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NASH FUNCTIONS OVER REAL SPECTRA

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In this note we construct a sheaf of rings over real spectra which generalizes the common sheaf of Nash functions over a Nash set and we use it to get some Nullstellensätze for the geometric case.

Introduction

Writing about the ring of continuous semialgebraic functions over a constructible set X of the real spectrum of an arbitrary ring A (his $\mathscr{C}(X)$), Brumfield says [3]: ''A more complicated subring of $\mathscr{C}(X)$ is the subring $N(X)$ of Nash functions on X. Roughly, $N(X)$ consists of sections which are locally elements of rings obtained by adjoining simple roots of polynomials to A. Clearly, $A \subset [\ldots] \subset N(X) \subset \mathscr{C}(X)$ (even though it is not clear what $N(X)$ is precisely)''. It is however obvious that $N(X)$ differs from $\mathscr{N}_A(X)$, the ring of global sections over X of the structure sheaf for A, since in the general case, different sections give the same function over X (see [11]).

In this note, we shall construct a sheaf of rings over the real spectrum of A which generalizes the common sheaf of Nash functions over an affine Nash set. In the first part we give the basic properties of this sheaf, in the second some basic properties of the geometric case (i.e. Nash functions over Nash sets) and in the last we deduce several kinds of stellensätze.

1. Two sheaves of Nash functions over the real spectrum

In the sequel, A denotes a commutative ring with unit, and $\mathscr{N}_A$ the structural sheaf over $\mathrm{Spec}_r A$ (see [11] or [1]). As usual, for two prime cones α, β we shall write $\beta \to \alpha$ if α specializes β i.e. if $\beta \subset \alpha$, and for supp(α) we shall understand the real prime ideal $\alpha \cap -\alpha$.

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Definition 1. For any open constructible set $U \subset \operatorname{Spec}_r A$ let (B, σ) be the objects of the inductive directed system where $\psi : A \to B$ is an étale ring morphism and σ is a continuous section over U of the local homeomorphism $\psi^* : \operatorname{Spec}_r B \to \operatorname{Spec}_r A$, and whose maps are the A-algebra homomorphisms $\varphi : B \to B'$ such that $\varphi^* \circ \sigma' = \sigma$.

The datum $U \dashrightarrow (\varinjlim B)_{(B,\sigma)}/\operatorname{rad}_r(0)$ gives us a separated presheaf and we shall consider its associated sheaf $\mathbf{N}_A$.

Remarks 1.1. (i) It follows directly from the definition that there is a surjective sheaf morphism $\mathcal{N}_A \xrightarrow{\varphi} \mathbf{N}_A$.

(ii) Since factorising by the real radical commutes with directed inductive limits, the stalk of $\mathbf{N}_A$ at α is $A_\alpha/\operatorname{rad}_r(0)$.

(iii) For the same reason, we have that $\mathbf{N}_A \cong \mathbf{N}_{A/\operatorname{rad}_r(0)}$.

(iv) The rings of global sections of $\mathbf{N}_A$ are real rings. In particular, $\mathbf{N}_A$ is a subsheaf of the sheaf of continuous semi-algebraic functions over $\operatorname{Spec}_r A$ (see [3, 8 or 13]).

Definition 2. We shall call the stalk of the sheaf $\mathbf{N}_A$ at the point α, i.e. $A_\alpha/\operatorname{rad}_r(0)$, the *strongly real localization* of A in α and we shall write it $A_{r(\alpha)}$. The canonical morphism from A to $A_{r(\alpha)}$ will be η_α.

Proposition 1.2. $\operatorname{Spec}_r A_{r(\alpha)} \cong \{\beta \in \operatorname{Spec}_r A \mid \beta \to \alpha\}$ *with the topology induced by* $\operatorname{Spec}_r A$.

Proof. By Remark 1.1(ii), $\operatorname{Spec}_r A_{r(\alpha)} \cong \operatorname{Spec}_r A_\alpha$ and now we use [11, 3.7]. □

Definition 3. We shall say that a ring A is *strongly real local* if A is local, henselian and its residue field is real closed, and in addition A is real.

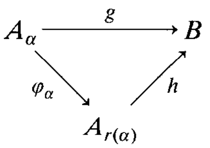
Proposition 1.3. *For every morphism f from A to a strongly real local ring B, there is a unique point $\alpha \in \operatorname{Spec}_r A$ and a unique local morphism $h : A_{r(\alpha)} \to B$ such that $f = h \circ \eta_\alpha$.*

Proof. The ring B is real closed local in the sense of [11], so, by [11, 4.3] there is a unique $g : A_\alpha \to B$ such that

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & \searrow \quad \nearrow_{g} & \\ & A_\alpha & \end{array}$$

commutes.

Let us consider $\varphi_\alpha : A_\alpha \to A_{r(\alpha)}$; since B is real, there is a unique local morphism h such that



commutes.

The union of these triangles gives us the desired result. □

Definition 4. A *strongly real locally ringed space* is a couple (X, Γ) where X is a topological space and Γ is a sheaf of rings over X whose stalks are strongly real local rings.

It is obvious that $(\mathrm{Spec}_r A, \mathbf{N}_A)$ is a locally strongly real ringed space.

Proposition 1.4 (Universal property of $\mathbf{N}_A$). *Let X be a topological space, (X, Γ) a strongly real locally ringed space, and $f: A \to \Gamma(X)$ a ring homomorphism. There is a unique pair (φ, h) with φ a continuous map from X to* $\mathrm{Spec}_r A$ *and h a local morphism from $\varphi^*(\mathbf{N}_A)$ to Γ such that $\Gamma(g) \circ \Theta_A = f$ where Θ_A denotes the canonical morphism $A \to \mathbf{N}_A(\mathrm{Spec}_r A)$.*

Proof. The same reasoning used before with [11, 4.8] in place of [11, 4.3]. □

Theorem 1.5 (Idempotence of $\mathbf{N}_A$). *Let U be an open constructible subset of* $\mathrm{Spec}_r A$; *U endowed with the restriction of $\mathbf{N}_A$ is isomorphic to* $\mathrm{Spec}_r(\mathbf{N}_A(U))$ *endowed with the sheaf* $\mathbf{N}_{\mathbf{N}_A(U)}$.

Proof. The proof given in [11, 5.1] remains valid in this case. □

2. The geometric case

In the rest of this note, $\mathbf{R}$ will be a real closed field, Ω an open semi-algebraic subset of $\mathbf{R}^n$, $\mathbf{N}(\Omega)$ the ring of Nash functions over Ω. Moreover V will be a Nash set in Ω, that is the zero-set of a finite number of elements of $\mathbf{N}(\Omega)$. Also, let I be the ideal of the Nash functions on Ω which vanish on V, and $A = \mathbf{N}(\Omega)/I$.

Let us consider the sheaf $\mathbf{N}_V$ of functions locally defined as restriction of Nash functions in open subsets of Ω, i.e. for every open constructible set U in $\mathrm{Spec}_r A$, let us consider the ring $\varinjlim_{W \subset U} \mathbf{N}(W)/{\equiv}$ where W is an open constructible subset of $\tilde{\Omega}$ containing U and $f \equiv g$ iff $\forall \alpha \in U$, $f(\alpha) = g(\alpha)$. The map sending each open constructible to this ring gives us a separated presheaf and we shall denote by $\mathbf{N}_V$ its associated sheaf. If S is an open semialgebraic subset of V, we shall consider the ring $\mathbf{N}_V(S) = \mathbf{N}_V(\tilde{S})$. See [4, §4] for first properties of the $\tilde{}$ operator.

Lemma 2.1. *Let A be a commutative ring, $I \subset A$ an ideal and $\alpha \in \mathrm{Spec}_r A$ such that $I \subset \mathrm{supp}(\alpha)$. Then $(A/I)_\alpha \cong A_\alpha / IA_\alpha$.*

Proof. Let us consider the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A_\alpha \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\psi} & A_\alpha \otimes_A A/I \cong A_\alpha/IA_\alpha \end{array}$$

Since φ is ind-étale, ψ is also. Thus, A_α/IA_α is a real strict localisation of A/I with residue field $k(\alpha)$ because the triangle

$$\begin{array}{ccc} A/I & \longrightarrow & A_\alpha/IA_\alpha \\ & \searrow \quad \swarrow & \\ & k(\alpha) & \end{array}$$

commutes. We have then, that $A_\alpha/IA_\alpha \cong (A/I)_\alpha$. $\square$

Proposition 2.2. *For every* $\alpha \in \operatorname{Spec}_r A$, *we have* $\mathbf{N}_{V,\alpha} = A_\alpha/\operatorname{rad}_r(0)$. *In particular, for* $x \in V$ $\mathbf{N}_{V,x} = {}^h(A_x)/\operatorname{rad}_r(0)$.

Proof.

$$\mathbf{N}_{V,\alpha} = \varinjlim_{\Omega \supset W \ni \alpha} (\mathbf{N}(W)/\equiv') = \Big(\varinjlim_{\Omega \supset W \ni \alpha} \mathbf{N}(W) \Big)/\equiv' = \mathbf{N}_{\Omega,\alpha}/\equiv'$$

where $f \equiv' g$ iff $\exists U$ open neighbourhood of α in V such that $\forall \beta \in U$, $f(\beta) = g(\beta)$. Using the real nullstellensatz for Nash germs in Ω (adapt [2, 8.6.5] for the germs) we can ensure that $f \equiv' g$ iff $(f-g) \in \operatorname{rad}_r(I^e)$ where I^e is the extension of I in $\mathbf{N}_{\Omega,\alpha}$.

So, we have that $\mathbf{N}_{V,\alpha} \cong \mathbf{N}_{\Omega,\alpha}/\operatorname{rad}_r(I^e) \cong (\mathbf{R}[X_1, \ldots, X_n]_\alpha)/\operatorname{rad}_r(I^e)$ and, by the lemma above, we get finally $\mathbf{N}_{V,\alpha} \cong (\mathbf{N}(\Omega)/I)_\alpha/\operatorname{rad}_r(0)$ i.e. $A_\alpha/\operatorname{rad}_r(0)$. $\square$

Corollary 2.3. *The sheaf* $\mathbf{N}_A$ *coincides with* $\mathbf{N}_V$.

Proof. There is an obvious sheaf morphism from $\mathbf{N}_A$ to $\mathbf{N}_V$, and this morphism induces isomorphisms in the stalks by Remark 1.1(ii). $\square$

Remark 2.4. Proposition 2.2 answers a question posed by Roy in [11, p. 417] because it is now clear that the stalk at x of the restriction of the structure sheaf coincides with the stalk at x of $\mathbf{N}_V$ iff x is a quasi-regular point (see [5] for more details).

Lemma 2.5. *Let* $S \subset \mathbf{R}^n$ *be an arbitrary semi-algebraic subset.*

There is a semi-algebraic partition $S_1, \ldots, S_r$ *of* S *such that the* S_j *are Nash diffeomorphic to polydisks* $(0,1)^{\dim S_j}$ *and, if* Z_j *is the Zariski closure of* S_j, S_j *is an open subset of the set of regular points of* Z_j.

Proof. This is [2, 8.1.12]. The last part of the statement does not appear there but is an easy consequence of the proof given. □

We now give a result that, following Coste, we call set-theoretic noetherianness of $\mathbf{N}_V(U)$ for an open semi-algebraic subset U of V. As usual, $z_U(f)$ (resp. $z_U(I)$) denotes the zero-set of f (resp. I) in U.

Proposition 2.6. *Let $I \subset \mathbf{N}_V(U)$ be an arbitrary ideal. We can find $f_1, \dots, f_p \in I$ such that $z_U(I) = z_U(f_1, \dots, f_p)$.*

Proof. The result is an immediate consequence of the fact that for any finitely generated ideal J, such that $J \subset I$ and $z_U(J) - z_U(I)$ is contained in a semi-algebraic set S of dimension k, there is a finitely generated ideal J' such that $J \subset J' \subset I$ and $z_U(J') - z_U(I)$ is contained in a semi-algebraic set of dimension lower than k.

Therefore we have only to see that the fact claimed above holds true. For this, let $S_1, \dots, S_r$ be the partition of S given by Lemma 2.5. We can suppose that $\forall j$ $S_j \not\subset z_U(I)$; then, there exists $g_j \in I$ such that g_j does not vanish identically on S_j, thus $z_U(g_j) \cap S_j = z_{S_j}(g_j)$ is a set of dimension lower than the one of S_j, by the identity principle. We now take $J' = J + (g_1, \dots, g_r)$. □

Corollary 2.7. *The maximal ideals of $\mathbf{N}_V(U)$ are the ideals of functions vanishing at points of U. In particular every ideal in $\mathbf{N}_V(U)$ is semi-real.*

Proof. It is enough to check that if I is a proper ideal of $\mathbf{N}_V(U)$, then $z_U(I)$ is not empty. For this, let $f_1, \dots, f_p \in I$ be such that $z_U(I) = z_U(f_1, \dots, f_p)$. If $z_U(I) = \emptyset$, then the function $f_1^2 + \dots + f_p^2$ is invertible in $\mathbf{N}_V(U)$, in which case I is not proper. □

Proposition 2.8. *Given $x \in U$ let $\mathbf{m}_x = \{f \in \mathbf{N}_V(U) \mid f(x) = 0\}$.*
We have $\mathbf{N}_{V,x} = {}^h(\mathbf{N}_V(U)_{\mathbf{m}_x}) / \mathrm{rad}_r(0) = {}^h(A_x)/\mathrm{rad}_r(0)$.

Proof. $\mathbf{N}_{V,x} = {}^h(A_x)/\mathrm{rad}_r(0)$ by Proposition 2.2.

Let us consider now the ring $\mathbf{N}_{\mathbf{N}_V(U),x}$. Proposition 2.2 tells us that this ring is ${}^h(\mathbf{N}_V(U)_{\mathbf{m}_x})$.

Now, using the operation ~ (cf. [4, §4]) and the idempotency of $\mathbf{N}_A$ we see that $\forall U' \subset U$, $\mathbf{N}_{\mathbf{N}_V(U)}(U') \cong \mathbf{N}_V(U')$, and taking inductive limits, we get $\mathbf{N}_{\mathbf{N}_V(U),x} \cong \mathbf{N}_{V,x}$.

In conclusion, $\mathbf{N}_{V,x} = {}^h(\mathbf{N}_V(U)_{\mathbf{m}_x})/\mathrm{rad}_r(0)$. □

3. The stellensätze

We shall use the results given above in order to prove some stellensätze for $\mathbf{N}_V(U)$. Some of them (real nullstellensatz, positivstellensatz), will be deduced from a substitution lemma combined with a formal stellensatz, and the others (Hilbert's 17th problem, central nullstellensatz), provided $\mathbf{N}_V(U)$ is an integral domain with adequate Krull dimension, from the structure of the space of orders of its quotient field. The ideas used in this paragraph are similar to those used in [7] where the stellensätze quoted above are obtained for the ring $\mathcal{N}_A(U)$. There is, however, a remarkable difference; since the sheaf $\mathcal{N}_A$ behaves better than $\mathbf{N}_V$ with respect to Krull dimension, no additional hypotheses are needed for the second group of stellensätze (see [7, Theorems 2.4 and 2.5]).

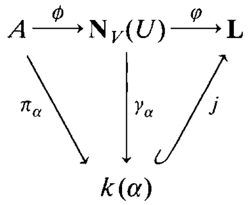
Theorem 3.1 (substitution lemma). *Let* $\mathbf{L}$ *be a real closed field containing* $\mathbf{R}$ *and* $\varphi : \mathbf{N}_V(U) \to \mathbf{L}$ *a ring morphism. Then*

(i) $(\varphi(X_1), \ldots, \varphi(X_n)) \in U_{\mathbf{L}}$;

(ii) $\forall g \in \mathbf{N}_V(U),\ \varphi(g) = g_{\mathbf{L}}(\varphi(X_1), \ldots, \varphi(X_n))$.

Proof. We know that if $\phi : A \to \mathbf{N}_V(U)$ is the canonical morphism, $\phi^* : \operatorname{Spec}_r \mathbf{N}_V(U) \to U$ is a homeomorphism (Theorem 1.5).

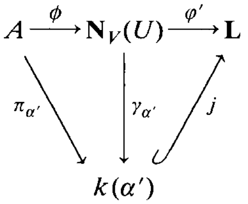
(i) So, the morphism $\varphi : \mathbf{N}_V(U) \to \mathbf{L}$ determines a prime cone of $\operatorname{Spec}_r \mathbf{N}_V(U)$, and so, there is $\alpha \in U$ such that the following diagram commutes:



Now, $(\varphi(X_1), \ldots, \varphi(X_n)) = j^n((\pi_\alpha(X_1), \ldots, \pi_\alpha(X_n))$, and since $(\pi_\alpha(X_1), \ldots, \pi_\alpha(X_n))$ belongs to $U_{k(\alpha)}$ because $\alpha \in U$, this point belongs to $U_{\mathbf{L}}$.

(ii) Let us consider the ring morphism $\varphi' : \mathbf{N}_V(U) \to \mathbf{L}$ defined by $\sigma'(g) = g_{\mathbf{L}}(\varphi(X_1), \ldots, \varphi(X_n))$.

The morphism φ' defines a point α' of U which makes the following diagram commutative:



Now, it is evident that $\varphi \circ \phi = \varphi' \circ \phi$. Then, those morphisms define the same prime cone in $\operatorname{Spec}_r A$, i.e. $\alpha = \alpha'$, and, of course $\gamma_\alpha = \gamma_{\alpha'}$ and $j = j'$. As $\varphi = j \circ \gamma_\alpha$ and $\varphi' = j' \circ \gamma_{\alpha'}$ we deduce $\varphi = \varphi'$. $\square$

Let us recall the following result:

Proposition 3.2 (formal theorem). *Let A be a commutative ring, and let $(a_j)_{j\in J}$, $(b_k)_{k\in K}$ and $(c_q)_{q\in Q}$ be arbitrary families of elements from A. We shall write P for the cone spanned by the a_j, M for the multiplicative monoid spanned by the b_k and I for the ideal spanned by the c_q. The following properties are equivalent:*

(i) *There does not exist any prime cone α of A such that*

$$\forall j\in J,\ a_j\in\alpha,\ \forall k\in K,\ b_k\notin \operatorname{supp}(\alpha)\quad \text{and}\quad \forall q\in Q,\ c_q\in\operatorname{supp}(\alpha).$$

(ii) *There does not exist any ring homomorphism $\varphi: A\to \mathbf{F}$, with $\mathbf{F}$ a real closed field such that*

$$\forall j\in J,\ \varphi(a_j)\geq 0,\ \forall k\in K,\ f(b_k)\neq 0\quad \text{and}\quad \forall q\in Q,\ f(c_q)=0.$$

(iii) *We can find $p\in P$, $b\in M$ and $c\in I$ such that $p+b^2+c=0$.*

Proof. [2, 4.4.1]. □

Theorem 3.3. *Let $(f_j)_{j=1,\dots,s}$, $(g_k)_{k=1,\dots,t}$, $(h_q)_{q=1,\dots,r}$ be finite families in $\mathbf{N}_V(U)$. We shall write P for the cone spanned by the $(f_j)_{j=1,\dots,s}$, M for the multiplicative monoid spanned by the $(g_k)_{k=1,\dots,t}$, and I for the ideal spanned by the $(h_q)_{q=1,\dots,r}$. The following properties are equivalent:*

(i) *The semi-algebraic set*

$$S=\{x\in V \mid \forall j=1,\dots,s, f_j(x)\geq 0,\ \forall k=1,\dots,t,\ g_k(x)\neq 0 \text{ and } \forall q=1,\dots,r,\ h_q(x)=0\}$$

is empty.

(ii) *There are $f\in P$, $g\in M$, and $h\in I$ such that $f+g^2+h=0$.*

Proof. (i) ⇒ (ii). It is enough to see that Proposition 3.2(ii) is valid.

By hypothesis, we have that $\mathbf{R}$ satisfies

$$\exists x_1,\dots,x_n\Big((x_1,\dots,x_n)\in U\wedge\Big(\bigwedge_{j\leq s} f_j(x_1,\dots,x_n)\geq 0\Big) \wedge\Big(\bigwedge_{k\leq t} g_k(x_1,\dots,x_n)\neq 0\Big)\wedge\Big(\bigwedge_{q\leq r} h_q(x_1,\dots,x_n)=0\Big)\Big).$$

Let us suppose that $\exists\,\varphi:\mathbf{N}_V(U)\to\mathbf{L}$ such that $\varphi(f_j)\geq 0$, $\varphi(g_k)>0$ and $\varphi(h_q)=0$. Using the substitution lemma we see that, if $\xi_i=\varphi(X_i)$, then $(\xi_1,\dots,\xi_n)\in U_{\mathbf{L}}$ and in addition $f_{j\mathbf{L}}(\xi_1,\dots,\xi_n)\geq 0$, $g_{k\mathbf{L}}(\xi_1,\dots,\xi_n)>0$ and $h_{q\mathbf{L}}(\xi_1,\dots,\xi_n)=0$.

But, using the Tarski's principle we get that $\mathbf{L}$ satisfies

$$\exists x_1,\dots,x_n\Big((x_1,\dots,x_n)\in U\wedge\Big(\bigwedge_{j\leq s} f_j(x_1,\dots,x_n)\geq 0\Big) \wedge\Big(\bigwedge_{k\leq t} g_k(x_1,\dots,x_n)\neq 0\Big)\wedge\Big(\bigwedge_{q\leq r} h_q(x_1,\dots,x_n)=0\Big)\Big)$$

which is a contradiction.

(ii) ⇒ (i). Let us suppose that $S \neq \emptyset$ and let $x \in S$. The morphism $\varphi : \mathbf{N}_V(U) \to \mathbf{R}$ defined by $\varphi(f) = f(x)$ satisfies $\varphi(f_j) \geq 0$, $\varphi(g_k) \neq 0$ and $\varphi(h_q) = 0$ which contradicts (iii) ⇒ (ii) of Proposition 3.2. □

Corollary 3.4 (real nullstellensatz). *Let $f \in \mathbf{N}_V(U)$ and $I \subset \mathbf{N}_V(U)$ be an ideal. If $z(f) \supset z(I)$, then $f \in \mathrm{rad}_r(I)$.*

Proof. Let $h_1, \ldots, h_r$ be such that $z_U(I) = z_U(h_1, \ldots, h_r)$ (Proposition 2.6). In this case the condition that $\forall x \in U$, $(x \in z_U(I) \Rightarrow f(x) = 0)$ merely says that the semi-algebraic set

$$\left\{ x \in U \mid f(x) \neq 0 \wedge \left(\bigwedge_{q \leq r} h_q(x) = 0 \right) \right\}$$

is empty.

Because of Theorem 3.3 this turns out to be equivalent with $\exists m \geq 1$, $\exists h \in I$ and $\exists s \in \sum (\mathbf{N}_V(U))^2$ such that $f^{2m} + s = h$, i.e., $f \in \mathrm{rad}_r(I)$. □

Corollary 3.5. *Let $f \in \mathbf{N}_V(U)$.*

(i) (Positivstellensatz). *$f(x) > 0$ $\forall x \in U$ iff $\exists g, h \in \sum (\mathbf{N}_V(U))^2$ such that $fg = 1 + h$.*

(ii) (Nichtnegativstellensatz). *$f(x) \geq 0$ $\forall x \in U$ iff $\exists m \geq 0$, $\exists g, h \in \sum (\mathbf{N}_V(U))^2$ such that $fg = f^{2m} + h$.* □

Before we begin the second part of this paragraph, let us recall some definitions.

If A is an arbitrary integral domain, with field of fractions k, $I \subset A$ an ideal, and ϱ the cone $A \cap \sum k^2$, we shall call central radical of I the ideal $\mathrm{rad}_{\mathrm{cent}}(I) = \{x \in A \mid \exists m \geq 0, \exists y \in \varrho \text{ such that } x^{2m} + y \in I\}$. The ideal I is said to be central iff it coincides with its central radical.

As usual we shall consider $\mathrm{Spec}_r k$ as a topological sub-space of $\mathrm{Spec}_r A$.

The following description of the central prime ideals is a particular case of a result from Saliba.

Proposition 3.6. *Let A be a integral domain with field of fractions k, and let $\mathfrak{p} \in \mathrm{Spec}\, A$.*

Then $\mathfrak{p}$ is a central ideal iff $\exists \alpha \in \mathrm{Spec}_r A$, $\exists \beta \in \mathrm{Spec}_r k$, such that $\beta \to \alpha$ and $\mathrm{supp}(\alpha) = \mathfrak{p}$.

Proof. [12, p. 28, Proposition 3.7]. □

Examples 3.7. Given the 'umbrellas' with equations $X^3 + Z(X^2 + Y^2) = 0$ and $X^3 + ZX^2 - Y^2 = 0$, let A and A' be their coordinate rings.

In both cases $\pi = (X, Y)$ is the ideal of the 'stick'. Following Proposition 3.6, the ideal $\mathfrak{p}$ is central in A' but not in A.

In the sequel we will suppose that the ring $A = \mathbf{N}(\Omega)/I$ is an integral domain and we shall denote by k its field of fractions.

A point x of V is a central point iff it belongs to the closure of the set of regular points of V. This amounts to saying that $\dim_r(V,x)=\dim_r V$ or else, that x which is a point in $\mathrm{Spec}_r A$ is the specialization of some point in $\mathrm{Spec}_r k$ (cf. [2, 7.6.1]). It is clear that the subset of central points of V, $\mathrm{Cent}(V)$ in the sequel, is a closed semi-algebraic subset of V.

If $U\subset V$ is an open semi-algebraic subset, we shall write $\mathrm{Cent}(U)$ for the intersection of U with $\mathrm{Cent}(V)$. We have the following result:

Lemma 3.8. $\tilde{U}\cap \mathrm{Spec}_r k\subset (\mathrm{Cent}(U))^{\sim}$.

Proof. The result is shown for algebraic sets during the course of the proof of [2, 7.6.3], but the same argument is valid here. □

Let us consider an open semi-algebraic subset U of V such that $\mathrm{Cent}(U)\neq\emptyset$ and $\mathbf{N}_V(U)$ is an integral domain. We shall denote by K the field of fractions of $\mathbf{N}_V(U)$. As Krull dimension does not behave well under quotient by the real radical, we shall consider the two following conditions:

(D) $\quad \dim \mathbf{N}_V(U)\leq \dim A.$

(C) $\quad \forall f\in \mathbf{N}_V(U)$, if $\dim(z(f))=\dim U$, then $f=0$ in $\mathbf{N}_V(U)$.

If ϕ is the canonical morphism $A\to \mathbf{N}_V(U)$, the idempotency result gives us the following proposition:

Proposition 3.9. *Let $U\subset V$ be an open semi-algebraic set such that* $\mathrm{Cent}(U)\neq\emptyset$.

(i) *If U satisfies* (C), *then $\mathbf{N}_V(U)$ is an integral domain.*

(ii) *If U satisfies* (D), *and $\mathbf{N}_V(U)$ is an integral domain, then U satisfies* (C).

Proof. (i) Trivial.

(ii) Let $f\in \mathbf{N}_V(U)$ such that $\dim(z(f))=\dim U$. We can find a nonempty semi-algebraic subset U' contained in $z(f)\cap \mathrm{Cent}(U)$.

Let $x\in U'$; there is $\alpha\in \mathrm{Spec}_r k$ such that $\alpha\to x$.

If the Krull dimension of A is d, we must have a chain of prime cones $\beta_0\to\cdots\to\beta_d=x$, with the β_j belonging to $\tilde{U}'$ (because U' is open) and $\mathrm{supp}(\beta_0)=\mathrm{supp}(\alpha)$ (cf. [4, 8.7]).

Because of the idempotency of $\mathbf{N}_A$, we find in $\mathrm{Spec}_r \mathbf{N}_V(U)$ a chain

$$(\phi^*)^{-1}(\beta_0)\to\cdots\to(\phi^*)^{-1}(\beta_d)=(\phi^*)^{-1}(x).$$

Thus, $\mathrm{depth}((\phi^*)^{-1}(\beta_0))\geq d$, and, as $\dim \mathbf{N}_V(U)\leq d$, we get that $\mathrm{ht}((\phi^*)^{-1}(\beta_0))=0$, that is $(\phi^*)^{-1}(\beta_0)\in \mathrm{Spec}_r K$. Since $(\phi^*)^{-1}(\beta_0)\to x$, $f((\phi^*)^{-1}(\beta_0))=0$ which implies that $f=0$. □

In the sequel, *U satisfies* $\mathrm{Cent}(U)\neq\emptyset$ *and condition* (C).

Remark 3.10. The hypothesis that, provided $\mathrm{Cent}(U)\neq\emptyset$, U satisfies condition (D) would follow from a general extension theorem (as well as the noetherianness and the excellence of this ring). But Efroymson has shown that we do not have such a theorem (see [9]). In the proof of the last proposition we have seen that $\mathrm{Cent}(U)\neq\emptyset$ implies that $\dim \mathbf{N}_V(U)\geq \dim A$, but the other inequality remains unknown.

Let us note that even if the truth of the general extension theorem for the sheaf $\mathcal{N}_A$ is not known, it is still possible to prove for this sheaf the result about the 'good' dimension of the rings of global sections as well as the noetherianness and the excellence of those rings (see [6]). As a consequence we have the same results for $\mathbf{N}_V(U)$ provided that the morphism induced on the global sections rings $\varphi_U:\mathcal{N}_A(\tilde{U})\to \mathbf{N}_V(\tilde{U})$ is surjective. This is the case, for instance, for curves and for open subsets of quasi-regular points (for all of this see [5]). The noetherianness in both cases is an improvement of results given in [10].

Proposition 3.11. (i) *Given* $x\in U$ *and* $\alpha\in \mathrm{Spec}_r k$ *such that* $\alpha\to x$, *then* $(\phi^*)^{-1}(\alpha)\in \mathrm{Spec}_r K$.

(ii) $\phi^*(\mathrm{Spec}_r K)\subset (\mathrm{Cent}(U))^{\tilde{}}$.

Proof. It is enough to see that $\mathrm{supp}((\phi^*)^{-1}(\alpha))=(0)$. For this, let $f\in \mathrm{supp}((\phi^*)^{-1}(\alpha))$; we have $f((\phi^*)^{-1}(\alpha))=0$, which implies $\dim(z(f))=\dim U$ and then, that $f=0$ as desired.

(ii) Let $\alpha\in\phi^*(\mathrm{Spec}_r K)$. Since $\mathrm{Cent}(U)\neq\emptyset$, U is Zariski-dense in V, and so, ϕ is injective. It follows that $\alpha\in \mathrm{Spec}_r k$, so $\alpha\in\tilde{U}\cap \mathrm{Spec}_r k$ and using Lemma 3.8, the result follows. □

We can now give the solution of Hilbert's 17th problem.

Theorem 3.12 (Hilbert's 17th problem). *Let* $f\in \mathbf{N}_V(U)$. *We have* $f\in\sum K^2$ *iff* $f\geq 0$ *on* $\mathrm{Cent}(U)$.

Proof. (⇒) Let us suppose that $f\in\sum K^2$, and let $x\in \mathrm{Cent}(U)$.

We know that there is $\beta\in \mathrm{Spec}_r k$ such that $\beta\to x$. So, following Proposition 3.11(i), $(\phi^*)^{-1}(\beta)\in \mathrm{Spec}_r K \Rightarrow f\in(\phi^*)^{-1}(\beta)$ (because all the orderings of K contain $\sum K^2$) $\Rightarrow f((\phi^*)^{-1}(\beta))\geq 0 \Rightarrow f(x)\geq 0$.

(⇐) If $f(\mathrm{Cent}(U))\geq 0 \Rightarrow f((\mathrm{Cent}(U))^{\tilde{}})\geq 0$. Following Proposition 3.11(ii), $f(\mathrm{Spec}_r K)\geq 0$, i.e. $f\in\alpha$ $\forall\alpha\in \mathrm{Spec}_r K$. Since the intersection of the orderings on K is $\sum K^2$, $f\in K^2$. □

Proposition 3.13. *Let* $\mathfrak{p}\subset \mathbf{N}_V(U)$ *be a prime ideal.*

The following properties are equivalent:

(i) $\mathfrak{p}$ *is a central ideal.*

(ii) $\mathfrak{p}=\mathcal{I}\,(z(\mathfrak{p})\cap \mathrm{Cent}(U))$.

Proof. (i) $\Rightarrow$ (ii). Let $f \in \mathfrak{p}$ and let $x \in (z(\mathfrak{p}) \cap \mathrm{Cent}(U))$. It is clear that $f(x) = 0$.

Let now $f \in \mathscr{I}(z(\mathfrak{p}) \cap \mathrm{Cent}(U))$. Since I is a central ideal, there are $\alpha \in \mathrm{Spec}_r\, \mathbf{N}_V(U)$ and $\beta \in \mathrm{Spec}_r K$ such that $\mathrm{supp}(\alpha) = \mathfrak{p}$ and $\beta \to \alpha$.

Then, $\alpha \in (z(\mathfrak{p}) \cap \mathrm{Cent}(U))^{\sim}$ (since $z_U(\mathfrak{p}) = z_U(h_1, \ldots, h_r)$ is semi-algebraic and we can apply the operation $\tilde{}$), thus $f(\alpha) = 0$, and hence $f \in \mathfrak{p}$ (since $\mathrm{supp}(\alpha) = \mathfrak{p}$).

(ii) $\Rightarrow$ (i). Let $f \in \mathrm{rad}_{\mathrm{cent}}(\mathfrak{p})$, then there are $m \geq 0$ and $q \in (\mathbf{N}_V(U) \cap \sum K^2)$ such that $f^{2m} + q \in \mathfrak{p}$. Since $q \in \sum K^2$, $q \geq 0$ over $\mathrm{Cent}(U)$ (Theorem 3.12) and, since $f^{2m} + q$ vanishes over $z(\mathfrak{p})$, f^{2m} must also vanish over $z(\mathfrak{p}) \cap \mathrm{Cent}(U)$. It follows that $f = 0$ over $z(\mathfrak{p}) \cap \mathrm{Cent}(U)$, and this clearly implies that $f \in \mathfrak{p}$. □

Theorem 3.14 (central nullstellensatz). *Let $I \subset \mathbf{N}_V(U)$ an arbitrary ideal.*

The following properties are equivalent:

(i) *I is a central ideal.*

(ii) $I = \mathscr{I}(z(I) \cap \mathrm{Cent}(U))$.

Proof. For an arbitrary integral domain it is easy to verify that $\mathrm{rad}_{\mathrm{cent}}(I)$ is the intersection of the central prime ideals which contain I. Then, we proceed as usual. □

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