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Compositions of analytic functions of the form $F_n(z) = F_{n-1}(f_n(z))$, $f_n(z) \to f(z)$

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Abstract: Frequently, in applications, a function is iterated in order to determine its fixed point, which represents the solution of some problem. In the variation of iteration presented in this paper fixed points serve a different purpose. The sequence $\{F_n(z)\}$ is studied, where $F_1(z) = f_1(z)$ and $F_n(z) = F_{n-1}(f_n(z))$, with $f_n \to f$. Many infinite arithmetic expansions exhibit this form, and the fixed point, α, of f may be used as a modifying factor $(z = \alpha)$ to influence the convergence behaviour of these expansions. Thus one *employs*, rather than *seeks* the fixed point of the function f.

Keywords: Limit periodic iteration, continued fractions, attractive fixed points.

Introduction

The purpose of this paper is to describe an important variation of iteration that appears frequently in the study of arithmetic infinite expansions and that may have applications in other kinds of functional approximations, as well as in investigations of dynamical behavior.

Consider a sequence of functions $\{f_n\}$ and an associated sequence of regions $\{D_n\}$ where

(i) For each n, f_n is analytic in D_n with $f_n(D_n)$ contained in D_{n-1}.
(ii) $\bigcap D_n \neq \emptyset$ and there is a domain D that lies in $\bigcap D_n$.
(iii) $f_n(z) \to f(z)$ on D.

For $z \in D$ define $F_1(z) = f_1(z)$ and $F_n(z) = F_{n-1}(f_n(z))$ for $n > 1$. The sequence $\{F_n(z)\}$ will be designated *limit periodic iteration*, since it represents a generalization of the compositional structure of limit periodic continued fractions (see Example 2).

We shall require two additional conditions in the theory developed in the present paper:

(iv) Let $f_n \to f$ uniformly on closed regions contained in D.
(v) There exists a simple closed contour Γ contained in D such that $f(\Omega)$ is contained in $\mathrm{Int}(\Omega)$, where $\Omega = \Gamma \cup \mathrm{Int}(\Gamma)$.

This last condition insures the existence of at least one point α in Ω such that $f(\alpha) = \alpha$.

Normal iteration occurs when $f_n \equiv f$. When, in addition, f is a contraction on Ω it follows that $F_n(z) \to \alpha$, the "attractive" fixed point of f.

We shall show that this behavior is paralleled in the more general setting in which $f_n \to f$. That is to say, $F_n \to \lambda$, a constant in D_1 for all $z \in \Omega$. Thus, in a certain sense, the best possible "accelerating factor" z to use in $\{F_n(z)\}$ is $z = \alpha$.

The theory developed in this paper generalizes results from the more confined framework of continued fractions (Magnus and Mandel [5], Gill [1], Thron and Waadeland [6]). The approach established in [5] is particularly helpful in proving Theorem 1, which extends (with slightly stronger hypotheses) a result obtained by Henrici [3, p.524].

It is of particular importance to observe that should λ exist, its value is highly dependent upon the characteristics of the f_n's. If one considers the more"natural" composition

$$G_n(z) = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1(z),$$

then it is not difficult to show that frequently $G_n(z) \to \alpha$, the attractive fixed point of f, regardless of the specific forms of the f_n's. Thus, with regard to variety of functional expansions it appears the $F_n(z)$ is of considerably greater importance that $G_n(z)$, and that the limit periodic format provides access to functions that are more sophisticated than those that can be expanded in simple iterative fashion.

The following elementary examples demonstrate limit periodic and reverse limit periodic "perturbations" of iterative expansions.

Example 1. $F(\zeta) = \zeta - \zeta^2 + \zeta^3 - \cdots = \zeta/(1+\zeta)$ if $|\zeta| < 1$. By setting $f_n(z) = 1 - \zeta z$ we may write $F(\zeta) = \lim_{n\to\infty} F_n(z)$ for all $z \in \mathbb{C}$. I.e., $F(\zeta) = \zeta\ \alpha(\zeta)$, where $\alpha(\zeta)$ is the attractive fixed point of $f(z) = 1 - \zeta z$. Now consider $f_n(z) = 1 - (1 - 1/n)\zeta z$ for $n > 1$ and $f_1(z) = \zeta z$. Then $\lim_{n\to\infty} F_n(z) = \mathrm{Ln}(1+\zeta)$, $|\zeta| < 1$, for all $z \in \mathbb{C}$. However, $\lim_{n\to\infty} G_n(z) = 1/(1+\zeta) = \alpha(\zeta)$ for all such z's. Convergence can be enhanced by using $z = \alpha(\zeta)$. For instance, $|F_{50}(0) - \ln 1.95| = 8.0 \times 10^{-4}$, whereas $|F_{50}(\alpha) - \mathrm{Ln}\ 1.95| = 7.6 \times 10^{-6}$.

Or, let $f_n(z) = 1 + (1/n)\zeta z$ for all n. Then $\lim_{n\to\infty} F_n(z) = e^{\zeta}$ for all $z \in \mathbb{C}$. Again, we find that $\lim_{n\to\infty} G_n(z) = \alpha(\zeta) = 1$.

Example 2. The approximants of the periodic continued fraction

$$F(\zeta) = \frac{\zeta/4|}{|1} + \frac{\zeta/4|}{|1}$$

can be interpreted as $\{F_n(0)\}$ where $f_n(z) = (\zeta/4)/(1+z) = f(z)$. It is well known that $F_n(z) \to \alpha(\zeta)$, where $\alpha(\zeta) = (\sqrt{1+\zeta} - 1)/2$, $\mathrm{Re}\sqrt{1+\zeta} > 0$, for all values of z with the exception of $z = -(\alpha(\zeta) + 1)$, [5].

By setting $f_n(z) = a_n(\zeta)/(1+z)$,where $a_n(\zeta) = n^2\zeta^2/(4n^2 - 1)$, we find that the resulting *limit periodic* continued fraction [4]

$$F\zeta) = \frac{\zeta^2/3|}{|1} + \frac{4\zeta^2/15|}{|1} + \cdots = \lim_{n\to\infty} F_n(z) = \zeta(\mathrm{Arctan}\ \zeta)^{-1} - 1$$

for $z = 0$ and $z = \alpha(\zeta)$, and convergence is improved by using $\alpha(\zeta)$, [6]. For example,

$$|F_4(0) - F(2.8)| = 1.2 \times 10^{-1} \quad \text{and} \quad |F_4(\alpha) - F(2.8)| = 6.1 \times 10^{-4}.$$

The convergence theorem

Let us assume that conditions (i) – (v), given in the introduction, are satisfied. Thus f has at least one fixed point in Ω.

Theorem 1. *Assuming* (i)–(v), *if* α, *a fixed point of* f *in* Ω, *is such that* $|f(z)-\alpha| < |z-\alpha|$ *for all* $z \in \Gamma$, *then* α *is the unique fixed point of* f *in* Ω, *and there exists a constant* λ *in* D_1 *where*

$$\lim_{n\to\infty} F_n(z_n) = \lambda \quad \textit{for all } \{z_n\} \textit{ in } \Omega.$$

Comment: In order to simplify the following proof it will be assumed that $z_n \equiv z$. The details can be easily adjusted for the more general case.

Proof. That α is the only fixed point of $f(z)$ follows from Rouche's theorem: let $G(z) = f(z) - \alpha$ and $F(z) = z - \alpha$. Then $|G(z)| < |F(z)|$ for $z \in \Gamma$. Hence, $G(z) + F(z) = f(z) - z$ and $F(z) = \alpha - z$ each have one zero inside Γ. Now, set $\Gamma^* = \{w : w = z - \alpha,\ z \in \Gamma\}$ and $\Omega^* = \Gamma^* \cup \operatorname{Int} \Gamma^*$. Then $h(w) = f(w+\alpha) - \alpha$ is analytic on Ω^*, with $h(0) = 0$ in Ω^*. Also, $|h(w)| < |w|$ on Γ^*. Setting $H(w) = h(w)/w$ for $w \neq 0$ and $H(0) = h'(0)$, we find that $\sup_{w\in\Omega^*} |H(w)| < 1$, so that $|f(z) - \alpha| < |z - \alpha|$ for all $z \in \Omega$, and $|f'(\alpha)| = k_1 < 1$.

Thus, there exists $K \in [0,1)$ such that $|f(z) - \alpha| < K|z - \alpha|$ for all $z \in \Omega$. Now, Ω is contained in D and $f_n(z) \to f(z)$ uniformly on compact subsets of D. Therefore, $f_n'(z) \to f'(z)$ uniformly on Ω. Since $|f'(\alpha)| = k_1 < 1$, there exists $\delta > 0$ such that $|f'(z)| < k_2 < 1$ if $|z - \alpha| < \delta$.

For all n sufficiently large

$$|f_n'(z)| - |f'(z)| \leqslant |f_n'(z) - f'(z)| < (1 - k_2)/2 \quad \text{when } |z - \alpha| < \delta.$$

Hence, $|f_n'(z)| < k_2 + (1 - k_2)/2 = k < 1$ for $|z - \alpha| < \delta$ and n large. Therefore, for z_1 and z_2 in $(|z - \alpha| < \delta)$ and n large,

$$|f_n(z_1) - f_n(z_2)| \leqslant \int_{z_2}^{z_1} |f_n'(z)| \cdot |dz| \leqslant k|z_1 - z_2|, \quad 0 \leqslant k < 1.$$

I.e., each f_n contracts in the vicinity of α when n is large.

Next, we show that z in Ω is drawn into $(|z - \alpha| < \delta)$ by $F_{n,n+m}(z) \equiv f_{n+1} \circ f_{n+2} \circ \cdots \circ f_{n+m}(z)$ for n and m sufficiently large.

Choose n_0 so large that $n \geqslant n_0$ implies $f_n(\Omega)$ is contained in Ω,

$$|f_n(z) - f(z)| < \epsilon = \delta(1 - K)/2 \quad \text{for all } z \in \Omega,$$

and

$$|f_n(z_1) - f_n(z_2)| \leqslant k|z_1 - z_2| \quad \text{for } z_1, z_2 \in (|z - \alpha| < \delta).$$

Then

$$\begin{aligned} |F_{n,n+m}(z) - \alpha| &\leqslant |f_{n+1}(F_{n+1,n+m}(z)) - f(F_{n+1,n+m}(z))| + |f(F_{n+1,n+m}(z) - \alpha| \\ &< \epsilon + K|F_{n+1,n+m}(z) - \alpha| \\ &\ \vdots \\ &< \epsilon + \epsilon K + \cdots + \epsilon K^{m-1} + K^m|z - \alpha| \\ &< \epsilon/(1 - K) + K^m \sup_{z\in\Omega}|z - \alpha| < \delta, \end{aligned}$$

when m is sufficiently large.

Thus, for all $n \geqslant n_0$ and all $z \in \Omega$ there exists m_0, where $m \geqslant m_0$ implies $|F_{n,n+m}(z) - \alpha| < \delta$. If $z \in (|z-\alpha| < \delta)$, then

$$|f_n(z) - \alpha| \leqslant |f_n(z) - f(z)| + |f(z) - \alpha| < \delta(1-K)/2 + K\delta < \delta,$$

so that $f_n(z) \in (|z-\alpha| < \delta)$.

In other words, $z \in \Omega$ can be pulled into $(|z-\alpha| < \delta)$ by certain composition chains of f_n's and kept there by the actions of additional f_n's if $n \geqslant n_0$.

We then have, for any $z \in \Omega$, fixing $n = n_0$ and $m = m_0$:

$$\begin{aligned} &|F_{n,n+m+p}(z) - F_{n,n+m+p+q}(z)| \\ &\quad = \left|F_{n,n+p-1}(F_{n+p,n+p+m}(z)) - F_{n,n+p-1}\left(F_{n+p,n+p+m}\left(F_{n+m+p+1,n+m+p+q}(z)\right)\right)\right| \\ &\quad \leqslant k^p(2\delta) \to 0 \quad \text{as } p \to \infty. \end{aligned}$$

Hence, $\{F_{n,n+m+p}(z)\}_{p=1}$ converges.

Since

$$\begin{aligned} &|F_{n,n+m+p}(z_1) - F_{n,n+m+p}(z_2)| \\ &\quad = \left|F_{n,n+p-1}(F_{n+p,n+m+p}(z_1)) - F_{n,n+p-1}(F_{n+p,n+m+p}(Z_2))\right| \\ &\quad \leqslant k^p(2\delta) \to 0 \quad \text{as } p \to \infty \quad \text{for } z_1, z_2 \in \Omega, \end{aligned}$$

we see that $\lim_{p\to\infty} F_{n,n+m+p}(z) = w_0$ for all $z \in \Omega$. Consequently, $\lim_{r\to\infty} F_r(z) = F_n(w_0) = \lambda$ for all $z \in \Omega$, and the proof of Theorem 1 is complete.

The use of $z = \alpha$ in computing λ is, in a sense, a best possible accelerating procedure since $F_n(\alpha) \equiv \lambda = \alpha$ if $f_n \equiv f$. In a more general setting it is still possible to demonstrate convergence acceleration.

Theorem 2. *If* (i) *The hypotheses of Theorem* 1 *are satisfied,* (ii) $f_n(\Omega)$ *is contained in* Ω *for all* n, (iii) *there exists* $0 < \rho < r < 1$ *such that*

$$\rho|z_1 - z_2| \leqslant |f_n(z_1) - f_n(z_2)| \leqslant r|z_1 - z_2| \quad \textit{for all } n \textit{ and all } z_1,\ z_2 \in \Omega,$$

and (iv) $|f_n(z) - f(z)| < \epsilon(n)$ *for all* $z \in \Omega$ *and all* n, *where* $\epsilon(n)\downarrow$ *and* $\epsilon(n) = o((\rho/r)^n)$, *then, for* $z \in (|z-\alpha| > \epsilon(1)/(1-K))$, *we have*

$$|F_n(\alpha) - \lambda| \leqslant \epsilon(n+1)(r/\rho)^n B(z)|F_n(z) - \lambda|.$$

where

$$B(z) = ((1-K)(|z-\alpha| - \epsilon(1)/(1-K)))^{-1} = ((1-K)|z-\alpha| - \epsilon(1))^{-1}.$$

Proof. We write

$$\begin{aligned} \left|\frac{F_n(\alpha) - \lambda}{F_n(z) - \lambda}\right| &= \left|\frac{F_n(\alpha) - F_n\left(\lim_{m\to\infty} F_{n+1,n+m}(\alpha)\right)}{F_n(z) - F_n\left(\lim_{m\to\infty} F_{n+1,n+m}(\alpha)\right)}\right| \\ &\leqslant \frac{r^n|\alpha - \lim_{m\to\infty} F_{n+1,n+m}(\alpha)|}{\rho^n|z - \lim_{m\to\infty} F_{n+1,n+m}(\alpha)|} \end{aligned}$$

where

$$\begin{aligned}
| F_{n+1,n+m}(\alpha) - \alpha | &\leqslant | f(F_{n+2,n+m}(\alpha)) - \alpha | + | f_{n+1}(F_{n+2,n+m}(\alpha)) - f(F_{n+2,n+m}(\alpha)) | \\
&\leqslant K | F_{n+2,n+m}(\alpha) - \alpha | + \epsilon(n+1) \\
&\leqslant K\big[| f(F_{n+3,n+m}(\alpha)) - \alpha | \\
&\qquad + | f_{n+2}(F_{n+3,n+m}(\alpha)) - f(F_{n+3,n+m}(\alpha)) | \big] + \epsilon(n+1) \\
&\leqslant K^2 | F_{n+3,n+m}(\alpha) - \alpha | + K\epsilon(n+2) + \epsilon(n+1) \\
&\ \ \vdots \\
&\leqslant K^m | \alpha - \alpha | + K^{m-1}\epsilon(n+m) + \cdots + K\epsilon(n+2) + \epsilon(n+1) \\
&\leqslant \epsilon(n+1)/(1-K) < \epsilon(1)/(1-K).
\end{aligned}$$

Hence

$$\begin{aligned}
\left| \frac{F_n(\alpha) - \lambda}{F_n(z) - \lambda} \right| &\leqslant \left(\frac{r}{\rho}\right)^n \cdot \frac{\epsilon(n+1)/(1-K)}{| z - \alpha | - | \alpha - \lim_{m\to\infty} F_{n+1,n+m}(\alpha) |} \\
&\leqslant \left(\frac{r}{\rho}\right)^n \cdot \frac{\epsilon(n+1)}{(1-K) | z - \alpha | - \epsilon(1)} \leqslant \epsilon(n+1) \cdot \left(\frac{r}{\rho}\right)^n \cdot B(z).
\end{aligned}$$

This completes the proof of Theorem 2. □

Suppose, now, that there exists a compact, connected set E contained in $\mathbb{C}$ such that for $\zeta \in E$, $f_n(z) = f_n(\zeta, z)$, $f(z) = f(\zeta, z)$, and $f_n \to f$ uniformly (jointly on E and Ω). Under these conditions $\alpha = \alpha(\zeta)$. We shall further stipulate that $| f(\zeta, z) - \alpha(\zeta) | < | z - \alpha(\zeta) |$ for all $z \in \Gamma$ and all $\zeta \in E$, and $f(\zeta, \Omega)$ is contained in $\text{Int}(\Omega)$ for all $\zeta \in E$.

By carefully following the proof of Theorem 1, keeping in mind that both E and Ω are compact, it is possible to infer the existence of $\delta > 0$ such that for all n sufficiently large, all $z_k \in N_\delta(\alpha(\zeta))$, and all $\zeta \in E$,

$$| f_n(\zeta, z_1) - f_n(\zeta, z_2) | \leqslant k | z_1 - z_2 | \quad \text{where } 0 \leqslant k < 1.$$

It then follows, after a rather tedious analysis, that $F_n(\zeta, z) \to \lambda(\zeta)$ uniformly on E for each $z \in \Omega$, so that $\lambda(\zeta)$ is analytic on E. These conditions are exhibited in the following example.

Example 3. Let $f_n(z) = 1 + \nu_n/(1+z) + (1 - 1/\zeta)z$, where $\nu_n \to 0$. Set $\Omega = (| z - 100 | \leqslant 100)$ and $E = (| \zeta - 100 | \leqslant 10)$. We find that $\alpha(\zeta) = \zeta$, and $| f(\zeta, z) - \alpha(\zeta) | < | z - \alpha(\zeta) |$ for all $\zeta \in E$ and $z \in \Gamma$. Furthermore, $| f(\zeta, z) - 100 | < 99.3$ which implies $f(\zeta, \Omega)$ lies in $\text{Int}(\Omega)$ for all $\zeta \in E$. Uniform convergence follows from $| f_n(\zeta, z) - f(\zeta, z) | \leqslant | \nu_n |$. For $\zeta = 100$ and $\nu_n = 1/2^n$ computation gives $| F_{400}(1) - \lambda | = 1.8$, $| F_{200}(50) - \lambda | = 6.7$, and $| F_{12}(100) - \lambda | = 5.0 \times 10^{-7}$.

Little is known about the conversion of a function to a limit periodic form other than a simple power series expansion (and its equivalent continued fraction), and a T-fraction expansion of a bounded function [2,7]. There is some evidence that continued fraction expansions of functions that satisfy certain Ricatti differential equations are limit periodic [4]. More information might support an argument for redefining the value of a limit periodic continued fraction: $\lim_{n\to\infty} F_n(\alpha)$, rather than $\lim_{n\to\infty} F_n(0)$.

Power series of functions are frequently limit periodic. Consider a function $G(\zeta)$ analytic in $(|\zeta| < R)$. Set $H_n = G^{(n)}(0)/nG^{(n-1)}(0)$ when $G^{(n-1)}(0) \neq 0$. If $G^{(n)}(0) \neq 0$ for all n and $\lim_{n\to\infty} H_n = L$, then $G(\zeta)$ has a limit periodic series expansion of the form $G(\zeta) = \lim_{n\to\infty} F_n(z)$, $z \in \mathbb{C}$, where $f_n(z) = 1 + H_n z\zeta \to 1 + Lz\zeta$. The expansion is valid in $(|\zeta| \leqslant R)$, with possible analytic continuation into $(|\zeta| < R^*)$, $R^* > R$, by using $z = \alpha(\zeta) = 1/(1 - L\zeta)$ [2].

An infinite product, $G(\zeta) = \prod g(n, \zeta)$, may be modified in a limit periodic setting by converting to log $G(\zeta)$ and writing $f_1(z) = z\log g(1, \zeta)$, $f_n(z) = 1 + zP(n, \zeta)$ for $n > 1$, where $P(n, \zeta) = \log g(n, \zeta)/\log g(n-1, \zeta)$. If $P(\zeta) = \lim_{n\to\infty} P(n, \zeta)$ exists and is not one, then $F_n(z)$ has limit periodic structure with $\alpha(\zeta) = 1/(1 - P(\zeta))$. E.g., the function $\prod(1 + \zeta^n)$ when modified with $\alpha(\zeta)$ shows a modest improvement in convergence of 10^{-2} for $n < 10$.

It is interesting to see the effect on the appearance of an expansion that occurs when α is used. For example, Euler's number γ, given by $\lim_{n\to\infty}[1 + 1/2 + \cdots + 1/n - \log n]$, may be determined by limit periodic analysis by setting $f_n(z) = 1 - n\log(1 + 1/n) + (n/(n+1))z$. Here $f(z) = z$, but $\alpha_n \to 1/2$ which we use as α to obtain

$$F_n(\alpha) = 1 + 1/2 + \cdots + 1/(n-1) + 1/2n - \log n.$$

Computation shows there is some improvement in convergence, simply by using $1/2n$ in place of $1/n$ in the standard definition of γ.

The last example demonstrates convergence of a "deviant" continued fraction. If $f(z) = \alpha\beta/(\alpha + \beta - z)$, where $|\alpha| < |\beta|$, then iteration of f produces a periodic continued fraction that displays the fixed points of $f(z)$ and converges to the attractor α. We modify the internal structure of this continued fraction to produce a rather bizarre arithmetic expansion.

Example 4. Let $f_n(z) = (2^{-n}z^2 + \alpha\beta)/(\alpha + \beta - z)$. Then $f_n \to f$ uniformly on $\Omega = (|z - 100| \leqslant 5.5)$, with $\alpha = 100$, $\beta = 110$. Formally, we compute $|F_{213}(0) - \lambda| = 1.0 \times 10^{-5}$ (the "continued fraction" approximant). Whereas, under the restraints of Theorem 1, $|F_{166}(95.5) - \lambda| = 1.0 \times 10^{-5}$ and $|F_{27}(\alpha) - \lambda| = 1.0 \times 10^{-5}$.

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