possibly infinite digraph with $V(D)$ and $A(D)$ being the sets of vertices and arcs of D, respectively. Often we shall write u_1u_2 instead of (u_1, u_2). An arc $u_1u_2 \in A(D)$ is called asymmetrical (resp. symmetrical) if $u_2u_1 \notin A(D)$ (resp. $u_2u_1 \in A(D)$). If S is a nonempty subset of $V(D)$ then the subdigraph $D[S]$ induced by S is the digraph with vertex set S and whose arcs are those arcs of D which join vertices of S.

A directed path is a finite or infinite sequence $(x_1, x_2, \ldots)$ of distinct vertices of D such that $(x_i, x_{i+1}) \in A(D)$ for each i. When D is infinite and the sequence is infinite we call the directed path an infinite outward path. Let S_1 and S_2 be subsets of $V(D)$. A finite directed path $(x_1, \ldots, x_n)$ will be called an S_1S_2-directed path whenever $x_1 \in S_1$ and $x_2 \in S_2$, in particular when the directed path is an arc, we will call it an S_1S_2-arc.

Definition 1.1. A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D.

A digraph D is called a kernel-perfect digraph when every induced subdigraph of D has a kernel.

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The concept of kernel was introduced by Von Neumann and Morgenstern [15] in the context of Game Theory. The problem of the existence of a kernel in a given digraph has been studied by several authors in particular by Richardson [16,17], Duchet and Meyniel [9], Duchet [7,8], Galeana-Sánchez and Neumann-Lara [10].

A digraph D is transitive whenever $(u, v) \in A(D)$ and $(v, w) \in A(D)$ implies $(u, w) \in A(D)$. A digraph is called *quasi-transitive* if whenever $(u, v) \in A(D)$ and $(v, w) \in A(D)$, then $(u, w) \in A(D)$ or $(w, u) \in A(D)$.

Quasi-transitive digraphs were introduced by Ghouilá-Houri [12] and have been studied by several authors for example Bang-Jensen and Huang [1–3], Huang [13], Skrien [19]. It was proved by Ghouilá-Houri [12] that an undirected graph can be oriented as a quasi-transitive digraph if and only if it can be oriented as a transitive digraph, namely a comparability graph. More information about comparability graphs can be found in [11,14].

In [6] Boros and Gurvich proved that if G is a perfect graph then any orientation of G in which each complete subdigraph has a kernel is kernel-perfect. It is well known that comparability graphs are perfect graphs (see for example [5]). Meyniel [9] observed that if D is a digraph such that every directed cycle of length 3 has at least two symmetrical arcs, then each complete subdigraph of D has a kernel.

We can conclude the following result.

Theorem 1.2. *If D is a finite quasi-transitive digraph such that every directed cycle of length* 3 *has at least two symmetrical arcs*, *then D is a kernel-perfect digraph.*

The result proved in this paper generalizes Theorem 1.2 and the following result of Sands et al. [18].

Theorem 1.3 (*Sands et al. [18]*). *Let D be a digraph whose arcs are colored with two colors. If D contains no monochromatic infinite outward path*, *then there exists a set S of vertices of D such that no two vertices of S are connected by a monochromatic directed path and for every vertex x not in S there is a monochromatic directed path from x to a vertex in S.*

We include the following definitions in order to understand Theorem 1.3 in terms of kernels.

We call the digraph D an m-colored digraph if the arcs of D are colored with m colors. A directed path is called monochromatic if all of its arcs are colored alike. A kernel by monochromatic paths in an m-colored digraph D is a set of vertices N which satisfies the following two conditions: (i) for every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them; and (ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path.

If D is an m-colored digraph then the closure of D, denoted $\mathscr{C}(D)$ is the digraph defined as follows: $V(\mathscr{C}(D)) = V(D)$ and $(u, v) \in A(\mathscr{C}(D))$ iff there exists a uv-monochromatic directed path contained in D.

Note that for any m-colored D, D has a kernel by monochromatic paths if and only if $\mathscr{C}(D)$ has a kernel.

In this terminology Theorem 1.3 asserts that if D is a 2-colored digraph, which contains no monochromatic infinite outward path, then $\mathscr{C}(D)$ has a kernel.

Now it is clear that Theorem 1.3 is equivalent to the following assertion. Let D be a digraph; D_1 and D_2 transitive subdigraphs of D such that $D = D_1 \cup D_2$ (recall that $D_1 \cup D_2$ is defined as follows: $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$ and $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$) and $A(D_1) \cap A(D_2) = \emptyset$. If D has no infinite outward path contained in D_i $(i = 1, 2)$, then D has a kernel.

Finally, we will introduce some notation. Two subdigraphs D_1 and D_2 of D are given (possibly $A(D_1) \cap A(D_2) \neq \emptyset$). For distinct vertices x, y of D, $x \overset{i}{\to} y$ will mean that the arc $(x, y) \in A(D_i)$ and $x \overset{i}{\to} S$ will mean that there exists an arc in D_i from x to a vertex in S, the negation of $x \overset{i}{\to} y$ (resp. $x \overset{i}{\to} S$) will be denoted by $x \overset{i}{\nrightarrow} y$ (resp. $x \overset{i}{\nrightarrow} S$), for $i = 1, 2$. When we do not know if the arc is in D_1 or in D_2 we write simply $x \to y$; and $x \nrightarrow y$ will mean that $(x, y) \notin A(D)$. A directed cycle of length 3 will be called a triangle.

2. Kernels in the union of two quasi-transitive digraphs

The main result of this section is Theorem 2.3. The proof is similar to that in Sands et al. [18].

Lemma 2.1. *Let D be a digraph such that every triangle has at least two symmetrical arcs. If D_1 is a quasi-transitive subdigraph of D and $(v_1, v_2, \ldots, v_n)$ is a sequence of vertices of D_1 such that $(v_i, v_{i+1}) \in A(D_1)$ and*

$(v_{i+1}, v_i) \notin A(D)$, then the sequence is an asymmetrical directed path of D contained in D_1, and for each $i \in \{1, \ldots, n-1\}$, $(v_i, v_j) \in A(D_1)$ and $(v_j, v_i) \notin A(D)$ for every $j \in \{i+1, \ldots, n\}$.

Proof. We proceed by induction on *n*. The result is obvious for $n \leqslant 2$. Assume the result is true for a sequence $(v_1, \ldots, v_n)$ which satisfies the hypothesis of Lemma 2.1. Consider a sequence $T = (v_1, \ldots, v_n, v_{n+1})$ such that for each $i \in \{1, \ldots, n\}$, $(v_i, v_{i+1}) \in A(D_1)$ and $(v_{i+1}, v_i) \notin A(D)$. Since $T' = (v_1, \ldots, v_n)$ satisfies the inductive hypothesis, we have that T' is an asymmetrical directed path contained in D_1 and for each $i \in \{1, \ldots, n-1\}$ $(v_i, v_j) \in A(D_1)$ and $(v_j, v_i) \notin A(D)$ for every $j \in \{i+1, \ldots, n\}$. So we only need to prove that for each $i \in \{1, \ldots, n-1\}$, $v_i \neq v_{n+1}$, $(v_i, v_{n+1}) \in A(D_1)$ and $(v_{n+1}, v_i) \notin A(D)$.

First assume by contradiction that $v_{n+1} = v_i$ for some $i \in \{1, \ldots, n-1\}$. It follows from the inductive hypothesis on T' that $(v_i, v_n) = (v_{n+1}, v_n) \in A(D_1)$ and thus $(v_{n+1}, v_n) \in A(D)$ contradicting our hypothesis on *T*. We conclude that *T* is an asymmetrical directed path of *D* contained in D_1. Now, we have from the inductive hypothesis on T' that for each $i \in \{1, \ldots, n-1\}$, $(v_i, v_n) \in A(D_1)$ and since $(v_n, v_{n+1}) \in A(D_1)$ and D_1 is a quasi-transitive digraph, we have that $(v_i, v_{n+1}) \in A(D_1)$ or $(v_{n+1}, v_i) \in A(D_1)$. If $(v_{n+1}, v_i) \in A(D)$ then $C_3 = (v_i, v_n, v_{n+1}, v_i)$ is a triangle and from the hypothesis on *D*, C_3 has at least two symmetrical arcs which is impossible as $(v_{n+1}, v_n) \notin A(D)$ (hypothesis on *T*) and $(v_n, v_i) \notin A(D)$ (inductive hypothesis). Thus $(v_{n+1}, v_i) \notin A(D)$ and $(v_i, v_{n+1}) \in A(D_1)$. □

Lemma 2.2. *Let D be a digraph such that every triangle has at least two symmetrical arcs, and D_1 be a quasi-transitive subdigraph of D which contains no asymmetrical (in D) infinite outward path. If $\emptyset \neq U \subseteq V(D)$ then there exists $x \in U$ such that for all $y \in U$ $(x, y) \in A(D_1)$ implies $(y, x) \in A(D)$.*

Proof. Suppose by contradiction that for each $x \in U$, there exists $y \in U$ such that $(x, y) \in A(D_1)$ and $(y, x) \notin A(D)$. Consider some $x_1 \in U$. Then there exists $x_2 \in U$ such that $(x_1, x_2) \in A(D_1)$ and $(x_2, x_1) \notin A(D)$. So for each $n \in \mathbb{N}$, given $x_n \in U$, there exists $x_{n+1} \in U$ such that $(x_n, x_{n+1}) \in A(D_1)$ and $(x_{n+1}, x_n) \notin A(D)$. It follows from Lemma 2.1 that $T_{n+1} = (x_1, x_2, \ldots, x_{n+1})$ is an asymmetrical directed path of *D* contained in D_1. Consider the sequence $T = (x_n)_{n \in \mathbb{N}}$; for each $n \in \mathbb{N}$, $(x_n, x_{n+1}) \in A(D_1)$, and for $n < m$ we have $\{x_n, x_m\} \subseteq V(T_m)$ and since T_m is a directed path we obtain $x_n \neq x_m$; hence *T* is an asymmetrical infinite outward path of *D* contained in D_1, a contradiction. □

Theorem 2.3. *Let D be a digraph such that $D = D_1 \cup D_2$ (possibly $A(D_1) \cap A(D_2) \neq \emptyset$), where D_i is a quasi-transitive subdigraph of D which contains no asymmetrical (in D) infinite outward path. If every triangle contained in D has at least two symmetrical arcs, then D is a kernel-perfect digraph.*

Proof. It suffices to prove that *D* has a kernel, as any induced subdigraph of *D* satisfies the hypothesis of Theorem 2.3.

For independent sets *S*, *T* of *D*, we write $S \leqslant T$ if and only if for each $s \in S$ there exists $t \in T$ such that either $s = t$ or ($s \xrightarrow{1} t$ and $t \not\twoheadrightarrow s$). Note that in particular $S \subseteq T$ implies $S \leqslant T$.

(1) The collection of all independent sets of vertices of *D* is partially ordered by $\leqslant$.

(1.1) $\leqslant$ is reflexive.

This follows from the fact $S \subseteq S$.

(1.2) $\leqslant$ is transitive.

Let *S*, *T* and *R* be independent sets of vertices of *D*, such that $S \leqslant T$ and $T \leqslant R$, and let $s \in S$. Since $S \leqslant T$ there exists $t \in T$ such that either $s = t$ or ($s \xrightarrow{1} t$ and $t \not\twoheadrightarrow s$) and $T \leqslant R$ implies that there exists $r \in R$ such that either $t = r$ or ($t \xrightarrow{1} r$ and $r \not\twoheadrightarrow t$). If $s = t$ or $t = r$, then $s = r$ or ($s \xrightarrow{1} r$ and $r \not\twoheadrightarrow s$) with $r \in R$. So we can assume $s \neq t$, $t \neq r$, ($s \xrightarrow{1} t$ and $t \not\twoheadrightarrow s$) and ($t \xrightarrow{1} r$ and $r \not\twoheadrightarrow t$). Since D_1 is a quasi-transitive digraph it follows from Lemma 2.1 on the sequence (s, t, r) that ($s \xrightarrow{1} r$ and $r \not\twoheadrightarrow s$).

(1.3) $\leqslant$ is antisymmetrical.

Let *S* and *T* be independent sets of vertices of *D* such that $S \leqslant T$ and $T \leqslant S$, and let $s \in S$. Since $S \leqslant T$ there exists $t \in T$ such that either $s = t$ or ($s \xrightarrow{1} t$ and $t \not\twoheadrightarrow s$). Suppose that $s \neq t$. The fact $T \leqslant S$ implies that there exists $s' \in S$ such that either $t = s'$ or ($t \xrightarrow{1} s'$ and $s' \not\twoheadrightarrow t$) . When $t = s'$ we obtain $s \xrightarrow{1} s'$ contradicting that *S* is an independent set; so $t \neq s'$ and ($t \xrightarrow{1} s'$ and $s' \not\twoheadrightarrow t$). Now applying Lemma 2.1 on the sequence (s, t, s'), we have $s \xrightarrow{1} s'$ contradicting that *S* is an independent set. We conclude $s = t$ and consequently $s \in T$ and $S \subseteq T$. Analogously it can be proved $T \subseteq S$.

Let $\mathscr{F}$ be the family of all nonempty independent sets S of vertices of D such that $S \xrightarrow{2} y$ implies $y \to S$ for all vertices y of D.

(2) $(\mathscr{F}, \leqslant)$ has maximal elements.

(2.1) $\mathscr{F} \neq \emptyset$.

Since D_2 is a quasi-transitive digraph which contains no asymmetrical infinite outward path, it follows from Lemma 2.2 (taking $U = V(D)$ and D_2 instead of D_1) that there exists a vertex $x \in V(D)$ such that $x \xrightarrow{2} y$ implies $y \to x$, for all vertices y of D, so $\{x\} \in \mathscr{F}$.

(2.2) Every chain in $(\mathscr{F}, \leqslant)$ is upper bounded.

Let $\mathscr{C}$ be a chain in $(\mathscr{F}, \leqslant)$, and define $S^\infty = \{s \in \bigcup_{S \in \mathscr{C}} S |$ there exists $S \in \mathscr{C}$ such that $s \in T$ whenever $T \in \mathscr{C}$ and $T \geqslant S\}$. (S^∞ consists of all vertices of D that belong to every member of $\mathscr{C}$ from some point on.)

We will prove that S^∞ is an upper bound of $\mathscr{C}$.

(2.2.1) $S^\infty \neq \emptyset$, and for each $S \in \mathscr{C}$, $S^\infty \geqslant S$.

Let $S \in \mathscr{C}$ and $t_0 \in S$. We will prove that there exists $t \in S^\infty$ such that either $t_0 = t$ or ($t_0 \xrightarrow{1} t$ and $t \not\to t_0$). If $t_0 \in S^\infty$ we are done. So assume $t_0 \notin S^\infty$. We proceed by contradiction; suppose that if $t \in V(D)$ with ($t_0 \xrightarrow{1} t$ and $t \not\to t_0$), then $t \notin S^\infty$. Take $T_0 = S$. Since $t_0 \notin S^\infty$ we have that there exists $T_1 \in \mathscr{C}$, $T_1 \geqslant T_0$ such that $t_0 \notin T_1$. Hence there exists $t_1 \in T_1$ such that $t_0 \xrightarrow{1} t_1$ and $t_1 \not\to t_0$. And our assumption implies $t_1 \notin S^\infty$. The fact $t_1 \notin S^\infty$ implies $t_1 \notin T_2$ for some $T_2 \in \mathscr{C}$, $T_2 \geqslant T_1$. Hence there exists $t_2 \in T_2$ such that $t_1 \xrightarrow{1} t_2$ and $t_2 \not\to t_1$. Since D_1 is a quasi-transitive digraph, it follows from Lemma 2.1 on the sequence $\tau_2 = (t_0, t_1, t_2)$ that τ_2 is an asymmetrical directed path of D contained in D_1, ($t_0 \xrightarrow{1} t_2$ and $t_2 \not\to t_0$); and $t_2 \notin S^\infty$. We may continue this way and we obtain, for each $n \in \mathbb{N}$, $T_n \in \mathscr{C}$, $t_n \in T_n$, ($t_0 \xrightarrow{1} t_n$ and $t_n \not\to t_0$) and $t_n \notin S^\infty$. Hence there exists $T_{n+1} \in \mathscr{C}$ such that $T_{n+1} \geqslant T_n$ and $t_n \notin T_{n+1}$. So there exists $t_{n+1} \in T_{n+1}$ with ($t_n \xrightarrow{1} t_{n+1}$ and $t_{n+1} \not\to t_n$).

Since D_1 is a quasi-transitive digraph, and ($t_n \xrightarrow{1} t_{n+1}$ and $t_{n+1} \not\to t_n$) for each $n \in \mathbb{N}$, it follows from Lemma 2.1 (on the sequence $\tau_{n+1} = (t_0, t_1, \ldots, t_{n+1})$) that τ_{n+1} is an asymmetrical directed path contained in D_1 and in particular ($t_0 \xrightarrow{1} t_{n+1}$ and $t_{n+1} \not\to t_0$). Our assumption implies $t_{n+1} \notin S^\infty$. Now consider the sequence $\tau = (t_n)_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$ we have ($t_n \xrightarrow{1} t_{n+1}$ and $t_{n+1} \not\to t_n$), and observe that for $n < m$, $\{t_n, t_m\} \subseteq V(\tau_m)$, and since τ_m is a directed path we have $t_n \neq t_m$. Hence τ is an asymmetrical infinite outward path contained in D_1, a contradiction. We conclude that there exists $t \in S^\infty$ such that ($t_0 \xrightarrow{1} t$ and $t \not\to t_0$).

(2.2.2) S^∞ is an independent set.

Let $s_1, s_2 \in S^\infty$ and suppose without loss of generality that $S_1, S_2 \in \mathscr{C}$ are such that $s_1 \in S_1$, $s_2 \in S_2$, $S_1 \leqslant S_2$, since $s_1 \in S^\infty$ we have $s_1 \in S$ whenever $S \in \mathscr{C}$ and $S \geqslant S_1$, so $s_1 \in S_2$, and since S_2 is independent, there is no arc in D between s_1 and s_2.

(2.2.3) $S^\infty \in \mathscr{F}$.

Suppose $S^\infty \xrightarrow{2} y$ with $y \in V(D)$, so there exists $s \in S^\infty$ with $s \xrightarrow{2} y$. Let $S \in \mathscr{C}$ such that $s \in T$ for all $T \in \mathscr{C}$, $T \geqslant S$. Since $S \in \mathscr{F}$ we have $y \to S$, so there exists $s' \in S$ with $y \to s'$. When $s' \in S^\infty$ we are done. When $s' \notin S^\infty$ we analyze the two possibilities; $y \xrightarrow{1} s'$ or $y \xrightarrow{2} s'$. First suppose $y \xrightarrow{2} s'$. Since $s \xrightarrow{2} y$ and D_2 is a quasi-transitive digraph it follows that $s \xrightarrow{2} s'$ or $s' \xrightarrow{2} s$ which is impossible as S is an independent set and $\{s, s'\} \subseteq S$. Now suppose $y \xrightarrow{1} s'$. Since $s' \in S$, $S \leqslant S^\infty$ by (2.2.1) and $s' \notin S^\infty$, there exists $t \in S^\infty$ such that $s' \xrightarrow{1} t$ and $t \not\to s'$. So we obtain $y \xrightarrow{1} t$ or $t \xrightarrow{1} y$ (as $y \xrightarrow{1} s'$, $s' \xrightarrow{1} t$ and D_1 is a quasi-transitive digraph). If $y \xrightarrow{1} t$ then $y \xrightarrow{1} S^\infty$ and we are done.If $t \xrightarrow{1} y$ then we obtain the triangle (y, s', t, y) and it follows from the hypothesis that it has two symmetrical arcs and since $t \not\to s'$ we have $s' \to y$ and $y \to t$, so $y \to S^\infty$.

We have proven that any chain in $\mathscr{F}$ has an upper bound in $\mathscr{F}$, and so by Zorn's Lemma, $(\mathscr{F}, \leqslant)$ contains maximal elements. Let S be a maximal element of $(\mathscr{F}, \leqslant)$.

(3) S is a kernel of D.

Since $S \in \mathscr{F}$, S is an independent set of vertices of D.

(3.1) For each $x \in (V(D) - S)$ there exists an xS-arc.

Suppose by contradiction there exists $x \in (V(D) - S)$ such that $x \not\to S$.

(3.1.1) There exists a vertex $x_0 \in V(D)$ such that $x_0 \nrightarrow S$ and x_0 satisfies: $x_0 \xrightarrow{2} y$ and $y \nrightarrow S$ imply $y \to x_0$ for all vertices $y \in V(D)$. Let $U = \{z \in V(D_2) - S | z \nrightarrow S\}$. When $U \neq \emptyset$, it follows from Lemma 2.2 (applied on D_2 and U) that there exists x_0 with the required properties. When $U = \emptyset$ it follows from our assumption that $z \nrightarrow S$, for some vertex z in $V(D_1) - (S \cup V(D_2))$, and we take x_0 to be any such vertex.

Note that the choice of x_0 implies $x_0 \nrightarrow S$ and since $S \in \mathscr{F}$, we also have $S \overset{2}{\nrightarrow} x_0$. Let $T = \{s \in S | s \overset{1}{\nrightarrow} x_0\}$, it follows from above that $T \cup \{x_0\}$ is an independent set of vertices of D.

(3.1.2) $T \cup \{x_0\} \in \mathscr{F}$.

Suppose $T \cup \{x_0\} \xrightarrow{2} y$ and $y \nrightarrow T$. We will prove $y \to x_0$. First we make the following observation.

(3.1.2.1) If $y \xrightarrow{1} (S - T)$ then $y \to x_0$.

Let $s \in (S - T)$ such that $y \xrightarrow{1} s$. Since $s \in (S - T)$ we have $s \xrightarrow{1} x_0$. Now the fact that D_1 is a quasi-transitive digraph implies $y \xrightarrow{1} x_0$ or $x_0 \xrightarrow{1} y$. If $x_0 \xrightarrow{1} y$ then (y, s, x_0, y) is a triangle which by the hypothesis has two symmetrical arcs, and since $x_0 \nrightarrow s$ it follows that $y \to x_0$.

We proceed to prove (3.1.2) by considering the two following cases:

Case a: $T \xrightarrow{2} y$.

Since $T \subset S$ we have $S \xrightarrow{2} y$ and the fact $S \in \mathscr{F}$ implies $y \to S$. So $y \to (S - T)$ (as we are assuming $y \nrightarrow T$).

When $y \xrightarrow{1} (S - T)$ it follows from (3.1.2.1) that $y \to x_0$.

When $y \xrightarrow{2} (S - T)$, since we have $T \xrightarrow{2} y$ and D_2 is a quasi-transitive digraph, we obtain $T \xrightarrow{2} (S - T)$ or $(S - T) \xrightarrow{2} T$ and this is impossible as $T \subseteq S$ and S is an independent set.

Case b: $x_0 \xrightarrow{2} y$.

We consider two possible subcases:

Case b.1: $y \nrightarrow S$.

Since $x_0 \xrightarrow{2} y$ and $y \nrightarrow S$, the choice of x_0 (see (3.1.1)) implies $y \to x_0$.

Case b.2: $y \to S$.

In this case we have $y \to (S - T)$ (as we are assuming $y \nrightarrow T$).

When $y \xrightarrow{2} (S - T)$, since $x_0 \xrightarrow{2} y$ and D_2 is a quasi-transitive digraph, we have $x_0 \xrightarrow{2} (S - T)$ or $(S - T) \xrightarrow{2} x_0$. Now recalling $x_0 \nrightarrow S$, we obtain $(S - T) \xrightarrow{2} x_0$ and since $S \in \mathscr{F}$ it follows $x_0 \to S$ which is impossible.

When $y \xrightarrow{1} (S - T)$ it follows from (3.1.2.1) that $y \to x_0$.

(3.1.3) $S < T \cup \{x_0\}$.

For $s \in (S - T)$ we have $s \xrightarrow{1} x_0$ and we have noted $x_0 \nrightarrow S$; hence $S \leqslant T \cup \{x_0\}$. Moreover since $x_0 \notin S$ (by construction in (3.1.1)) we have $S < T \cup \{x_0\}$.

Clearly propositions (3.1.2) and (3.1.3) contradict that S is a maximal element of $(\mathscr{F}, \leqslant)$. □

Remark 2.4. The condition that D_i has no infinite outward path in Theorem 2.3 is necessary.

Consider the following digraph D' with $V(D') = \{u_n | n \in \mathbb{N}\}$ and $A(D') = \{(u_n, u_m) | n, m \in \mathbb{N}$ and $n < m\}$, $D_1 = D'$, $D_2 = D'$ and $D = D_1 \cup D_2$.

Remark 2.5. The following digraph D is the union of two quasi-transitive finite digraphs; each triangle in D has at least one symmetrical arc and D has no kernel.

$V(D_1) = \{u_0, u_1, u_2, u_3\}$,

$V(D_2) = V(D_1) \cup \{w\}$,

$A(D_1) = \{(u_i, u_{i+1}) | i \in \{0, 1, 2, 3\} \pmod 4\} \cup \{(u_0, u_2), (u_2, u_0), (u_1, u_3), (u_3, u_0)\}$,

$A(D_2) = \{(w, u_i) | i \in \{0, 1, 2, 3\}\}$,

$D = D_1 \cup D_2$.

Remark 2.6. Clearly $\vec{C}_5$ the directed cycle of length 5 is the union of two finite digraphs, $\vec{C}_5$ has no triangle and $\vec{C}_5$ has no kernel.

We conclude that the conditions on Theorem 2.3 are tight.

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