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Symmetry and optimal control in economics

Philippe Askenazy

CNRS and Cepremap, École Normale Supérieure, 48 Boulevard Jourdan, 75014 Paris, France

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Abstract

This paper provides a new simple version of Noether's theorem. From symmetries of dynamic optimal programs, this theorem gives invariant quantities along optimal paths. It is suited to optimal control programs especially for economic models. Applications in growth economics are given.

Keywords: Conservation laws; Noether's theorem; Maximum principle; Economic growth models

1. Introduction

The existence of symmetries helps to reduce the dimension of analytic problems. This method has been extensively used in physics or mechanics (e.g., to solve systems of partial differential equations; see [1] or [21]). It has also potential applications in economics. For example, Russell and Farris [14] discuss the integrability of some classes of functions of good demand, exploiting symmetries (actually Lie groups) that act on these functions; Samuelson [15] and Sato [17] study symmetries of technical change and optimal growth models.

The Noether theorem is the main tool to exploit symmetries of dynamic optimal programs. This theorem gives conservation laws, i.e., invariant quantities along optimal paths. As in physics, conservation laws can add necessary conditions of optimality for economic problems; they can thus reduce the dimension of optimal dynamical systems (of control and state variables). From an econometric viewpoint, because they offer stationary quantities instead of dynamic differential equations, they can help to validate some theories [16].

E-mail address: philippe.askenazy@cepremap.ens.fr.

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Finally, conservation laws can have direct interpretations such as the income/wealth law in economics (cf. [18] or [24]) or the conservation of energy in physics.

In economics, optimal control methods with the maximum principle of Pontryagin et al. [12] are preferred to variational approaches because it is more general and better-adapted to a modeling of optimization under constrained and control variables. However, standard versions of Noether's theorem sometimes used for economic problems (cf. [17,19]) only apply in the approach of calculus of variation and primary versions for optimal control can seem complex [20]. The aim of this note is thus twofold: to provide a tractable and operational version of Noether's theorem in an optimal control framework and to give first applications in economics.

The outline of the note will be the following. Section 2 presents basic notions of geometry. The basic optimal control version of Noether's theorem is given in Section 3. The paper concludes with direct applications in growth economics.

2. A simple program and basic notions

We take a very simple program (P) as a benchmark:

$$\max \int_0^T U\big(s(t), c(t), t\big)\, dt, \tag{1}$$

where the state variable s takes its values in S, an open subset of $\mathbb{R}^n$, the control variable c takes its values in a subset C of $\mathbb{R}^m$, and the function U is continuous and continuously differentiable with respect to s and t. The dynamical constraint is

$$\dot{s}(t) = f\big(s(t), c(t), t\big), \tag{2}$$

where f is continuously differentiable over S and $[0, T]$, and continuous as well as its partial derivatives with respect to s and t over $S \times C \times [0, T]$. Finally, the initial and final constraints are simply

$$s(0) = s_0, \qquad s(T) = s_T. \tag{3}$$

We will denote by $H(s, c, \lambda, t) = p_0 U(s, c, t) + \lambda f(s, c, t)$ the Hamiltonian of program (P), where λ is the co-state variable (of dimension n).

Remark 1. For normal programs (in the sense used by Clarke [7]), p_0 can be normalized $p_0 = 1$. This is generally the case for economic programs, but it is not always possible (see [6] for examples).

The results presented in this note can be extended to more general programs if the latter maintain a minimum of regularity that guarantees the use of the maximum principle. Thus, one could soften some regularity constraints by using the extensions of Pontryagin's theorem developed by Warga [23] or Clarke [7]. Boundary constraints can also be replaced by inequalities, can be free…, if they verify the constraint qualifications (cf. [8]). Finally, one can (and we will do it in the last section) apply Noether's theorems to infinite horizon programs (see [6] for correct assumptions that give a maximum theorem in this case).

We will give a set of definitions of standard geometric notions, necessary for the presentation of the Noether theorem. Let us consider a differential manifold M and its tangent space TM. The reader can take M as the Euclidian space $\mathbb{R}^n$ or S; then, TM is identified as the space of the partial derivatives over $\mathbb{R}^n$, which is isomorphous to $\mathbb{R}^n \times \mathbb{R}^n$.

Definition 1. A vector field X is a map which associates to $s \in M$, an element of the tangent space TM such that the natural projection of this element on M is s.

Let us take $M = \mathbb{R}^n$. Let $(s_1, \dots, s_n, X_1(s_1, \dots, s_n), \dots, X_n(s_1, \dots, s_n))$ denote the canonical coordinates of the vector field X in $\mathbb{R}^n \times \mathbb{R}^n$. If g is a differentiable function of $s \in M$, we define Xg as

$$Xg = \sum_{i=1,\dots,n} X_i \frac{\partial g}{\partial s_i}; \tag{4}$$

in this case $X(s)$ identifies with a derivation.

Definition 2. We call an integral curve of X, from $p \in M$, any trajectory defined over a maximal temporal domain such that $\forall i$, $\dot{s}_i(t) = X_i(s(t))$ and $s(0) = p$.

Assume henceforth that there exists $\epsilon > 0$ such that for all $p \in M$, there is an integral curve $s(t)$ of X from p, defined at least for $t \in]-\epsilon, +\epsilon[$. Let us take $\Psi_t(p) = s(t)$. We thus define a family of maps Ψ_u, $u \in]-\epsilon, +\epsilon[$ from M into M. It is straightforward that this family forms a local u-parameter group of maps on M. The local group (Ψ_u) is said to be generated by X.

Conversely, if we consider a sufficiently regular one-parameter group of maps (Ψ_u), then

$$X(p) = \frac{d\Psi_u}{du}(p)\bigg|_{u=0} \tag{5}$$

defines a vector field on M that generates (Ψ_u).

Finally, $T\Psi_u$ will denote the tangent map of Ψ_u. Let us recall that for $M = S$ or $\mathbb{R}^n$, $T\Psi$ is the map that associates to a tangent vector $(x_1, \dots, x_n)$ at a point s of M, the vector $(y_1, \dots, y_n)$ tangent at Ψs,

$$y_i = \sum_{j=1}^{n} \frac{\partial \Psi_i}{\partial s_j}(s) x_j. \tag{6}$$

Example 1. Consider $M = \mathbb{R}^n$. Let X be the vector field on $\mathbb{R}^n$ generated by the matrix $n \times n$ A, i.e., $X(s) = As$ for all $s \in M$. The one-parameter group generated by X is thus the set of the endomorphisms of $\mathbb{R}^n$, $\Psi_u = \exp(uA)$, $u \in \mathbb{R}$. If x is a tangent vector at the point s then $T\Psi_u x = \exp(uA)x$.

3. Noether's theorem in optimal control

Let us first consider the autonomous version of the program (P), i.e., we maximize

$$V = \int_0^T U\big(s(t), c(t)\big)\, dt \tag{7}$$

under the dynamical constraint

$$\dot{s}(t) = f\big(s(t), c(t)\big), \quad c(t) \in C. \tag{8}$$

Getting inspiration from the notion of symmetry (or invariance) for a Lagrangian (cf. [21]), we will define a "symmetry" for a problem in optimal control.

Definition 3. We call a symmetry for the autonomous program, a local one-parameter group (Ψ_u) (generated by the vector field X) acting on S such as $\forall u$, $\forall c \in C$ and $\forall s \in S$,

(1) there exists c_u such that

$$T\Psi_u\big(f(\Psi_{-u}s, c)\big) = f(s, c_u), \tag{9}$$

and

(2) $$U(\Psi_u s, c_u) = U(s, c). \tag{10}$$

We are now able to give Noether's theorem in optimal control.

Theorem 1 (Noether's theorem in optimal control). *If the local group (Ψ_u) is a symmetry for the autonomous optimal program, then along the optimal path (s^*, c^*, λ^*),*

$$\sum_i \lambda_i^* X_i \tag{11}$$

is invariant.

Proof. The derivative of (10) with respect to u gives

$$\frac{dU(\Psi_u s, c_u)}{du} = 0. \tag{12}$$

Using the chain rule for $u = 0$, it comes that

$$\sum_i \left\{ X_i \frac{\partial U}{\partial s_i}\big(s(t), c(t)\big) + \sum_l \frac{\partial U}{\partial c_l}\big(s(t), c(t)\big) \frac{d(c_u)_l}{du}(t) \right\} = 0. \tag{13}$$

Therefore

$$\frac{dU}{du}\big(s(t), c_u(t)\big)\Big|_{u=0} = -(XU)\big(s(t), c(t)\big). \tag{14}$$

Let us take the vector

$$(T\Psi_u)\big(f(\Psi_{-u}s, c)\big) = f(s, c_u).$$

Its derivation with respect to u shows that

$$\frac{df}{du}\big(s(t),c_u(t)\big)\bigg|_{u=0}=\left(\sum_j\frac{\partial X_i}{\partial s_j}\big(s(t)\big)f_j\big(s(t),c(t)\big)-\frac{\partial f_i}{\partial s_j}\big(s(t),c(t)\big)X_j\big(s(t)\big)\right)_i. \tag{15}$$

Indeed,

$$\begin{aligned}\frac{d}{du}f(\Psi_{-u}s,c)&=\left(\frac{d}{du}\left[\sum_j\frac{\partial\Psi_u^i}{\partial s_j}\times f_j(\Psi_u s,c)\right]\right)_i\\&=\left(\sum_j\left[\frac{d}{du}\left(\frac{\partial\Psi_u^i}{\partial s_j}\right)\times f_j(\Psi_u s,c)+\frac{\partial\Psi_u^i}{\partial s_j}\times\frac{d}{du}f_j(\Psi_u s,c)\right]\right)_i,\end{aligned} \tag{16}$$

and

$$\frac{d}{du}\left(\frac{\partial\Psi_u^i}{\partial s_j}\right)\bigg|_{u=0}=\frac{\partial}{\partial s_j}\left(\frac{d}{du}\Psi_u^i\right)\bigg|_{u=0}=\frac{\partial}{\partial s_j}X_i, \tag{17}$$

$$\frac{d}{du}f_j(\Psi_u s,c)\bigg|_{u=0}=\sum_k-X_k\frac{\partial f_j}{\partial s_k}(s,c). \tag{18}$$

The second member of (15) is the Lie bracket of X and of f[1] noted $[X,f](s(t),c(t))$ (see [21] or [4]). The reader can consider here that this is just a notation. Now, let us take place on the optimal path (s^*,c^*,λ^*) at a given date t. Consider the function of u, $\mathcal{H}(u)=p_0U(s^*,c_u^*)+\lambda^*f(s^*,c_u^*)$. According to the previous derivatives at $u=0$,

$$\frac{d\mathcal{H}(u)}{du}\bigg|_{u=0}=-p_0(XU)\big(s^*(t),c^*(t)\big)+\lambda^*(t)[X,f]\big(s^*(t),c^*(t)\big). \tag{19}$$

But the left member of this relation is merely the temporal derivative at t of $\sum_i X_i\lambda_i^*$ along the optimal path. Indeed, the maximum principle implies that

$$\dot\lambda_i^*=-\frac{\partial H}{\partial s_i}=-\frac{\partial U}{\partial s_i}p_0-\lambda^*\frac{\partial f}{\partial s_i}. \tag{20}$$

Besides

$$\dot X_i=\sum_{j=1}^n\frac{\partial X_i}{\partial s_i}\dot s_j=\sum_{j=1}^n\frac{\partial X_i}{\partial s_i}f_j. \tag{21}$$

And therefore, using the chain rule, it comes that

$$\frac{d}{dt}\lambda_i^*X_i=-p_0\frac{\partial U}{\partial s_i}X_i-X_i\lambda^*\frac{\partial f}{\partial s_i}+\lambda_i^*\left(\sum_j\frac{\partial X_i}{\partial s_j}f_j\right), \tag{22}$$

[1] Because f is equal to the temporal derivative of the state variable, f should be seen not as a function but as an element of the tangent space.

thus

$$\left.\frac{d\mathcal{H}(u)}{du}\right|_{u=0}=\frac{d}{dt}\sum_{i}^{n}X_i\lambda_i^*. \tag{23}$$

Finally, in order to prove the theorem, it is sufficient to show that $(d\mathcal{H}(u)/du)|_{u=0}=0$.

The maximum theorem states that the optimal control variable maximizes almost everywhere the Hamiltonian. One has therefore a.e.

$$p_0U(s^*,c^*)+\lambda^*f(s^*,c^*)\geqslant p_0U(s^*,c_u)+\lambda^*f(s^*,c_u). \tag{24}$$

Because $\mathcal{H}(0)=H(s^*,c^*,\lambda^*)$, $\mathcal{H}(u)$ reaches its maximum at $u=0$, i.e.,

$$\left.\frac{d\mathcal{H}(u)}{du}\right|_{u=0}=0.\quad\square \tag{25}$$

Before giving some economic examples in Section 4, we will see how the theorem can be generalized for non-autonomous programs. The method is inspired by the approach used in [21] or [2, p. 95, Exercise 4], for the extension of Noether's theorem to non-autonomous Lagrangian systems.

Take the non-autonomous program (P) described in Section 2. We parameterize time and consider that it is a new state variable (next to s). Consider the program (P′):

$$\max\int_0^T U\big(s(z),c(z),t(z)\big)v\,dz \tag{26}$$

under

$$\frac{ds}{dz}=vf\big(s(z),c(z),t(z)\big) \tag{27}$$

and

$$\frac{dt}{dz}=v \tag{28}$$

with the new control variable $v\in\,]1/2,2[$ (one could take any open set around 1); and the constraints $t(z=0)=0$ and $t(z=T)=T$. Program (P′) is thus autonomous. Now, note that if $(s^*(t),c^*(t))$ is an optimal solution of program (P), then necessarily $((s^*(z),t=z),(c^*(z),1))$ is an optimal solution of (P′). Therefore, if a symmetry Ψ_u exists for program (P′), with generators $X=((X_i)_{i=1,\dots,n},X_{n+1})$, then, by applying Noether's theorem along the optimal path, one finds the invariance of $\sum_{i=1,\dots,n}X_i\lambda_i^*+X_{n+1}\lambda_{n+1}^*$, where λ_i are for $i=1,\dots,n$ the co-state variables for (P′) identical to those of (P) along the optimal path and where λ_{n+1} is the co-state variable associated with time in program (P′). An one-parameter group of symmetries for (P′) will be said a symmetry for the non-autonomous program (P).

The Hamiltonian of the autonomous program (P′) is

$$\begin{aligned}\widetilde{H}\big((s,t),(c,v),(\lambda,\lambda_{n+1})\big)&=p_0vU(s,c,t)+\lambda f(s,c,t)v+\lambda_{n+1}v\\&=v\big\{p_0U(s,c,t)+\lambda f(s,c,t)+\lambda_{n+1}\big\}.\end{aligned} \tag{29}$$

It is therefore v-linear. But the maximum principle applied to (P′) induces that this Hamiltonian is maximal when the control v is equal to 1 on the optimal path. Hence $\widetilde{H}$ is necessarily null along this path. Consequently,

$$U(s^*, c^*, t) + \lambda^* p_0 f(s^*, c^*, t) + \lambda^*_{n+1} = 0, \tag{30}$$

i.e.,

$$\lambda^*_{n+1}(t) = -H\big(s^*(t), c^*(t), \lambda^*, t\big), \tag{31}$$

where H is the Hamiltonian of program (P). The invariant quantity can be rewritten $\sum_{i=1,\dots,n} X_i \lambda^*_I - X_{n+1} H(s^*(t), c^*(t), \lambda^*, t)$. A corollary of Noether's theorem for a non-autonomous program follows.

Corollary 1. *If Ψ_u is a symmetry for the non-autonomous program* (P) *then along the optimal path* (s^*, c^*, λ^*)*,*

$$\left\{ \sum_{i\in[1,n]} \lambda^*_i X_i \right\} - H(s^*, c^*, \lambda^*, t) X_{n+1} \tag{32}$$

is invariant, where H is the Hamiltonian of program (P)*.*

4. Applications in economics

We will now give applications of the previous version of the Noether theorem to basic models of economic growth.

4.1. Ramsey's program without preference for present

The Ramsey model is the first step of the analyses of optimal economic growth (cf. [3]). Consider a central planner that maximizes welfare of a representative consumer during a period $[0, T]$,

$$\int_0^T U\big(c(t)\big)\, dt, \tag{33}$$

where c is the consumption and U is the current utility of the consumer. The accumulation of production capital k is

$$\dot{k}(t) = f\big(k(t)\big) - \delta k(t) - c(t), \tag{34}$$

where f is the production function and δ is the depreciation of capital. This equation basically expresses that total net output is split between consumption and investment $I(t) = \dot{k}(t)$. The functions f and U are taken sufficiently regular. We can interpret this program as a non-autonomous program and parameterize time with z,

$$\max \int_0^T U\big(c(z)\big)\frac{dt}{dz}\,dz, \tag{35}$$

$$\frac{dk(z)}{dz} = v\big[f\big(k(z)\big) - \delta k(z) - c(z)\big], \tag{36}$$

and

$$\frac{dt}{dz} = v, \quad v \in]1/2, 2[. \tag{37}$$

Let us consider the one-parameter group G acting on $\mathbb{R}^2$,

$$\forall u \in \mathbb{R}, \quad u \cdot t = t + u \quad \text{and} \quad u \cdot k = k.$$

The group G is a symmetry for the parameterized program. The non-autonomous version of Noether's theorem induces the well-known result that the Hamiltonian H is constant along the optimal path. Recall that H can be interpreted as the income in terms of welfare (cf. [22]).

4.2. N-dimensional optimal growth

We will show with the following example that the Noether theorem can provide "non-obvious" conservation laws and can simplify the study of optimal growth programs.

Consider an $(n+1)$-sectors economy. The sector 0 produces final consumption goods c using n capital goods. For $i > 0$, the output of the sector i is capital goods of type i (e.g., computers, instruments, structures, or human capital).

Let $s_{i,j}$[2] be the share of capital stock k_j devoted to the sector i; $\forall j$, $\sum_i s_{i,j} = 1$ and $s_{i,j}$ is non-negative. Let δ_i denote the depreciation of capital stock k_i. The accumulation of capital verifies $\forall i > 0$,

$$\dot{k}_i = f_i(s_{i,1}k_1, \ldots, s_{i,n}k_n) - \delta_i k_i. \tag{38}$$

Consumption goods are also produced using capital inputs

$$c = f_0(s_{0,1}k_1, \ldots, s_{0,n}k_n). \tag{39}$$

The functions f_j are sufficiently regular (see Section 2), convex, and 1-homogeneous maps.

The central planner maximizes intertemporal welfare

$$\max \int_0^\infty e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma}\,dt, \tag{40}$$

where $\rho > 0$ is the rate of time preference (or the subjective discount rate) and σ is the intertemporal elasticity of substitution. The current utility $U = c^{1-\sigma}/(1-\sigma)$ has a constant relative risk aversion σ.

[2] Here s does not designate state variables but controls *shares*.

This general framework includes numerous models of multi-sectors endogenous growth [5,9,13] and central planner versions of growth models [3,10].

Let $\mathcal{H}(k_i, s_{i,j}, \lambda_i, t)$ be the Hamiltonian of this program. Again, we parameterize time. The program becomes

$$\max \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} v \, dz, \tag{41}$$

under $\forall i > 0$, $dk_i/dz = v[f(s_{i,j}) - \delta_i k_i]$ and $dt/dz = v \in]1/2, 2[$.

Consider the one-parameter group (Ψ_u) acting on $\mathbb{R}^{n+1}$ by $\forall u \in \mathbb{R}$,

$$\Psi_u \cdot \big(t, (k_i)\big) = \big(t + u, e^{\rho u/(1-\sigma)}(k_i)\big). \tag{42}$$

Taking $v_u = v$ and $\forall i, j$, $s_{(i,j),u} = s_{i,j}$, the group (Ψ_u) is a symmetry for the program. Hence, the Noether theorem yields the following conservation law

$$-\mathcal{H} + \frac{\rho}{1-\sigma} \sum_{i=1}^{i=n} k_i \lambda_i \text{ is invariant along interior optimal paths.}$$

Therefore, one dynamical variable can be expressed as a function of the other current control, co-state and state variables. Because of the symmetry (Ψ_u), the dimension of the dynamical system is reduced.

For example, this result can be applied to the Mankiw et al. [10] framework. Consider an economy with the aggregated production function

$$Q(t) = K(t)^\alpha H(t)^\beta \big(A(t)L(t)\big)^\gamma,$$

where K is the physical capital, H is the human capital, A is the exogenous technical progress, L is the labor, and $\alpha + \beta + \gamma = 1$. The production includes consumption goods and investments in human or physical capital. The growth rate of the population is n. Let us assume that the technological progress is Dixit–Stiglitz, $A = \theta K^\mu H^{1-\mu}$, where θ is a constant. Small letters will refer to values per capita. Thus $q = \theta^\gamma k^{1-\beta'} h^{\beta'}$, where $\beta' = \beta + \gamma(1-\mu)$. Assume that the accumulations of capitals verify

$$\dot{k} = s_1 q - k(\delta_K + n) \tag{43}$$

and

$$\dot{h} = s_2 q - h(\delta_H + n), \tag{44}$$

where the parameters δ are the rates of depreciation of physical and human capitals and s are the saving rates or the shares of production devoted to physical investments and education or training. The amount of consumption goods is thus

$$c = (1 - s_1 - s_2)q. \tag{45}$$

We introduce a central planner who sets the investments s_1 and s_2; he/she maximizes the welfare $\int_0^\infty e^{-\rho t}(c^{1-\sigma}/(1-\sigma))\,dt$. Let $\mathcal{H}_{\mathcal{MRW}}(k, h, s_1, s_2, \lambda_1, \lambda_2, t)$ be the Hamiltonian of this program.

In this case, the conservation law becomes $-\mathcal{H}_{\mathcal{MRW}} + (\rho/(1-\sigma))(k\lambda_1 + h\lambda_2)$ is constant along interior optimal paths. Let us recall that the maximum principle implies that $\partial\mathcal{H}_{\mathcal{MRW}}/\partial s_i = 0$, i.e., $\lambda_i = -e^{-\rho t}(\partial U/\partial q s_i) = e^{-\rho t}c^{-\sigma}$; therefore

$$\mathcal{H}_{\mathcal{MRW}} - \frac{\rho e^{-\rho t}}{1-\sigma}c^{-\sigma}(h+k) \tag{46}$$

is invariant.

In economic terms, the Hamiltonian $\mathcal{H}_{\mathcal{MRW}} = e^{-\rho t}U + \lambda_1\dot{k} + \lambda_2\dot{h}$ represents the discounted utility measure of the sum of consumption and investment, i.e., the discounted utility measure of income. Furthermore, the marginal increase in the total value of the program (beginning at date t) due to a marginal increase in k or h is equal to $e^{\rho t}\lambda$ (see [8]); hence, the co-state variables λ can be interpreted as the shadow discounted price of capital. The quantity $\lambda(h+k)$ is then the discounted "value" of human and physical capital stocks. This drives an economic interpretation of the conservation law (46),

$$\text{discounted “income”} - \frac{\rho}{1-\sigma} \times \text{ discounted “value” of capital stocks } = \text{ constant.}$$

The economic path should also verify the transversality conditions $\lim_{+\infty}\lambda_1 k = \lim_{+\infty}\lambda_2 h = 0$, i.e., the discounted value of capital stocks vanishes when t tends to infinity (cf. [8,11]). Under this condition, the left side of the conservation law tends to zero while it is constant; this quantity is thus null along the path. Consequently, the conservation law becomes

$$\text{current “income” } = \frac{\rho}{1-\sigma} \times \text{ current “value” of the capital stocks.}$$

Finally, because

$$\mathcal{H}_{\mathcal{MRW}} = e^{-\rho t}c^{-\sigma}\left[\frac{c}{1-\sigma} + s_1 q + s_2 q - k(\delta_K + n) - h(\delta_H + n)\right]$$

and $c = (1 - s_1 - s_2)q$, the conservation law can be written as

$$\frac{c}{1-\sigma} + s_1 q + s_2 q - k(\delta_K + n) - h(\delta_H + n) = \frac{\rho}{1-\sigma}(h+k).$$

This relation gives the value of the total current saving rate $\tilde{s} = s_1 + s_2$,

$$\sigma\tilde{s}q = q - \big[(1-\sigma)(\delta_K + n) + \rho\big]k - \big[(1-\sigma)(\delta_H + n) + \rho\big]h. \tag{47}$$

Here, the "forward looking" control variable $\tilde{s}$ explicitly depends only on the current values of the capital stocks.

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