Available online at www.sciencedirect.com

SCIENCE DIRECT

Topology 45 (2006) 65–99

TOPOLOGY

www.elsevier.com/locate/top

Eigenvalues of products of unitary matrices and Lagrangian involutions

Elisha Falbel[a], Richard A. Wentworth[b,*]

[a] *Institut de Mathématiques, Université Pierre et Marie Curie, Case 82, 4, Place Jussieu, F-75252 Paris, France*
[b] *Department of Mathematics, John Hopkins University, Baltimore, MD 21218, USA*

Received 7 December 2004; accepted 8 June 2005

Abstract

This paper introduces a submanifold of the moduli space of unitary representations of the fundamental group of a punctured sphere with fixed local monodromy. The submanifold is defined via products of involutions through Lagrangian subspaces. We show that the moduli space of Lagrangian representations is a Lagrangian submanifold of the moduli of unitary representations.

Keywords: Unitary representations; Parabolic bundles; Lagrangian involutions

1. Introduction

Let $\operatorname{spec}(A)$ denote the set of eigenvalues of a unitary $n \times n$ matrix A. An old problem asks the following question: what are the possible collections of eigenvalues $\operatorname{spec}(A_1), \ldots, \operatorname{spec}(A_\ell)$ which arise from matrices satisfying $A_1 \cdots A_\ell = \mathbf{I}$, $\ell \geqslant 3$? (A review of related problems and recent developments can be found in [7]). For an equivalent formulation in terms of representations, let Γ_ℓ denote the free group on $\ell - 1$ generators with presentation

$$\Gamma_\ell = \langle \gamma_1, \ldots, \gamma_\ell : \gamma_1 \cdots \gamma_\ell = 1 \rangle \tag{1}$$

* Corresponding author. Tel.: +1 4105167398; fax: +1 4105165549.
E-mail addresses: falbel@math.jussieu.fr (E. Falbel), wentworth@jhu.edu (R.A. Wentworth).

0040-9383/$ - see front matter © 2005 Published by Elsevier Ltd.
doi:10.1016/j.top.2005.06.003

and let $U(n)$ denote the group of unitary $n \times n$ matrices. We shall say that a collection of conjugacy classes $C_1, \ldots, C_\ell$ in $U(n)$ is *realized* by a unitary representation if there is a homomorphism $\rho : \Gamma_\ell \to U(n)$ with $\rho(\gamma_s) \in C_s$ for each $s = 1, \ldots, \ell$.

A natural subclass of linear representations of Γ_ℓ consists of those generated by reflections through linear subspaces. In the case of unitary representations, one may consider Lagrangian planes L and their associated involutions σ_L. Given a pair of Lagrangian subspaces L_1, L_2 in $\mathbb{C}^n$, the product $\sigma_{L_1}\sigma_{L_2}$ is an element of $U(n)$. Moreover, *any* unitary matrix may be obtained in this way (cf. Proposition 3.3 below). For Lagrangians $L_1, \ldots, L_\ell$, one can define a unitary representation of Γ_ℓ via $\gamma_s \mapsto \sigma_{L_s}\sigma_{L_{s+1}}$, for $s = 1, \ldots, \ell - 1$, and $\gamma_\ell \mapsto \sigma_{L_\ell}\sigma_{L_1}$. We shall call these *Lagrangian representations* (see Definition 3.3). There is a natural equivalence relation obtained by rotating every Lagrangian by an element of $U(n)$, and this corresponds to conjugation of the representation. We will say that a given collection of conjugacy classes is *realized* by a Lagrangian representation if the homomorphism ρ of the previous paragraph may be chosen to be Lagrangian.

At first sight, Lagrangian representations may seem very special. The main result of this paper is that in fact they exist in abundance. We will prove

Theorem 1 (*cf. Section 5 and Propositions 3.5 and 4.3*). *If there exists a unitary representation of Γ_ℓ realizing a given collection of conjugacy classes in $U(n)$, then there also exists a Lagrangian representation realizing the same conjugacy classes.*

We also study the global structure of the moduli space of Lagrangian representations. Let $\mathfrak{a}$ denote a specification of ℓ conjugacy classes $C_1, \ldots, C_\ell$, and let $\mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$ denote the set of equivalence classes of irreducible representations $\rho : \Gamma_\ell \to U(n)$ with each $\rho(\gamma_s) \in C_s$. Note that for generic choices of $\mathfrak{a}$, all representations are irreducible. Then $\mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$ is a smooth manifold which carries a symplectic structure coming from its realization as the reduction of a quasi-Hamiltonian G-space (cf. [1]; for a brief description, see Section 3.3). We refer to this as the *natural* symplectic structure. Let $\mathscr{L}\ \mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n)) \subset \mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$ denote the subset of irreducible Lagrangian representations. Then we have

Theorem 2. *With respect to the natural symplectic structure*

$$\mathscr{L}\ \mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n)) \subset \mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$$

is a smoothly embedded Lagrangian submanifold.

Characterizations of which conjugacy classes are realized by products of unitary matrices have been given in [3,5,2,15]. We will give a brief review in Section 2.2 below. The basic result is that the allowed region is given by a collection of affine inequalities on the log eigenvalues. The “outer walls” of the allowed region correspond to spectra realized only by reducible representations. In general, there are also “inner walls” corresponding to spectra that are realized by both reducible and irreducible representations. The open chambers complementary to these walls correspond to spectra that are realized only by irreducible representations. The term “generic” used above refers to spectra in the open chambers.

This structure suggests a proof of Theorem 1 via induction on the rank and deformation theory, and this is the approach we shall take. In Section 3, we prove some elementary facts about configurations of pairs and triples of Lagrangian subspaces in $\mathbb{C}^n$. We define Lagrangian representations and discuss their relationship to unitary representations. In particular, we show that the Lagrangian representation space is isotropic with respect to the natural symplectic structure. In Section 4, after briefly reviewing the case of unitary representations, we develop the deformation theory of Lagrangian representations in more detail. We introduce two methods to produce a family of Lagrangian representations from a given one. We call these deformations *twisting* and *bending* (see Definitions 4.1 and 4.2), and they are in part motivated by the geometric flows studied by Kapovich and Millson [13]. We prove that twisting and bending deformations, applied to an irreducible Lagrangian representation, span all possible variations of the conjugacy classes (see Proposition 4.3). As a consequence, if there is a single point interior to one of the chambers described above that is realized by a Lagrangian representation, then all points in the chamber are also realized by Lagrangians (see Corollary 4.1). This reduces the existence problem to ruling out the possibility of isolated chambers realized by unitary representations, but not by Lagrangians. To achieve this we make a detailed analysis of the wall structure in Section 5. A basic fact is that any reducible Lagrangian representation may be perturbed to an irreducible one. Hence, inductively, any chamber having an outer wall as a face is necessarily populated by Lagrangian representations. A topological argument that exploits an estimate (Proposition 4.4) on the codimension of the set of reducible representations shows that inner walls may also be "crossed" by Lagrangian representations.

It should be apparent from this description that our proof of Theorem 1 is somewhat indirect. A more precise description of the obstructions to deformations of reducible unitary and Lagrangian representations is desirable. In [6] Lagrangians were used to give a geometrical explanation of the inequalities for $U(2)$ representations in terms of spherical polygons. For higher rank it is tempting to look for a similar geometrical interpretation of the inequalities, though we have not obtained such at present. Unitary representations of surface groups are related to stability of holomorphic vector bundles through the famous theorem of Narasimhan and Seshadri [18] and its generalization to punctured surfaces by Mehta and Seshadri [17]. A challenging problem is to give an analytic description of those holomorphic structures which give rise to Lagrangian representations.

We conclude this introduction by pointing out an alternative interpretation of the result in Theorem 1. Let us say that matrices $A_1, \ldots, A_\ell \in U(n)$ are *pairwise symmetrizable* if for each $s = 1, \ldots, \ell$, there is $g_s \in U(n)$ so that both $g_s A_s g_s^{-1}$ and $g_s A_{s+1} g_s^{-1}$ are symmetric (where $A_{\ell+1} = A_1$). Also, throughout the paper, for unitary matrices A and B, $A \sim B$ indicates that A and B are conjugate. We then have the following reformulation of Theorem 1.

Theorem 3. *Given $n \times n$ unitary matrices $\{A_s\}_{s=1}^{\ell}$, $A_1 \cdots A_\ell = \mathbf{I}$, there exists a possibly different collection of unitary matrices $\{B_s\}_{s=1}^{\ell}$, $B_1 \cdots B_\ell = \mathbf{I}$, $A_s \sim B_s$ for $s = 1, \ldots, \ell$, such that $B_1, \ldots, B_\ell$ are pairwise symmetrizable.*

See Section 3.2 for the proof.

2. Unitary representations

2.1. The space of conjugacy classes

We begin with some notation. Given integers $n \geqslant 1$ and $\ell \geqslant 3$:

- Let $\overline{\mathscr{M}}_\ell(n)$ denote the set of all $\ell \times n$ matrices $\mathfrak{a} = (\alpha_j^s)$, $1 \leqslant s \leqslant \ell$, $1 \leqslant j \leqslant n$, where for each s, $\alpha^s = (\alpha_1^s, \ldots, \alpha_n^s)$ satisfies $0 \leqslant \alpha_1^s \leqslant \cdots \leqslant \alpha_n^s \leqslant 1$.

- Let $\overline{\mathscr{A}}_\ell(n)$ be the quotient of $\overline{\mathscr{M}}_\ell(n)$ defined by the following equivalence: identify a point of the form $\alpha^s=(\alpha^s_1,\dots,\alpha^s_k,1,\dots,1)$, $\alpha^s_k<1$, with $\tilde\alpha^s=(0,\dots,0,\tilde\alpha^s_{n-k+1},\dots,\tilde\alpha^s_n)$, where $\tilde\alpha^s_{n-k+i}=\alpha^s_i$, $i=1,\dots,k$.
- Let $\mathscr{A}_\ell(n)\subset\overline{\mathscr{A}}_\ell(n)$ be the open subset where all inequalities are strict: $0<\alpha^s_1<\cdots<\alpha^s_n<1$, for each s.

For each $\mathfrak{a}\in\overline{\mathscr{A}}_\ell(n)$ we define the *index* as follows: choose the representative of $\mathfrak{a}$ where $0\leqslant\alpha^s_1\leqslant\cdots\leqslant\alpha^s_n<1$, for each s, and set

$$I(\mathfrak{a})=\sum_{s=1}^{\ell}\sum_{j=1}^{n}\alpha^s_j. \tag{2}$$

We define $\overline{\mathscr{A}}^{\mathbb{Z}}_\ell(n)=\{\mathfrak{a}\in\overline{\mathscr{A}}_\ell(n):I(\mathfrak{a})\text{ is an integer}\}$, $\mathscr{A}^{\mathbb{Z}}_\ell(n)=\mathscr{A}_\ell(n)\cap\overline{\mathscr{A}}^{\mathbb{Z}}_\ell(n)$.

Definition 2.1. For a nonnegative integer I, define the *open M-plane* by

$$\mathscr{P}_{I,\ell}(n)=\{\mathfrak{a}\in\mathscr{A}^{\mathbb{Z}}_\ell(n):I(\mathfrak{a})=I\}.$$

The closure $\overline{\mathscr{P}}_{I,\ell}(n)$ of $\mathscr{P}_{I,\ell}$ in $\overline{\mathscr{A}}^{\mathbb{Z}}_\ell(n)$ will be called the *closed M-plane*. Finally, let

$$\overline{\mathscr{P}}^*_{I,\ell}(n)=\{\mathfrak{a}\in\overline{\mathscr{P}}_{I,\ell}:I(\mathfrak{a})=I\}.$$

Observe that $\overline{\mathscr{P}}_{I,\ell}(n)$ is a closed connected cell. Notice also that the closed M-planes are not disjoint, whereas of course $\overline{\mathscr{P}}^*_{I,\ell}(n)\cap\overline{\mathscr{P}}^*_{J,\ell}(n)=\emptyset$ if $I\neq J$. We therefore have a *disjoint* union

$$\overline{\mathscr{A}}^{\mathbb{Z}}_\ell(n)=\bigcup_{0\leqslant I\leqslant n\ell-1}\overline{\mathscr{P}}^*_{I,\ell}(n).$$

For each s choose a partition m^s of $\{1,\dots,n\}$, i.e. a set of integers $0=m^s_0<m^s_1<\cdots<m^s_{l_s}=n$. Here, l_s is the *length* of the partition. Specifying l_s numbers $0\leqslant\hat\alpha^s_1<\cdots<\hat\alpha^s_{l_s}<1$ along with a partition of length l_s uniquely determines a point in $\mathfrak{a}=(\alpha^s_j)\in\overline{\mathscr{A}}_\ell(n)$, where $\alpha^s_i=\hat\alpha^s_j$ for $m^s_{j-1}<i\leqslant m^s_j$. Conversely, given a point $\mathfrak{a}\in\overline{\mathscr{A}}_\ell(n)$ with the distinct entries $0\leqslant\hat\alpha^s_1<\cdots<\hat\alpha^s_{l_s}<1$, a partition of length l_s is determined by the multiplicities $\mu^s_j=m^s_j-m^s_{j-1}$ of the $\hat\alpha^s_j$. We shall say that α^s *has the multiplicity structure of* m^s.

Let $\mathfrak{m}=(m^1,\dots,m^\ell)$ be a choice of ℓ partitions. In addition, choose a (possibly empty subset) $z\subset\{1,\dots,\ell\}$ of cardinality $|z|$. This data leads to the following refinement of the M-plane.

$$\begin{aligned}\mathscr{P}_{I,\ell}(n,\mathfrak{m},z)=\{\mathfrak{a}=(\alpha^s_j)\in\overline{\mathscr{P}}^*_{I,\ell}(n):\ &\alpha^s\text{ has multiplicity structure }m^s\text{ for all }s,\\ &\text{and }\hat\alpha^s_1=0\text{ if and only if }s\in z\};\end{aligned}$$

$$\overline{\mathscr{P}}_{I,\ell}(n,\mathfrak{m},z)=\text{ the closure of }\mathscr{P}_{I,\ell}(n,\mathfrak{m},z)\text{ in }\overline{\mathscr{A}}^{\mathbb{Z}}_\ell(n);$$

$$\overline{\mathscr{P}}^*_{I,\ell}(n,\mathfrak{m},z)=\overline{\mathscr{P}}_{I,\ell}(n,\mathfrak{m},z)\cap\overline{\mathscr{P}}^*_{I,\ell}(n).$$

Next, notice that there is a natural partial ordering on multiplicities: if $\mathfrak{p}=(p^1,\dots,p^\ell)$ and $\mathfrak{m}=(m^1,\dots,m^\ell)$, we say that $\mathfrak{p}\leqslant\mathfrak{m}$ if for each $s=1,\dots,\ell$ the partition p^s is a subset of m^s. We then have a stratification by the cells $\mathscr{P}_{I,\ell}(n, \mathfrak{m}, z)$ in the sense that

$$\overline{\mathscr{P}}^*_{I,\ell}(n, \mathfrak{m}, z) = \bigcup_{\mathfrak{p} \leqslant \mathfrak{m}, z \subset \tilde{z} \subset \{1,\ldots,\ell\}} \mathscr{P}_{I,\ell}(n, \mathfrak{p}, \tilde{z}).$$

In particular,

$$\overline{\mathscr{P}}^*_{I,\ell}(n) = \bigcup_{\mathfrak{m}, z \subset \{1,\ldots,\ell\}} \mathscr{P}_{I,\ell}(n, \mathfrak{m}, z)$$

There is a similar, though slightly more complicated, stratification of $\overline{\mathscr{P}}_{I,\ell}(n, \mathfrak{m}, z)$ which involves strata of lower index. To describe this, consider the limit $\bar{\mathfrak{a}}$ in $\overline{\mathscr{A}}^{\mathbb{Z}}_{\ell}(n)$ of points in $\mathscr{P}_{I,\ell}(n, \mathfrak{m}, z)$ where $\hat{\alpha}^{s_0}_{l_{s_0}} \to 1$, for some $s_0 \in \{1, \ldots, \ell\}$, but the $\hat{\alpha}^{s}_{l_s}$ remain bounded away from 1 for $s \neq s_0$. From the defining equivalence $\overline{\mathscr{M}}_{\ell}(n) \to \overline{\mathscr{A}}_{\ell}(n)$ and the convention (2) for the index, it follows that

$$\bar{I} = I(\bar{\mathfrak{a}}) = I - (n - m^{s_0}_{l_{s_0}-1}) < I.$$

Furthermore, we may define a new collection of partitions $\bar{\mathfrak{m}}$, $\bar{m}^s(\bar{l}_s) = m^s(l_s)$ for $s \neq s_0$, and

$$\text{if } s_0 \in z \quad \text{then} \quad \begin{cases} \bar{m}^{s_0}_i = m^{s_0}_i + (n - m^{s_0}_{l_{s_0}-1}), & 1 \leqslant i \leqslant l_{s_0} - 1, \\ \bar{l}_{s_0} = l_{s_0} - 1, \\ \bar{z} = z; \end{cases}$$

$$\text{if} \quad s_0 \notin z \quad \text{then} \quad \begin{cases} \bar{m}^{s_0}_1 = n - m^{s_0}_{l_{s_0}-1}, \\ \bar{m}^{s_0}_{i+1} = m^{s_0}_i + (n - m^{s_0}_{l_{s_0}-1}), & 1 \leqslant i \leqslant l_{s_0} - 1, \\ \bar{l}_{s_0} = l_{s_0}, \\ \bar{z} = z \cup \{s_0\}. \end{cases}$$

With these definitions, it is clear that $\bar{\mathfrak{a}} \in \mathscr{P}_{\bar{I},\ell}(n, \bar{\mathfrak{m}}, \bar{z})$. A stratification of $\overline{\mathscr{P}}_{I,\ell}(n, \mathfrak{m}, z)$ is then obtained by adding, in addition to sets of the form $\mathscr{P}_{I,\ell}(n, \mathfrak{p}, \tilde{z})$, all sets $\mathscr{P}_{\bar{I},\ell}(n, \bar{\mathfrak{m}}, \bar{z})$ derived from these strata in the manner described above.

2.2. Inequalities for unitary representations

Let Γ_ℓ be as in (1), and fix an integer $n \geqslant 1$. We will denote the *$U(n)$-representation variety* of Γ_ℓ by

$$\mathrm{Hom}(\Gamma_\ell, U(n)) = \{\text{homomorphisms } \rho : \Gamma_\ell \to U(n)\}.$$

We denote the subspaces of irreducible and reducible homomorphisms by $\mathrm{Hom}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$ and $\mathrm{Hom}^{\mathrm{red.}}(\Gamma_\ell, U(n))$, respectively. The group $U(n)$ acts on $\mathrm{Hom}(\Gamma_\ell, U(n))$ (say, on the left) by conjugation. We define the *moduli space of representations* to be the quotient

$$\mathrm{Rep}(\Gamma_\ell, U(n)) = U(n) \backslash \mathrm{Hom}(\Gamma_\ell, U(n)).$$

Following the notation for homomorphisms, subsets of equivalence classes of irreducible and reducible homomorphisms are denoted by $\mathrm{Rep}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$ and $\mathrm{Rep}^{\mathrm{red.}}(\Gamma_\ell, U(n))$, respectively. With the presentation of Γ_ℓ given in (1), to each $[\rho] \in \mathrm{Rep}(\Gamma_\ell, U(n))$ we associate conjugacy classes $\rho(\gamma_1), \ldots, \rho(\gamma_\ell)$.

In this section, we give a brief description of which collections of ℓ conjugacy classes are realized by unitary representations in this way.

Given $A \in U(n)$, we may express its eigenvalues as $(\exp(2\pi i\alpha_1), \ldots, \exp(2\pi i\alpha_n))$, with $0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n < 1$, and this expression is unique. We will therefore write: $\mathrm{spec}(A) = \alpha = (\alpha_1, \ldots, \alpha_n)$. The spectrum determines and is determined uniquely by the conjugacy class of A. If $A_1, \ldots, A_\ell \in U(n)$, $A_1 \cdots A_\ell = \mathbf{I}$, and $\mathrm{spec}(A_s) = \alpha^s$, then by taking determinants we see that the index $I(\alpha_j^s)$ defined in (2) is an *integer*. As in the introduction, we may recast this in terms of representations. For $\rho \in \mathrm{Hom}(\Gamma_\ell, U(n))$, we set $A_s = \rho(\gamma_s)$, and there is a well-defined integer $I = I(\rho)$ associated to ρ. Clearly, $I(\rho)$ depends only on the conjugacy class of the representation, so it is actually well-defined for $[\rho] \in \mathrm{Rep}(\Gamma_\ell, U(n))$.

Definition 2.2. Given $\rho \in \mathrm{Hom}(\Gamma_\ell, U(n))$, the integer $I(\rho)$ is called the *index* of the representation. We define the *spectral projection*

$$\pi : \mathrm{Hom}(\Gamma_\ell, U(n)) \longrightarrow \overline{\mathscr{A}}_\ell^{\mathbb{Z}}(n) \quad : \quad \rho \longmapsto [\mathrm{spec}(\rho(\gamma_1)), \ldots, \mathrm{spec}(\rho(\gamma_\ell))].$$

Then π factors through a map (also denoted π) on $\mathrm{Rep}(\Gamma_\ell, U(n))$. We denote the fibers of π over $\mathfrak{a} \in \overline{\mathscr{A}}_\ell^{\mathbb{Z}}(n)$ by

$$\mathrm{Hom}_{\mathfrak{a}}(\Gamma_\ell, U(n)) = \pi^{-1}(\mathfrak{a}) \subset \mathrm{Hom}(\Gamma_\ell, U(n)),$$

$$\mathrm{Rep}_{\mathfrak{a}}(\Gamma_\ell, U(n)) = \pi^{-1}(\mathfrak{a}) \subset \mathrm{Rep}(\Gamma_\ell, U(n)).$$

The image of π is our main focus in this section.

Definition 2.3. Let $\overline{\mathscr{U}}^*_{I,\ell}(n) = \pi(\mathrm{Hom}(\Gamma_\ell, U(n))) \cap \overline{\mathscr{P}}^*_{I,\ell}(n)$. For each collection of multiplicities $\mathfrak{m} = (m^s)$ and subsets $z \subset \{1, \ldots, \ell\}$, we set

$$\mathscr{U}_{I,\ell}(n, \mathfrak{m}, z) = \overline{\mathscr{U}}^*_{I,\ell}(n) \cap \mathscr{P}_{I,\ell}(n, \mathfrak{m}, z).$$

Definition 2.4. Denote the interior points of $\mathscr{U}_{I,\ell}(n, \mathfrak{m}, z)$ in $\mathscr{P}_{I,\ell}(n, \mathfrak{m}, z)$ by $\mathring{\mathscr{U}}_{I,\ell}(n, \mathfrak{m}, z)$. A stratum $\mathscr{P}_{I,\ell}(n, \mathfrak{m}, z)$ is called *nondegenerate* if either

$$\mathscr{U}_{I,\ell}(n, \mathfrak{m}, z) = \emptyset,$$

or

$$\mathring{\mathscr{U}}_{I,\ell}(n, \mathfrak{m}, z) \neq \emptyset.$$

The regions $\mathscr{U}_{I,\ell}(n, \mathfrak{m}, z)$ have the following simple description (cf. [5, Theorem 3.2, 3,2,15]).

Theorem 2.1. *There is a finite collection $\Phi_{I,\ell}(n)$ of affine linear functions of the $\{\alpha_j^s\}$ such that*

$$\overline{\mathscr{U}}^*_{I,\ell}(n) = \{\mathfrak{a} \in \overline{\mathscr{P}}^*_{I,\ell}(n) : \phi(\mathfrak{a}) \leqslant 0 \quad \textit{for all } \phi \in \Phi_{I,\ell}(n)\}.$$

Moreover, the sets $\Phi_{I,\ell}(n)$, as I varies, are compatible with the stratification described in the previous section.

Definition 2.5. For each $\phi \in \Phi_{I,\ell}(n)$ we define the *outer wall* associated to ϕ by

$$W_\phi = \{\mathfrak{a} \in \mathscr{P}_{I,\ell}(n, \mathfrak{m}, z) : \phi(\mathfrak{a}) = 0\}.$$

We denote the union of all outer walls by

$$\mathscr{W}_{I,\ell}(n, \mathfrak{m}, z) = \bigcup_{\phi \in \Phi_{I,\ell}(n)} W_\phi.$$

It follows that $\mathscr{U}_{I,\ell}(n, \mathfrak{m}, z)$ is the closure in $\mathscr{P}_{I,\ell}(n, \mathfrak{m}, z)$ of a convex connected component of $\mathscr{P}_{I,\ell}(n, \mathfrak{m}, z) \backslash \mathscr{W}_{I,\ell}(n, \mathfrak{m}, z)$. The representations with $\pi(\rho) \in \mathscr{W}_{I,\ell}(n, \mathfrak{m}, z)$ are reducible (see Proposition 2.1). Indeed, the functions ϕ defining the walls are all of the following type. Fix an integer $1 \leqslant k < n$. Choose $\wp_{(k)} = (\wp^1_{(k)}, \ldots, \wp^\ell_{(k)})$, where for each $s = 1, \ldots, \ell$, $\wp^s_{(k)}$ is a subset of $\{1, \ldots, n\}$ of cardinality k. We define a relative index by

$$I(\mathfrak{a}, \wp_{(k)}) = \sum_{s=1}^{\ell} \sum_{\alpha^s_j \in \wp^s_{(k)}} \alpha^s_j. \tag{3}$$

Notice that for $\mathfrak{a} \in \overline{\mathscr{U}}^*_{I,\ell}(n)$ the value of $I(\mathfrak{a}, \wp_{(k)})$ may a priori be any real number less than I. Suppose $\rho \in \mathrm{Hom}_I(\Gamma_\ell, U(n))$ is reducible. Hence, there is a reduction $\rho : \Gamma_\ell \to U(k) \times U(n-k)$ for some $1 \leqslant k < n$. The set of eigenvectors of $\rho(\gamma_s)$ lying in the $U(k)$ factor gives a collection of subsets $\wp^s_{(k)}$. Moreover, it follows, again by taking determinants that the relative index $I(\pi(\rho), \wp_{(k)})$ is equal to some integer K, $0 \leqslant K \leqslant I$. We will say that the reducible representation is *compatible* with $(K, \wp_{(k)})$ if the pair $(K, \wp_{(k)})$ arises from some reduction of ρ. The functions $\phi \in \Phi_{I,\ell}(n)$ are all of the form $\phi(\mathfrak{a}) = I(\mathfrak{a}, \wp_{(k)}) - K$, for various choices of partitions $\wp_{(k)}$ and integers K.

It is not necessarily the case, however, that every reducible ρ projects via π to an *outer* wall. Nevertheless, we see that there is still a hyperplane associated to any reducible. This motivates the following

Definition 2.6. Let $\Psi_{I,\ell}(n)$ be the finite collection of affine linear functions of the form $\psi(\mathfrak{a}) = I(\mathfrak{a}, \wp_{(k)}) - K$, for partitions $\wp_{(k)}$ and positive integers K, such that there is some reducible ρ compatible with $(K, \wp_{(k)})$ for which $\pi(\rho) \in \overset{\circ}{\mathscr{U}}_{I,\ell}(n, \mathfrak{m}, z)$, for some $\mathfrak{m}$, z. For $\psi \in \Psi_{I,\ell}(n)$ we define the *inner wall* associated to ψ by

$$V_\psi = \{\mathfrak{a} \in \mathscr{P}_{I,\ell}(n, \mathfrak{m}, z) : \psi(\mathfrak{a}) = 0\}.$$

We denote the union of all inner walls by

$$\mathscr{V}_{I,\ell}(n, \mathfrak{m}, z) = \bigcup_{\psi \in \Psi_{I,\ell}(n)} V_\psi.$$

Hence, the distinction between the two types of walls is that there are points of $\mathscr{U}_{I,\ell}(n, \mathfrak{m}, z)$ on either side of an inner wall, whereas $\mathscr{U}_{I,\ell}(n, \mathfrak{m}, z)$ lies on only one side of each outer wall.

The precise determination of the functions in $\Phi_{I,\ell}(n)$ is quite involved. In Section 6, we give the result for $\Phi_{I,3}(2)$ and $\Phi_{I,3}(3)$. One way to view the origin of these conditions is via the notion of stable and semistable parabolic structures on holomorphic vector bundles over $\mathbb{C}P^1$. We will require very few details of this theory; the interested reader may consult the references cited above. The following two results are consequences of this holomorphic description. First, we have

Proposition 2.1. *Let* $\rho \in \mathrm{Hom}_I(\Gamma_\ell, U(n))$ *with* $\pi(\rho) \in \mathscr{P}_{I,\ell}(n, \mathfrak{m}, z)$.

(1) *If* $\pi(\rho) \in \mathscr{W}_{I,\ell}(n, \mathfrak{m}, z)$, *then* ρ *is reducible*.
(2) *If* ρ *is reducible, then* $\pi(\rho) \in \mathscr{W}_{I,\ell}(n, \mathfrak{m}, z) \cup \mathscr{V}_{I,\ell}(n, \mathfrak{m}, z)$.
(3) *If* $\pi(\rho) \in \mathring{\mathscr{U}}_{I,\ell}(n, \mathfrak{m}, z)$, *there is an irreducible representation* $\tilde{\rho}$ *with* $\pi(\tilde{\rho}) = \mathfrak{a}$.

Proof. Part (1) follows from the fact that an irreducible representation corresponds to a stable parabolic structure. And if a parabolic structure is stable for a given set of weights, it is also stable for a sufficiently small neighborhood of weights (an alternative, purely representation theoretic proof of this follows from the arguments in Section 4 below). Part (2) is by definition. Part (3) is immediate from [5, Theorem 3.23], since if the strict inequalities are satisfied there exists a stable parabolic structure. Stable structures, as mentioned, correspond to irreducible representations. □

Next, we give sharp bounds on the index.

Theorem 2.2. *For any representation* $\rho : \Gamma_\ell \to U(n)$ *we have*

$$n - N_0(\rho) \leqslant I(\rho) \leqslant n(\ell - 1) + N_0(\rho) - N_1(\rho),$$

where $N_0(\rho)$ *is the number of trivial representations appearing in the decomposition of* ρ *into irreducibles, and* $N_1(\rho)$ *is the total multiplicity of the eigenvalue* 0 *among* $\alpha^s = \rho(\gamma_s)$ *for all* $s = 1, \ldots, \ell$. *Moreover, these bounds are sharp*.

Proof. The case $n = 1$ is straightforward. For $n \geqslant 2$, we first show that $I(\rho) \geqslant n - N_0(\rho)$. Since both sides of this inequality are additive on reducibles, an inequality $I(\rho) \geqslant n$ for irreducible representations proves the result in general by induction. Hence, suppose $\rho : \Gamma_\ell \to U(n)$ is an irreducible representation with $\pi(\rho) = (\alpha_j^s)$ and $I(\rho) < n$. Associated to ρ is a stable parabolic bundle on $\mathbb{C}P^1$ with weights $(\hat{\alpha}_j^s)$ whose underlying holomorphic bundle E has degree $-I(\rho)$ (cf. [17]). By the well-known theorem of Grothendieck, $E \to \mathbb{C}P^1$ is holomorphically split into a sum of line bundles: $E = \mathscr{O}(d_1) \oplus \cdots \oplus \mathscr{O}(d_n)$, where $\mathscr{O}(d)$ denotes the (unique up to isomorphism) holomorphic line bundle of degree d on $\mathbb{C}P^1$. By assumption $\sum_{j=1}^n d_j = \deg E = -I(\rho) > -n$. Hence, there is some $d_j \geqslant 0$. But then E contains a subbundle $\mathscr{O}(d_j)$ with nonnegative parabolic degree. This contradicts parabolic stability, and hence also the assumption $I(\rho) < n$. Thus, the inequality $I(\rho) \geqslant n$ for irreducibles holds. Next, notice that to any representation $\rho : \Gamma_\ell \to U(n)$ we may associate a dual representation $\rho^* : \Gamma_\ell \to U(n)$ defined by: $\rho^*(\gamma_s) = \rho(\gamma_{\ell+1-s})^{-1}$, $s = 1, \ldots, \ell$. Using the convention (2) it follows that $I(\rho^*) = n\ell - I(\rho) - N_1(\rho)$, where $N_1(\rho)$ is defined in the statement of the theorem. Combining this with the previous result $I(\rho) \geqslant n$, we see that $I(\rho) \leqslant n(\ell - 1) - N_1(\rho)$, for ρ irreducible. This argument generalizes to the case where ρ contains trivial factors as well. This completes the proof of the inequality. To prove that the bounds are sharp we need only remark that both sides of the inequalities are additive on reducibles and that the bounds are evidently sharp for the case $n = 1$. □

In Section 3, we will indicate a “Lagrangian” proof of this result for the case $\ell = 3$ (see Proposition 3.2). We conclude this section with one more

Definition 2.7. A connected component of

$$\mathscr{U}_{I,\ell}(n, \mathfrak{m}, z)\backslash\{\mathscr{W}_{I,\ell}(n, \mathfrak{m}, z) \cup \mathscr{V}_{I,\ell}(n, \mathfrak{m}, z)\}$$

will be called a *chamber*.

Remark 2.1.

(1) From the description given above the chambers of $\mathscr{P}_{I,\ell}(n, \mathfrak{m}, z)$ are convex subsets and their boundaries are unions of convex subsets in the intersections of the inner and outer walls.
(2) By Proposition 2.1 (2), if $\pi(\rho)$ is in a chamber then ρ is irreducible.

3. Lagrangian representations

3.1. Linear algebra of Lagrangians in $\mathbb{C}^n$

We denote by $\Lambda(n)$ the $(n/2)(n + 1)$-dimensional manifold of subspaces of $\mathbb{C}^n$ that are Lagrangian with respect to the standard hermitian structure. Fixing a preferred Lagrangian $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$, we observe that $\Lambda(n) = U(n)/O(n)$, where the orthogonal group $O(n) \subset U(n)$ is the stabilizer of L_0 for the action $L_0 \mapsto gL_0$. Define the involution $\sigma_0(z) \to \bar{z}$. Then to each Lagrangian $L = gL_0 = [g] \in \Lambda(n)$ one associates a canonical skew-symplectic complex anti-linear involution $\sigma_L : \mathbb{C}^n \to \mathbb{C}^n$ given by $\sigma_L = g\sigma_0 g^{-1}$, whose set of fixed points is precisely the Lagrangian L. We will set O_L= the stabilizer of L, with Lie algebra $\mathfrak{o}_L$. Note that O_L is simply the conjugate of $O(n)$ by g. Let $\mathfrak{u}(n)$ denote the Lie algebra of $U(n)$ with the Ad-invariant inner product $\langle X, Y\rangle = -\mathrm{Tr}(XY)$. We have the following useful

Lemma 3.1. *For a Lagrangian L*: $\mathrm{Ad}_{\sigma_L}|_{\mathfrak{o}_L} = \mathbf{I}$, *and* $\mathrm{Ad}_{\sigma_L}|_{\mathfrak{o}_L^\perp} = -\mathbf{I}$.

Proof. For $X \in \mathfrak{u}(n)$, $\mathrm{Ad}_{\sigma_L}(X)$ is by definition the derivative at $t = 0$ of the curve $\sigma_L e^{tX}\sigma_L \in U(n)$. In the case $L = \mathbb{R}^n$, σ_L is just complex conjugation, and then $\mathrm{Ad}_{\sigma_L} X = \bar{X}$. Using the orthogonal decomposition $\mathfrak{u}(n) = i\mathbb{R}^n \oplus \mathfrak{o}(n) \oplus \mathfrak{s}(n)$, into diagonal, real orthogonal and symmetric skew-hermitian matrices, the result follows immediately. □

For $g \in U(n)$, let $Z(g)$ denote the centralizer of g with Lie algebra $\mathfrak{z}(g)$. The relationship between the stabilizers of a pair of Lagrangians is given precisely by the following

Proposition 3.1. *Let L_1, L_2 be two Lagrangian subspaces with stabilizers O_1, O_2, and let $g = \sigma_1\sigma_2$ be the composition of the corresponding Lagrangian involutions. Let $\mathfrak{o}_1$, $\mathfrak{o}_2$ denote the Lie algebras of O_1 and O_2. Then*

(1) $O_1 \cap O_2 \subset Z(g)$;
(2) *there is an orthogonal decomposition* $\mathfrak{z}(g) = (\mathfrak{o}_1 + \mathfrak{o}_2)^\perp \oplus (\mathfrak{o}_1 \cap \mathfrak{o}_2)$;
(3) $2\dim(\mathfrak{o}_1 \cap \mathfrak{o}_2) = \dim \mathfrak{z}(g) - n$.

Proof. Observe first that $\mathfrak{z}(g) = \mathrm{Ker}(\mathbf{I} - \mathrm{Ad}_g) = \mathrm{Ker}(\mathbf{I} - \mathrm{Ad}_{\sigma_1\sigma_2})$. Using Lemma 3.1, we obtain: $(\mathfrak{o}_1 + \mathfrak{o}_2)^\perp \oplus (\mathfrak{o}_1 \cap \mathfrak{o}_2) \subset \mathfrak{z}(g)$. Let P denote the orthogonal projection to $\mathfrak{o}_1 \cap \mathfrak{o}_2$, and let $P_1 = (1/2)(\mathbf{I} + \mathrm{Ad}_{\sigma_1})$ and $P_2 = (1/2)(\mathbf{I} + \mathrm{Ad}_{\sigma_2})$ denote the projections to $\mathfrak{o}_1$ and $\mathfrak{o}_2$, respectively. If $X \in \mathfrak{z}(g)$, then $\mathrm{Ad}_{\sigma_1} X = \mathrm{Ad}_{\sigma_2} X$, which implies $P_1 X = P_2 X$. Hence, $P|_{\mathfrak{z}(g)} = P_1|_{\mathfrak{z}(g)} = P_2|_{\mathfrak{z}(g)}$. In particular, if $X \in \mathfrak{z}(g) \cap (\mathfrak{o}_1 \cap \mathfrak{o}_2)^\perp$, then $P_1 X = P_2 X = 0$, and $X \in (\mathfrak{o}_1 + \mathfrak{o}_2)^\perp$. This proves (2). Finally, (3) follows from (2). $\square$

Corollary 3.1. *If $g = \sigma_1\sigma_2$ is regular (i.e. $\mathfrak{z}(g)$ is isomorphic to $i\mathbb{R}^n$), then*

(1) $O_1 \cap O_2 = \{\mathbf{I}\}$,
(2) $O_1 \cap Z(g) = O_2 \cap Z(g) = \{\mathbf{I}\}$.

That is: $\mathfrak{u}(n) = i\mathbb{R}^n \oplus \mathfrak{o}_1 \oplus \mathfrak{o}_2$ (*not necessarily orthogonal*).

Definition 3.1. We define three maps:

$$\tau_1 : \Lambda(n) \longrightarrow U(n) : L \longmapsto \sigma_L\sigma_0;$$

$$\tau_2 : \Lambda^2(n) \longrightarrow U(n) : (L_1, L_2) \longmapsto \sigma_{L_1}\sigma_{L_2};$$

$$\tau_3 : \Lambda^3(n) \longrightarrow U^2(n) : (L_1, L_2, L_3) \longmapsto (\tau_2(L_1, L_2), \tau_2(L_2, L_3)).$$

Lemma 3.2. *We have the following*:

(1) $\tau_1([g]) = gg^T$;
(2) $\tau_2(L_1, L_2) = \tau_1(L_1)\overline{\tau_1(L_2)}$, *and* $\tau_2(L, L) = \mathbf{I}$;
(3) $\tau_2(L_1, L_3) = \tau_2(L_1, L_2)\tau_2(L_2, L_3)$.

We prove some elementary facts about each of these maps. Let $S(n)$ denote the space of symmetric $n \times n$ complex matrices.

Proposition 3.2. *The map $\tau_1 : \Lambda(n) \to U(n)$ is an embedding with image $U(n) \cap S(n)$.*

Proof. The fact that the image consists of symmetric matrices is the statement Lemma 3.2 (1). We prove that τ_1 is injective. If $\tau_1([g]) = \tau_1([h])$, then: $gg^T = hh^T$; hence $h^{-1}g \in U(n) \cap O(n, \mathbb{C})$. But $U(n) \cap O(n, \mathbb{C}) = O(n)$, so we conclude that $g \in hO(n)$, and $[g] = [h]$. To prove τ_1 is an embedding we compute its derivative. Any variation of L is determined up to first order by a variation of the involution σ_L of the form $\sigma_{L(t)} = e^{tX}\sigma_L e^{-tX}$, where $X \in \mathfrak{u}(n)$. Then: $\dot{\sigma}_L = [X, \sigma_L]$, so $\dot{\sigma}_L\sigma_L \in \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\sigma_L})$. In particular, $\dot{\sigma}_L\sigma_L = 0 \iff X \in \mathfrak{o}_L \iff L(t) \equiv L$. With this understood, we have $\dot{\tau}_1(L)\tau_1^{-1}(L) = (\dot{\sigma}_L\sigma_0)(\sigma_0\sigma_L) = \dot{\sigma}_L\sigma_L$. Hence, by the discussion above, τ_1 is an immersion. One may show that the image is all of $S(n)$ either by noticing that dimensions agree, or directly using the following result, whose proof is straightforward.

Lemma 3.3. *If $g \in U(n) \cap S(n)$ there is $h \in O(n)$ such that hgh^{-1} is diagonal.*

Now take g and h as in the lemma. Clearly, there exists $k \in U(n)$ such that $kk^T = hgh^{-1}$. Then: $\tau_1(hk) = g$. $\square$

Proposition 3.3. $\tau_2 : \Lambda^2(n) \to U(n)$ *is surjective and is equivariant with respect to the diagonal action on the domain and the conjugation action in the target. Over the regular elements of* $U(n)$ (*i.e. those whose eigenvalues have multiplicity one*) τ_2 *is a fibration with fiber the torus* T^n. *The general fiber is*: $\tau_2^{-1}(g) = Z(g) \cap S(n)$, *where* $Z(g)$ *is the centralizer of* g.

Proof. Equivariance is an easy computation. As a consequence, it suffices to prove the remaining statements for a diagonal $g \in U(n)$. For such a g we can solve $g = \tau_2([g_1], [g_2])$, and we may even assume g_1 and g_2 are diagonal. Let $g = h_1 h_2$ with $h_1 = \tau_1([g_1])$ and $h_2 = \overline{\tau_1([g_2])}$. Since h_2 is determined by h_1 and τ_1 is an embedding, it suffices to find all possible h_1. Note that since g is diagonal and h_1, h_2 are symmetric, $h_1, h_2 \in Z(g) \cap S(n)$. Conversely, if $h_1 \in Z(g) \cap S(n)$, then by Proposition 3.2, $h_1 \in \mathrm{Im}(\tau_1)$. Since $h_2 = h_1^{-1} g$, we obtain $h_2^T = g^T (h_1^{-1})^T = g h_1^{-1} = h_1^{-1} g = h_2$. We conclude that h_2 is also symmetric, and hence $h_2 \in \mathrm{Im}(\tau_1)$. Thus, $\tau_2^{-1}(g)$ is diffeomorphic to $Z(g) \cap S(n)$. □

Note that $Z(g) \cap S(n) = S(n_1) \cap U(n_1) \times \cdots \times S(n_k) \cap U(n_k)$, where n_i, for $1 \leqslant i \leqslant k$, are the multiplicities of the eigenvalues of g. Finally, we determine the image of τ_3.

Definition 3.2. A pair $k_1, k_2 \in U(n)$ is said to be *symmetrizable* if there is $g \in U(n)$ such that both $g k_1 g^{-1}, g k_2 g^{-1} \in S(n)$. The set of symmetrizable pairs will be denoted by $\mathrm{Sym}_2(n)$.

Proposition 3.4. *The image of* τ_3 *is precisely the set of symmetrizable pairs*: $\mathrm{Sym}_2(n) \subset U^2(n)$.

Proof. Clearly if $\tau_3([g_1], [g_2], [g_3]) = (h_1, h_2)$, then $\tau_3([g_2^{-1} g_1], L_0, [g_2^{-1} g_3]) = (g_2^{-1} h_1 g_2, g_2^{-1} h_2 g_2)$. But $g_2^{-1} h_1 g_2 = \tau_2([g_2^{-1} g_1], L_0) = \overline{\tau_1([g_2^{-1} g_1])}$ and $g_2^{-1} h_2 g_2 = \tau_2(L_0, [g_2^{-1} g_3]) = \overline{\tau_1([g_2^{-1} g_3])}$ which are symmetric. Therefore $(h_1, h_2) \in \mathrm{Sym}_2(n)$. Conversely, suppose $(h_1, h_2) \in \mathrm{Sym}_2(n)$, and let g be a matrix such that $g h_1 g^{-1}, g h_2 g^{-1} \in S(n)$. We can solve

$$\tau_2([g_1], L_0) = \tau_1([g_1]) = g h_1 g^{-1}, \quad \tau_2(L_0, [g_2]) = \overline{\tau_1([g_2])} = g h_2 g^{-1}.$$

Then $\tau_3([g_1], L_0, [g_2]) = (g h_1 g^{-1}, g h_2 g^{-1})$. Since τ_3 is equivariant, acting by g^{-1} gives the result. □

3.2. The space of Lagrangian representations

We now define the main object of study in this paper. Fix an integer $\ell \geqslant 3$. Given the presentation (1), a representation $\rho \in \mathrm{Hom}(\Gamma_\ell, U(n))$ is equivalent to a choice of ℓ matrices whose product is the identity. By Lemma 3.2 (2) and (3), we therefore have a map

$$\begin{aligned} &\tilde{\varphi} : \Lambda^\ell(n) \longrightarrow \mathrm{Hom}(\Gamma_\ell, U(n)), \\ &(L_1, \ldots, L_\ell) \longmapsto (\tau_2(L_1, L_2), \tau_2(L_2, L_3), \ldots, \tau_2(L_\ell, L_1)). \end{aligned} \tag{4}$$

$U(n)$ acts diagonally on the left of $\Lambda^\ell(n)$, and by Proposition 3.3, $\tilde{\varphi}$ is equivariant with respect to this action and the left action by conjugation of $U(n)$ on $\mathrm{Hom}(\Gamma_\ell, U(n))$. Hence, we have an induced map

$$\varphi : U(n) \backslash \Lambda^\ell(n) \longrightarrow \mathrm{Rep}(\Gamma_\ell, U(n)).$$

Given $\lambda = (L_1, \ldots, L_\ell) \in \Lambda^\ell(n)$, let $Z(\lambda) = O_{L_1} \cap \cdots \cap O_{L_s} \subset U(n)$ denote the stabilizer, and let $\mathfrak{z}(\lambda)$ be its Lie algebra. Similarly, for $\rho \in \mathrm{Hom}(\Gamma_\ell, U(n))$, let $Z(\rho)$ denote its stabilizer with Lie algebra

$\mathfrak{z}(\rho)$. Because of the equivariance of $\tilde{\varphi}$, $Z(\lambda) \subset Z(\rho)$, where $\rho = \tilde{\varphi}(\lambda)$, but the two groups are not equal. For example, the center $U(1)$ is always in $Z(\rho)$ but never in $Z(\lambda)$. The precise relationship is given by the following

Lemma 3.4. *Given $\lambda \in \Lambda^\ell(n)$, then* $\mathrm{Ker}(D\tilde{\varphi}_\lambda) \subset \mathfrak{u}(n)$, *where* $\mathfrak{u}(n) \to T_\lambda \Lambda^\ell(n)$ *via the $U(n)$ action. If $\rho = \tilde{\varphi}(\lambda)$, then* $\mathfrak{z}(\rho) = \mathrm{Ker}(D\tilde{\varphi}_\lambda) \oplus \mathfrak{z}(\lambda)$.

Proof. Let $\sigma_s = \sigma_{L_s}$, with $\sigma_{\ell+1} = \sigma_1$. Then: $\tilde{\varphi}(\lambda) = (\gamma_1, \ldots, \gamma_\ell)$, where $\gamma_s = \sigma_s \sigma_{s+1}$ (see Definition 3.1 and (4)). Let $\dot{\lambda}$ be a tangent vector to $\Lambda^\ell(n)$ at λ. Expressing the components of the image $D\tilde{\varphi}_\lambda(\dot{\lambda}) = (X_1, \ldots, X_s)$ as elements of $\mathfrak{u}(n)$, we have: $X_s = \dot{\gamma}_s \gamma_s^{-1}$. Hence,

$$X_s = (\dot{\sigma}_s \sigma_{s+1} + \sigma_s \dot{\sigma}_{s+1}) \sigma_{s+1} \sigma_s = \dot{\sigma}_s \sigma_s + \sigma_s \dot{\sigma}_{s+1} \sigma_{s+1} \sigma_s. \tag{5}$$

Since σ_s is an involution, we conclude from the equation above that $\dot{\lambda} \in \mathrm{Ker}(D\tilde{\varphi}_\lambda)$ if and only if $\sigma_s \dot{\sigma}_s = \sigma_{s+1} \dot{\sigma}_{s+1}$, for all $s = 1, \ldots, \ell$. As in the proof of Proposition 3.2, $\sigma_s \dot{\sigma}_s \in \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\sigma_s})$. If we let O_s denote the stabilizer of the Lagrangian corresponding to σ_s, and if $\mathfrak{o}_s$ is the Lie algebra of O_s, then the kernel of $D\tilde{\varphi}_\lambda$ is determined by an element in

$$\begin{aligned} \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\sigma_1}) \cap \cdots \cap \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\sigma_\ell}) &= \mathfrak{o}_1^\perp \cap \cdots \cap \mathfrak{o}_\ell^\perp = (\mathfrak{o}_1 + \cdots + \mathfrak{o}_\ell)^\perp \\ &= (\mathfrak{o}_1 + \mathfrak{o}_2 + \mathfrak{o}_2 + \mathfrak{o}_3 + \cdots + \mathfrak{o}_{\ell-1} + \mathfrak{o}_\ell)^\perp \\ &= (\mathfrak{o}_1 + \mathfrak{o}_2)^\perp \cap \cdots \cap (\mathfrak{o}_{\ell-1} + \mathfrak{o}_\ell)^\perp. \end{aligned}$$

By Proposition 3.1 (2) $(\mathfrak{o}_s + \mathfrak{o}_{s+1})^\perp \subset \mathfrak{z}(\gamma_s)$. Since

$$\mathfrak{z}(\rho) = \mathfrak{z}(\gamma_1) \cap \cdots \cap \mathfrak{z}(\gamma_{\ell-1}) = (\mathfrak{o}_1 \cap \cdots \cap \mathfrak{o}_\ell) \oplus (\mathfrak{o}_1 + \mathfrak{o}_2)^\perp \cap \cdots \cap (\mathfrak{o}_{\ell-1} + \mathfrak{o}_\ell)^\perp,$$

and $\mathfrak{z}(\lambda) = \mathfrak{o}_1 \cap \cdots \cap \mathfrak{o}_\ell$, the result follows. $\square$

We take the opportunity to point out a fact about the image of $D\tilde{\varphi}_\lambda$.

Lemma 3.5. *Let $(X_1, \ldots, X_\ell) \in \mathrm{Im}(D\tilde{\varphi}_\lambda)$, with λ as above. Then: $X_s \in (\mathfrak{o}_s \cap \mathfrak{o}_{s+1})^\perp$ for each $s = 1, \ldots, \ell$.*

Proof. From Lemma 3.1 and the proof of Lemma 3.4, we have

$$\dot{\sigma}_s \sigma_s \in \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\sigma_s}) = \mathfrak{o}_s^\perp, \quad \dot{\sigma}_{s+1} \sigma_{s+1} \in \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\sigma_{s+1}}) = \mathfrak{o}_{s+1}^\perp.$$

Now if $Z \in \mathfrak{o}_s \cap \mathfrak{o}_{s+1}$, then by (5) and Lemma 3.1 again,

$$\langle Z, X_s \rangle = \langle Z, \mathrm{Ad}_{\sigma_s}(\dot{\sigma}_{s+1}\sigma_{s+1}) \rangle = \langle \mathrm{Ad}_{\sigma_s} Z, \dot{\sigma}_{s+1}\sigma_{s+1} \rangle = \langle Z, \dot{\sigma}_{s+1}\sigma_{s+1} \rangle = 0. \qquad \square$$

Definition 3.3. A representation $\rho \in \mathrm{Hom}(\Gamma_\ell, U(n))$ is called a *Lagrangian representation* if it is in the image of $\tilde{\varphi}$. We denote the space of *Lagrangian representations* by

$$\mathscr{L}\ \mathrm{Hom}(\Gamma_\ell, U(n)) = \mathrm{Im}(\tilde{\varphi}) \subset \mathrm{Hom}(\Gamma_\ell, U(n)).$$

Similarly, the image of φ is the *moduli space of Lagrangian representations*.

$$\mathscr{L}\ \mathrm{Rep}(\Gamma_\ell, U(n)) = \mathrm{Im}(\varphi) \subset \mathrm{Rep}(\Gamma_\ell, U(n)).$$

We also set

$$\mathscr{L}\ \mathrm{Hom}_{\mathfrak{a}}(\Gamma_\ell, U(n)) = \mathscr{L}\ \mathrm{Hom}(\Gamma_\ell, U(n)) \cap \mathrm{Hom}_{\mathfrak{a}}(\Gamma_\ell, U(n)),$$

$$\mathscr{L}\ \mathrm{Rep}_{\mathfrak{a}}(\Gamma_\ell, U(n)) = \mathscr{L}\ \mathrm{Rep}(\Gamma_\ell, U(n)) \cap \mathrm{Rep}_{\mathfrak{a}}(\Gamma_\ell, U(n)).$$

From general considerations of group actions, $\mathrm{Rep}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$ is a smooth (open) manifold, since the isotropy $Z(\rho)$ of an irreducible representation ρ is just the center of $U(n)$. Let: $\Lambda^n_{\mathrm{irr.}}(n) = \tilde{\varphi}^{-1}(\mathrm{Hom}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$. Then for Lagrangian representations we have the following

Proposition 3.5. (1) *For $\lambda \in \Lambda^\ell(n)$ and $\rho = \tilde{\varphi}(\lambda)$, the fiber $\tilde{\varphi}^{-1}(\rho) \simeq Z(\rho)/Z(\lambda)$. In particular, $\mathscr{L}\ \mathrm{Hom}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$ is an embedded submanifold of dimension*

$$\dim(\mathscr{L}\ \mathrm{Hom}^{\mathrm{irr.}}(\Gamma_\ell, U(n))) = \frac{(\ell-1)}{2}n^2 + \frac{\ell}{2}n - 1,$$

and: $\tilde{\varphi} : \Lambda^\ell_{\mathrm{irr.}}(n) \to \mathscr{L}\ \mathrm{Hom}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$ *is a circle bundle.*

(2) *$U(n)$ acts freely on $\Lambda^n_{\mathrm{irr.}}(n)$. Moreover,*

$$\varphi : U(n)\backslash\Lambda^\ell_{\mathrm{irr.}}(n) \longrightarrow \mathscr{L}\ \mathrm{Rep}^{\mathrm{irr.}}(\Gamma_\ell, U(n)) \subset \mathrm{Rep}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$$

is an embedding with

$$\dim(\mathscr{L}\ \mathrm{Rep}^{\mathrm{irr.}}(\Gamma_\ell, U(n))) = \frac{(\ell-2)}{2}n^2 + \frac{\ell}{2}n.$$

Proof. We determine the fiber of $\tilde{\varphi}$. Suppose $\rho = \tilde{\varphi}(\lambda) = \tilde{\varphi}(\lambda')$, where $\lambda = (L_1, \ldots, L_\ell)$ and $\lambda' = (L'_1, \ldots, L'_\ell)$. By Propositions 3.2 and 3.3, $L'_1 = hL_1$ and $L'_2 = hL_2$ for $h \in Z(\rho(\gamma_1)) \cap S(n)$. Applying the result to each pair L_s, L_{s+1}, we see that in fact $h \in Z(\rho(\gamma_1)) \cap \cdots \cap Z(\rho(\gamma_{\ell-1})) \cap S(n)$. In particular, $h \in Z(\rho)$. Conversely, by equivariance, $Z(\rho)$ acts on the fiber of $\tilde{\varphi}$ with $Z(\lambda)$. The remaining statements follow from Lemma 3.4. □

We will denote the restriction of the spectral projection to the Lagrangian representations also by $\pi : \mathscr{L}\ \mathrm{Hom}(\Gamma_\ell, U(n)) \to \mathscr{A}^{\mathbb{Z}}_\ell(n)$. By analogy with Definition 2.3, we have

Definition 3.4. Let $\overline{\mathscr{L}}^*_{I,\ell}(n) = \pi(\mathscr{L}\ \mathrm{Hom}(\Gamma_\ell, U(n))) \cap \overline{\mathscr{P}}^*_{I,\ell}(n)$. For each collection of multiplicities $\mathfrak{m} = (m^s)$, and subsets $z \subset \{1, \ldots, \ell\}$, we set: $\mathscr{L}_{I,\ell}(n, \mathfrak{m}, z) = \overline{\mathscr{L}}^*_{I,\ell}(n) \cap \mathscr{P}_{I,\ell}(n, \mathfrak{m}, z)$.

From the definition we have: $\overline{\mathscr{L}}^*_{I,\ell}(n) \subset \overline{\mathscr{U}}^*_{I,\ell}(n)$. The goal of this paper is to prove that in fact $\overline{\mathscr{L}}^*_{I,\ell}(n) = \overline{\mathscr{U}}^*_{I,\ell}(n)$. Assuming Theorem 1, however, we may now give the

Proof of Theorem 3. By Theorem 1, the conjugacy classes of $A_1, \ldots, A_\ell$ may be realized by a Lagrangian representation. Hence, we may find B_i as in the statement of Theorem 3 such that $B_i = \sigma_{L_i}\sigma_{L_{i+1}}$ for Lagrangians $L_1, \ldots, L_\ell$, where $L_{\ell+1} = L_1$. In particular, the pair (B_i, B_{i+1}) is in the image of τ_3 for each i. The result then follows from Proposition 3.4. □

3.3. The symplectic structure

The purpose of this section is to show that the tangent space to the Lagrangian representations for fixed conjugacy classes is isotropic with respect to the natural symplectic form. We begin with a brief review of quasi-Hamiltonian reduction. For more details, see [1]. Let (M, ω) be a manifold equipped with a 2-form ω, G a Lie group with Lie algebra $\mathfrak{g}$ and $G \times M \to M$ a Lie group action preserving ω. In order to define a G-valued moment map we assume the existence of an Ad-invariant inner product $\langle\, ,\, \rangle$ on $\mathfrak{g}$. Let θ^{R} and θ^{L} be the right and left Maurer–Cartan forms on G. That is, for $V \in T_g G$, $\theta^{\mathrm{L}}_g(V) = g^{-1}V \in \mathfrak{g}$ and $\theta^{\mathrm{R}}_g(V) = Vg^{-1} \in \mathfrak{g}$ ($g^{-1}\,\mathrm{d}g$ and $\mathrm{d}g g^{-1}$ in matrix groups). Let χ be the bi-invariant closed Cartan 3-form defined by

$$\chi = \tfrac{1}{2}\,\langle \theta^{\mathrm{L}}, [\theta^{\mathrm{L}}, \theta^{\mathrm{L}}]\rangle = \tfrac{1}{2}\,\langle \theta^{\mathrm{R}}, [\theta^{\mathrm{R}}, \theta^{\mathrm{R}}]\rangle.$$

Definition 3.5. A *quasi-Hamiltonian G-space* (M, G, ω, μ) is a manifold equipped with a 2-form ω that is invariant under the action of G and an equivariant moment map $\mu : M \to G$ satisfying

(1) $\mathrm{d}\omega = -\mu^*\chi$
(2) $\imath_{\xi^\#}\omega = \frac{1}{2}\langle \mu^*(\theta^{\mathrm{L}} + \theta^{\mathrm{R}}), \xi\rangle$
(3) $\ker \omega_x = \{\xi^\#(x) \mid \xi \in \ker(\mathbf{I} + \mathrm{Ad}_{\mu(x)})\}$.

Here, $\xi^\#$ denotes the vector field on M induced by $\xi \in \mathfrak{g}$ and the action of G. The following theorem is proved in [1].

Theorem 3.1. *Let (M, G, ω, μ) be a quasi-Hamiltonian space as above. Let $\imath : \mu^{-1}(\mathbf{I}) \to M$ be the inclusion and $p : \mu^{-1}(\mathbf{I}) \to M^{\mathrm{red.}} = \mu^{-1}(\mathbf{I})/G$ the projection on the orbit space. Then there exists a unique symplectic form ω^{red} on the smooth stratum of the reduced space M^{red} such that $p^*\omega^{\mathrm{red}} = \imath^*\omega$ on $\mu^{-1}(\mathbf{I})$.*

This formulation of symplectic reduction is well-adapted to computations on the representation space of the free group with fixed conjugacy classes. Let $\mathrm{Hom}_{\mathfrak{a}}(\Gamma_\ell, U(n))$ and $\mathrm{Rep}_{\mathfrak{a}}(\Gamma_\ell, U(n))$ be as in Definition 2.2. Then $\mathrm{Hom}_{\mathfrak{a}}(\Gamma_\ell, U(n))$ is naturally contained in $M_{\mathfrak{a}} = C_1 \times \cdots \times C_\ell$ where $\{C_s\}$ are the conjugacy class of $U(n)$ prescribed by $\mathfrak{a}$. Moreover, $\mathrm{Hom}_{\mathfrak{a}}(\Gamma_\ell, U(n)) = \mu^{-1}(\mathbf{I})$, where $\mu(\gamma_1, \ldots, \gamma_\ell) = \gamma_1\gamma_2\cdots\gamma_\ell \in U(n)$, and $\mathrm{Rep}_{\mathfrak{a}}(\Gamma_\ell, U(n)) = \mu^{-1}(\mathbf{I})/U(n)$. To describe the form ω, we require

Definition 3.6. Let (M_1, ω_1, μ_1) and (M_2, ω_2, μ_2) be two quasi-Hamiltonian G-spaces. Then $M_1 \times M_2$ is also a quasi-Hamiltonian G-space, called the *fusion product of* M_1 and M_2. The moment map is given by $\mu_1\mu_2 : M_1 \times M_2 \to G$, and the 2-form is given by $\omega = \omega_1 + \omega_2 + \langle \mu_1^*\theta^{\mathrm{L}} \wedge \mu_2^*\theta^{\mathrm{R}}\rangle$.

Explicitly, we have

$$\langle \mu_1^*\theta^{\mathrm{L}} \wedge \mu_1^*\theta^{\mathrm{R}}\rangle((v_1, v_2), (w_1, w_2)) = \tfrac{1}{2}\,(\langle \mu_1^*\theta^{\mathrm{L}}(v_1), \mu_2^*\theta^{\mathrm{R}}(w_2)\rangle - \langle \mu_1^*\theta^{\mathrm{L}}(w_1), \mu_2^*\theta^{\mathrm{R}}(v_2)\rangle).$$

To find the expression of the fusion product for a product conjugacy classes, recall that the fundamental vector field corresponding to $\xi \in \mathfrak{g}$ at a point γ is

$$\xi^\# = \xi\gamma - \gamma\xi = (\mathbf{I} - \mathrm{Ad}_\gamma)\xi\gamma = \gamma(\mathrm{Ad}_{\gamma^{-1}} - \mathbf{I})\xi.$$

The 2-form on a conjugacy class C is given by

$$\omega_\gamma(\xi^\#, \eta^\#) = \tfrac{1}{2}\,(\langle \mathrm{Ad}_\gamma \xi, \eta\rangle - \langle \mathrm{Ad}_\gamma \eta, \xi\rangle).$$

For the product of two conjugacy classes C_1 and C_2, let $\mu_i : C_i \to G$ be the tautological embeddings. Then

$$\begin{aligned}\mu_1^*\theta^{\mathrm{L}}(\xi_1^\#) &= \theta^{\mathrm{L}}(\mu_{1*}\xi_1^\#) = \theta^{\mathrm{L}}(\xi_1^\#) = \theta^{\mathrm{L}}(\gamma_1(\mathrm{Ad}_{\gamma_1^{-1}} - \mathbf{I})\xi_1)\\ &= \gamma_1^{-1}\gamma_1(\mathrm{Ad}_{\gamma_1^{-1}} - \mathbf{I})\xi_1 = (\mathrm{Ad}_{\gamma_1^{-1}} - \mathbf{I})\xi_1.\end{aligned}$$

Similarly, $\mu_2^*\theta^{\mathrm{R}}(\eta_2^\#) = (\mathbf{I} - \mathrm{Ad}_{\gamma_2})\eta_2$. Using these formulas, the 2-form on the product $C_1 \times C_2$ of two conjugacy classes is

$$\begin{aligned}\omega_{(\gamma_1,\gamma_2)}((\xi_1^\#, \xi_2^\#), (\eta_1^\#, \eta_2^\#)) = {}& \tfrac{1}{2}\,(\langle \mathrm{Ad}_{\gamma_1}\xi_1, \eta_1\rangle - \langle \mathrm{Ad}_{\gamma_1}\eta_1, \xi_1\rangle)\\ &+ \tfrac{1}{2}\,(\langle \mathrm{Ad}_{\gamma_2}\xi_2, \eta_2\rangle - \langle \mathrm{Ad}_{\gamma_2}\eta_2, \xi_2\rangle)\\ &+ \tfrac{1}{2}\,\langle (\mathbf{I} - \mathrm{Ad}_{\gamma_1})\xi_1, \mathrm{Ad}_{\gamma_1}(\mathbf{I} - \mathrm{Ad}_{\gamma_2})\eta_2\rangle - \{\xi \leftrightarrow \eta\}\end{aligned}$$

where $\xi \leftrightarrow \eta$ means that the previous terms are repeated with ξ and η interchanged, keeping the indices unchanged. In general, for the product $C_1 \times \cdots \times C_\ell$ we obtain

$$\begin{aligned}&\omega_{(\gamma_1,\ldots,\gamma_\ell)}((\xi_1^\#, \ldots, \xi_\ell^\#), (\eta_1^\#, \ldots, \eta_\ell^\#))\\ &\quad= \frac{1}{2}\left\{\sum_{s=0}^{\ell} \langle \mathrm{Ad}_{\gamma_s}\xi_s, \eta_s\rangle + \sum_{t=1}^{\ell-1} \langle (\mathbf{I} - \mathrm{Ad}_{\gamma_1})\xi_1 + \mathrm{Ad}_{\gamma_1}(\mathbf{I} - \mathrm{Ad}_{\gamma_2})\xi_2\right.\\ &\qquad\left. + \cdots + \mathrm{Ad}_{\gamma_1\cdots\gamma_{t-1}}(\mathbf{I} - \mathrm{Ad}_{\gamma_t})\xi_t, \mathrm{Ad}_{\gamma_1\cdots\gamma_t}(\mathbf{I} - \mathrm{Ad}_{\gamma_{t+1}})\eta_{t+1}\rangle\right\} - \{\xi \leftrightarrow \eta\}\\ &\quad= \frac{1}{2}\left\{\sum_{s=0}^{\ell} \langle \mathrm{Ad}_{\gamma_s}\xi_s, \eta_s\rangle + \cdots + \sum_{0 \leqslant s < t \leqslant \ell-1} \langle \mathrm{Ad}_{\gamma_1\cdots\gamma_s}(\mathbf{I} - \mathrm{Ad}_{\gamma_{s+1}})\xi_{s+1},\right.\\ &\qquad\left. \mathrm{Ad}_{\gamma_1\cdots\gamma_t}(\mathbf{I} - \mathrm{Ad}_{\gamma_{t+1}})\eta_{t+1}\rangle\right\}\\ &\qquad - \{\xi \leftrightarrow \eta\}.\end{aligned}$$

Proposition 3.6. *The product of conjugacy classes of a compact Lie group G, $C_1 \times \cdots \times C_\ell$ is a quasi-Hamiltonian space equipped with the moment map which is the product of the embeddings in G and the following 2-form*:

$$\begin{aligned}&\omega_{(\gamma_1,\ldots,\gamma_\ell)}((\xi_1^\#, \ldots, \xi_\ell^\#), (\eta_1^\#, \ldots, \eta_\ell^\#))\\ &\quad= \frac{1}{2}\left\{\sum_{s=0}^{\ell} (\mathrm{Ad}_{\gamma_s}\xi_s, \eta_s) + \sum_{0 \leqslant s < t \leqslant \ell-1} (\mathrm{Ad}_{\gamma_1\cdots\gamma_s}(\mathbf{I} - \mathrm{Ad}_{\gamma_{s+1}})\xi_{s+1}, \mathrm{Ad}_{\gamma_1\cdots\gamma_t}(\mathbf{I} - \mathrm{Ad}_{\gamma_{t+1}})\eta_{t+1})\right\}\\ &\qquad - \{\xi \leftrightarrow \eta\}.\end{aligned}$$

Proposition 3.7. *The moduli space of moduli of Lagrangian representations*

$$\mathscr{L}\ \mathrm{Rep}_{\mathfrak{a}}(\Gamma_\ell, U(n)) \subset \mathrm{Rep}_{\mathfrak{a}}(\Gamma_\ell, U(n))$$

is isotropic with respect to the symplectic structure defined by Proposition 3.6 *and Theorem* 3.1.

Proof. Let

$$X_s = \dot\sigma_s\sigma_s + \mathrm{Ad}_{\sigma_s}(\dot\sigma_{s+1}\sigma_{s+1}), \quad Y_s = \dot\rho_s\rho_s + \mathrm{Ad}_{\sigma_s}(\dot\rho_{s+1}\rho_{s+1}).$$

where $\rho_s = \sigma_s$ (see (5)). By the assumption of fixed conjugacy classes, we have

$$\dot\sigma_s\sigma_s = \xi_s - \mathrm{Ad}_{\sigma_s}\xi_s, \quad \dot\rho_s\rho_s = \eta_s - \mathrm{Ad}_{\sigma_s}\eta_s,$$

$$\dot\sigma_{s+1}\sigma_{s+1} = \xi_s - \mathrm{Ad}_{\sigma_{s+1}}\xi_s, \quad \dot\rho_{s+1}\rho_{s+1} = \eta_s - \mathrm{Ad}_{\sigma_{s+1}}\eta_s.$$

In particular,

$$\mathrm{Ad}_{\sigma_s}X_s = \dot\sigma_{s+1}\sigma_{s+1} - \dot\sigma_s\sigma_s, \quad \mathrm{Ad}_{\sigma_s}Y_s = \dot\rho_{s+1}\rho_{s+1} - \dot\rho_s\rho_s. \tag{6}$$

It follows that

$$\begin{aligned}\langle \mathrm{Ad}_{\gamma_s}\xi_s, \eta_s\rangle &= \langle \mathrm{Ad}_{\sigma_{s+1}}\xi_s, \mathrm{Ad}_{\sigma_s}\eta_s\rangle = \langle \xi_s - \dot\sigma_{s+1}\sigma_{s+1}, \eta_s - \dot\rho_s\rho_s\rangle \\ &= \langle \xi_s, \eta_s\rangle + \langle \dot\sigma_{s+1}\sigma_{s+1}, \dot\rho_s\rho_s\rangle - \langle \xi_s, \dot\rho_s\rho_s\rangle - \langle \eta_s, \dot\sigma_{s+1}\sigma_{s+1}\rangle.\end{aligned}$$

Notice that since $\dot\rho_s\rho_s$ is in the (-1)-eigenspace of Ad_{σ_s},

$$2\langle \xi_s, \dot\rho_s\rho_s\rangle = \langle \xi_s - \mathrm{Ad}_{\sigma_s}\xi_s, \dot\rho_s\rho_s\rangle = \langle \dot\sigma_s\sigma_s, \dot\rho_s\rho_s\rangle.$$

Similarly, $2\langle \eta_s, \dot\sigma_{s+1}\sigma_{s+1}\rangle = \langle \dot\rho_{s+1}\rho_{s+1}, \dot\sigma_{s+1}\sigma_{s+1}\rangle$. Because of the symmetry upon interchanging σ and ρ, these terms cancel, and we are left with

$$\sum_{s=1}^{\ell}\langle \mathrm{Ad}_{\gamma_s}\xi_s, \eta_s\rangle - \langle \mathrm{Ad}_{\gamma_s}\eta_s, \xi_s\rangle = \sum_{s=1}^{\ell}\langle \dot\sigma_{s+1}\sigma_{s+1}, \dot\rho_s\rho_s\rangle - \langle \dot\sigma_s\sigma_s, \dot\rho_{s+1}\rho_{s+1}\rangle. \tag{7}$$

For the second term, notice that for a Lagrangian representation $\gamma_1\cdots\gamma_s = \sigma_1\sigma_{s+1}$. Hence,

$$\begin{aligned}\sum_{0\leqslant s<t\leqslant \ell-1}\langle \mathrm{Ad}_{\gamma_1\cdots\gamma_s}X_{s+1}, \mathrm{Ad}_{\gamma_1\cdots\gamma_t}Y_{t+1}\rangle &= \sum_{0\leqslant s<t\leqslant \ell-1}\langle \mathrm{Ad}_{\sigma_{s+1}}X_{s+1}, \mathrm{Ad}_{\sigma_{t+1}}Y_{t+1}\rangle \\ &= \sum_{1\leqslant s<t\leqslant \ell}\langle \mathrm{Ad}_{\sigma_s}X_s, \mathrm{Ad}_{\sigma_t}Y_t\rangle.\end{aligned}$$

Using (6) (and recalling the convention that $\rho_{\ell+1} = \rho_1$) we have

$$\begin{aligned}
&\sum_{0\leqslant s<t\leqslant \ell-1} \langle \mathrm{Ad}_{\gamma_1\cdots\gamma_s} X_{s+1}, \mathrm{Ad}_{\gamma_1\cdots\gamma_t} Y_{t+1}\rangle \\
&\quad= \sum_{1\leqslant s<t\leqslant \ell} \langle \dot\sigma_{s+1}\sigma_{s+1} - \dot\sigma_s\sigma_s, \dot\rho_{t+1}\rho_{t+1} - \dot\rho_t\rho_t\rangle \\
&\quad= \sum_{1\leqslant s\leqslant \ell-1} \langle \dot\sigma_{s+1}\sigma_{s+1} - \dot\sigma_s\sigma_s, \dot\rho_1\rho_1 - \dot\rho_{s+1}\rho_{s+1}\rangle \\
&\quad= \sum_{1\leqslant s\leqslant \ell-1} \langle \dot\sigma_s\sigma_s, \dot\rho_{s+1}\rho_{s+1}\rangle - \langle \dot\sigma_{s+1}\sigma_{s+1}, \dot\rho_{s+1}\rho_{s+1}\rangle + \langle \dot\sigma_{s+1}\sigma_{s+1} - \dot\sigma_s\sigma_s, \dot\rho_1\rho_1\rangle \\
&\quad= \langle \dot\sigma_\ell\sigma_\ell - \dot\sigma_1\sigma_1, \dot\rho_1\rho_1\rangle + \sum_{1\leqslant s\leqslant \ell-1} \langle \dot\sigma_s\sigma_s, \dot\rho_{s+1}\rho_{s+1}\rangle - \langle \dot\sigma_{s+1}\sigma_{s+1}, \dot\rho_{s+1}\rho_{s+1}\rangle \\
&\quad= \sum_{1\leqslant s\leqslant \ell} \langle \dot\sigma_s\sigma_s, \dot\rho_{s+1}\rho_{s+1}\rangle - \langle \dot\sigma_s\sigma_s, \dot\rho_s\rho_s\rangle
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{0\leqslant s<t\leqslant \ell-1} \langle \mathrm{Ad}_{\gamma_1\cdots\gamma_s} X_{s+1}, \mathrm{Ad}_{\gamma_1\cdots\gamma_t} Y_{t+1}\rangle - \langle \mathrm{Ad}_{\gamma_1\cdots\gamma_s} Y_{s+1}, \mathrm{Ad}_{\gamma_1\cdots\gamma_t} X_{t+1}\rangle \\
&\quad= \sum_{s=1}^{\ell} \langle \dot\sigma_s\sigma_s, \dot\rho_{s+1}\rho_{s+1}\rangle - \langle \dot\sigma_{s+1}\sigma_{s+1}, \dot\rho_s\rho_s\rangle.
\end{aligned}$$

The proposition now follows by comparing this with (7). □

3.4. The Maslov index

In this section, we briefly digress to explain the relationship between the quantity $I(\rho)$, which we have called the index of a representation, and the usual Maslov index of a triple of Lagrangians, in the case ρ is a Lagrangian representation. The diagonal action of the symplectic group acting on triple of Lagrangian subspaces (L_1, L_2, L_3) in $\mathbb{C}^n$ has a finite number of orbits. To classify the orbits, one introduces the notion of an *inertia index* (or *Maslov index*) of a Lagrangian triple (cf. [14, p. 486]).

Definition 3.7. The *inertia index* $\tau(\lambda)$ of a triple $\lambda = (L_1, L_2, L_3)$ of Lagrangian subspaces of $\mathbb{C}^n$ is the signature of the quadratic form q defined on the $3n$ (real) dimensional vector space $L_1 \oplus L_2 \oplus L_3$ by: $q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$, where ω is the standard symplectic form on $\mathbb{C}^n$.

In order to state the symplectic classification of triples of Lagrangians, we need the following data. For $\mathfrak{d} = (n_0, n_{12}, n_{23}, n_{31}, \tau) \in \mathbb{N}^4 \times \mathbb{Z}$, let $C_{\mathfrak{d}}$ denote the set of all $\lambda = (L_1, L_2, L_3)$ satisfying $\tau(\lambda) = \tau$, $\dim(L_1 \cap L_2 \cap L_3) = n_0$, and $\dim(L_j \cap L_k) = n_{jk}$. For the following result, see [14, p. 493].

Proposition 3.8. *$C_{\mathfrak{d}}$ is non-empty if and only if $\mathfrak{d} = (n_0, n_{12}, n_{23}, n_{31}, \tau)$ satisfies the conditions*

(1) $0 \leqslant n_0 \leqslant n_{12}, n_{23}, n_{31} \leqslant n$.

(2) $n_{12} + n_{23} + n_{31} \leqslant n + 2n_0$.
(3) $|\tau| \leqslant n + 2n_0 - (n_{12} + n_{23} + n_{31})$.
(4) $\tau \equiv n - (n_{12} + n_{23} + n_{31}) \bmod 2\mathbb{Z}$.

If λ and λ' are two triples of Lagrangian subspaces of $\mathbb{C}^n$, there exists a symplectic map $\psi \in Sp(\mathbb{C}^n)$ such that $\psi(L_1) = L_1'$, $\psi(L_2) = L_2'$ and $\psi(L_3) = L_3'$, if and only if $n_0 = n_0'$, $n_{12} = n_{12}'$, $n_{23} = n_{23}'$, $n_{31} = n_{31}'$ and $\tau = \tau'$.

Using this classification one may show

Proposition 3.9 (*Falbel et al.* [6, Theorem 4.4]). *Let $\lambda = (L_1, L_2, L_3)$, $\rho = \tilde{\varphi}(\lambda)$, and $n_{jk} = \dim(L_j \cap L_k)$. Then*

$$\tau(\lambda) = 3n - 2I(\rho) - (n_{12} + n_{23} + n_{31}).$$

This relationship between τ and I gives an alternative proof of Theorem 2.2 for the case $\ell = 3$ (and assuming Theorem 1).

Corollary 3.2. *Let λ be a triple of Lagrangian subspaces of $\mathbb{C}^n$, $\rho = \tilde{\varphi}(\lambda)$. Then*

$$n - N_0(\rho) \leqslant I(\rho) \leqslant 2n + N_0(\rho) - N_1(\rho).$$

Proof. This follows from Propositions 3.8 and 3.9, and the fact that $N_0(\rho) = n_0$, and $N_1(\rho) = n_{12} + n_{23} + n_{31}$. $\square$

The Maslov index generalizes to multiple Lagrangians as follows. Let $L_1, \ldots, L_\ell$, $\ell \geqslant 3$, be a collection of Lagrangian subspaces in $\mathbb{C}^n$. We define

$$\tau(L_1, \ldots, L_\ell) = \tau(L_1, L_2, L_3) + \tau(L_1, L_3, L_4) + \cdots + \tau(L_1, L_{\ell-1}, L_\ell).$$

For the next result, set $I(L_1, \ldots, L_\ell) = I(\tilde{\varphi}(L_1, \ldots, L_\ell))$.

Proposition 3.10. *Let $L_1, \ldots, L_\ell$, $\ell \geqslant 4$, be a collection of Lagrangian subspaces in $\mathbb{C}^n$. Write $n_{1i} = \dim(L_1 \cap L_i)$, then*

$$I(L_1, \ldots, L_\ell) = I(L_1, L_2, L_3) + I(L_1, L_3, L_4) + \cdots + I(L_1, L_{\ell-1}, L_\ell) - \sum_{i=3}^{\ell-1} (n - n_{1i}).$$

Proof. Observe that if $\operatorname{spec}(\sigma_{L_1}\sigma_{L_3}) = (0, \ldots, 0, \alpha_{n_{13}+1}, \ldots, \alpha_n)$ then

$$\operatorname{spec}(\sigma_{L_3}\sigma_{L_1}) = (0, \ldots, 0, 1 - \alpha_n, \ldots, 1 - \alpha_{n_{13}+1}).$$

Summing all the angles in both spectra gives us: $n - n_{13}$. This implies that

$$I(L_1, L_2, L_3, L_4) = I(L_1, L_2, L_3) + I(L_1, L_3, L_4) - (n - n_{13}).$$

The general case follows by induction. $\square$

A relationship between τ and I still exists. Indeed, this follows directly from the previous result and Proposition 3.9.

Proposition 3.11. *For* $\ell \geqslant 3$, $\tau(L_1, \ldots, L_\ell) = n\ell - 2I(L_1, \ldots, L_\ell) - (n_{12} + n_{23} + \cdots + n_{\ell 1})$.

It is not immediately clear how to prove the analogue of Proposition 3.8 for $\ell \geqslant 4$, since the invariants no longer necessarily classify ℓ-tuples of Lagrangians. On the other hand, we can *use* Theorem 2.2, along with Proposition 3.11, to prove bounds on the generalized Maslov index.

Theorem 3.2. *For any* ℓ*-tuple of Lagrangians,*

$$|\tau(L_1, \ldots, L_\ell)| \leqslant n(\ell - 2) + 2n_0 - (n_{12} + n_{23} + \cdots + n_{\ell 1}).$$

4. Deformations of unitary and Lagrangian representations

4.1. The deformation space

For an algebraic group G and a finitely presented group Γ, let $\mathrm{Hom}(\Gamma, G)$ be the space homomorphisms of Γ into G. If Γ has generators $\{\gamma_1, \ldots, \gamma_\ell\}$, then $\mathrm{Hom}(\Gamma, G)$ is given the structure of an algebraic variety as the common locus of inverse images of the identity in G^ℓ for a finite number of functions $r_i : G^\ell \to G$. The tangent space to G^ℓ is identified with $\mathfrak{g}^\ell$, where $\mathfrak{g}$ is the Lie algebra of G, by right invariant vector fields. If ρ_t is a path of representations, $\rho_0 = \rho$, then differentiating ρ_t on a word $\gamma_{i_1} \cdots \gamma_{i_m}$, and using $X_k = \dot{\rho}_0(\gamma_{i_k})\rho_0^{-1}(\gamma_{i_k})$, we obtain the cocycle relation

$$X_1 + \mathrm{Ad}_{\rho(\gamma_{i_1})} X_2 + \cdots + \mathrm{Ad}_{\rho(\gamma_{i_1} \cdots \gamma_{i_{m-1}})} X_m = 0.$$

This formula implies the following observation of Weil [20].

Proposition 4.1. *The Zariski tangent space* $T_\rho\, \mathrm{Hom}(\Gamma, G)$ *is isomorphic to* $Z^1(\Gamma, \mathfrak{g})$.

In order to analyze deformations fixing conjugacy classes we compute the derivative of the curve $t \to \mathrm{Ad}_{e^{tX}} \rho(\gamma)$ to obtain $\{X - \mathrm{Ad}_{\rho(\gamma)} X\}\rho(\gamma)$. Identifying this with $X - \mathrm{Ad}_{\rho(\gamma)} X \in \mathfrak{g}$, we obtain a boundary in the group cohomology.

We apply these general considerations to the case of $U(n)$ representations of the free groups $\Gamma = \Gamma_\ell$ with presentation as in (1). $\mathrm{Hom}(\Gamma_\ell, U(n))$ is a smooth manifold of dimension $(\ell - 1)n^2$, with tangent space at a representation ρ given by

$$X_1 + \mathrm{Ad}_{\rho(\gamma_1)} X_2 + \cdots + \mathrm{Ad}_{\rho(\gamma_1 \cdots \gamma_{\ell-1})} X_\ell = 0. \tag{8}$$

We will be concerned with $g \in U(n)$ with fixed multiplicity for the eigenvalues. As in Section 3, let $\mathfrak{z}(g)$ be the Lie algebra of the centralizer of g. Then $\mathfrak{z}(g) = \mathfrak{u}(\mu_1) \times \cdots \times \mathfrak{u}(\mu_l)$, where: $\mu_1, \ldots, \mu_l$ are the multiplicities of the eigenvalues of g. We define $\mathfrak{z}^{ab.}(g) \subset \mathfrak{z}(g) = \mathfrak{u}(1) \times \cdots \times \mathfrak{u}(1)$, to be the subalgebra consisting of elements that are block diagonal with respect to this decomposition. Alternatively, it is the maximal abelian ideal of $\mathfrak{z}(g)$. We have the following

Lemma 4.1. *Let m be a multiplicity structure as in Section* 2.1. *Let $U(n,m)$ denote the set of all $g\in U(n)$ with multiplicity structure m. Then $U(n,m)$ is a smooth submanifold with tangent bundle* (*identified with a subspace of* $\mathfrak{u}(n)$) *given by*: $\mathfrak{u}(n,m)=\mathfrak{z}^{ab\cdot}(g)\oplus\mathfrak{z}(g)^{\perp}$. *Similarly, if $U(n,m,0)$ is the set of all $g\in U(n,m)$ with $0\in\operatorname{spec}(g)$, then $U(n,m,0)$ is a smooth submanifold with tangent bundle given by $\mathfrak{u}(n,m,0)=\mathfrak{z}^{ab\cdot,0}(g)\oplus\mathfrak{z}(g)^{\perp}$, where the superscript indicates that the first $\mathfrak{u}(1)$ factor is zero.*

Proof. It suffices the prove the statement concerning the tangent space. But small deformations of the eigenvalues are obtained by $g(t)=e^{tX}g$ for $X\in\mathfrak{z}^{ab\cdot}(g)$. Conjugating by an arbitrary unitary matrix, we find

$$\mathfrak{u}(n,m)=\{X+(\mathbf{I}-\mathrm{Ad}_g)Y : X\in\mathfrak{z}^{ab\cdot}(g),\ \ Y\in\mathfrak{u}(n)\}.$$

Since $\mathfrak{z}(g)^{\perp}=\operatorname{Im}(\mathbf{I}-\mathrm{Ad}_g)$, the result follows. The reasoning for $U(n,m,0)$ is similar. □

Now we prove

Proposition 4.2 (*cf. Mehta and Seshadri [17, Section 5]*). *Let $\rho:\Gamma_\ell\to U(n)$ be irreducible with $\pi(\rho)=\mathfrak{a}\in\mathscr{U}_{I,\ell}(n,\mathfrak{m},z)$. Then near ρ, $\mathrm{Rep}^{\mathrm{irr}\cdot}_{\mathfrak{a}}(\Gamma_\ell,U(n))$ is a smooth manifold of dimension*

$$\dim(\mathrm{Rep}^{\mathrm{irr}\cdot}_{\mathfrak{a}}(\Gamma_\ell,U(n)))=(\ell-2)n^2+2-\sum_{s=1}^{\ell}\sum_{j=1}^{l_s}(\mu^s_j)^2$$

Here, μ^s_j denotes the multiplicity $m^s_j-m^s_{j-1}$ of the jth distinct eigenvalue of $\rho(\gamma_s)$, $j=1,\dots,l_s$, (*see Section* 2.1). *Moreover, the spectral projection*

$$\pi:\mathrm{Rep}^{\mathrm{irr}\cdot}(\Gamma_\ell,U(n))\cap\pi^{-1}(\mathscr{U}_{I,\ell}(n,\mathfrak{m},z))\longrightarrow\mathscr{P}_{I,\ell}(n,\mathfrak{m},z),$$

is locally surjective and is a fibration near ρ.

Proof. We fix the conjugacy classes of $\rho(\gamma_s)$ for $s\geqslant 2$ and determine the variation in $\rho(\gamma_1)$. The space $\mathrm{Hom}_{\mathfrak{a}^{(1)}}(\Gamma_\ell,U(n))$ of representations ρ', $\rho'(\gamma_s)\simeq g_s$ for $s\geqslant 2$, is clearly a manifold of dimension

$$\dim(\mathrm{Hom}_{\mathfrak{a}^{(1)}}(\Gamma_\ell,U(n)))=\sum_{s=2}^{\ell}\dim(U(n)/Z(g_s))=(\ell-1)n^2-\sum_{j=2}^{l_s}(\mu^s_j)^2, \tag{9}$$

where we have used that: $\dim\ Z(g_s)=\sum_{j=1}^{l_s}(\mu^s_j)^2$. We compute the derivative of the map

$$\pi^{(1)}:\mathrm{Hom}_{\mathfrak{a}^{(1)}}(\Gamma_\ell,U(n))\longrightarrow U(n):\rho\mapsto\rho(\gamma_1).$$

Note that $\pi^{(1)}$ takes values in a single fiber of the determinant map. By (8), the tangent space to $\mathrm{Hom}_{\mathfrak{a}^{(1)}}(\Gamma_\ell,U(n))$ at ρ is given by $(X_1,\dots,X_\ell)\in\mathfrak{g}^\ell$ satisfying the conditions $X_s\in\operatorname{Im}(\mathbf{I}-\mathrm{Ad}_{\rho(\gamma_s)})$, for $s\geqslant 2$, and

$$X_1\in V^{(1)}=\mathrm{Ad}_{\rho(\gamma_1)}\operatorname{Im}(\mathbf{I}-\mathrm{Ad}_{\rho(\gamma_2)})+\cdots+\mathrm{Ad}_{\rho(\gamma_1\gamma_2\cdots\gamma_{\ell-1})}\operatorname{Im}(\mathbf{I}-\mathrm{Ad}_{\rho(\gamma_\ell)}).$$

We claim that $V^{(1)} = z(\rho)^\perp$. Indeed,

$$\begin{aligned}(V^{(1)})^\perp &= \{\mathrm{Ad}_{\rho(\gamma_1)}\mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_2)}) + \cdots + \mathrm{Ad}_{\rho(\gamma_2\cdots\gamma_{\ell-1})}\mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_\ell)})\}^\perp \\ &= \mathrm{Ad}_{\rho(\gamma_1)}\{(\mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_2)}))^\perp \cap \mathrm{Ad}_{\rho(\gamma_2)}(\mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_3)}))^\perp \\ &\quad \cap \cdots \cap \mathrm{Ad}_{\rho(\gamma_2\cdots\gamma_{\ell-1})}(\mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_\ell)}))^\perp\}.\end{aligned}$$

Now $\mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_2)})^\perp = \mathfrak{z}(\gamma_2)$, and therefore

$$(\mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_2)}))^\perp \cap \mathrm{Ad}_{\rho(\gamma_2)}(\mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_3)}))^\perp = \mathfrak{z}(\gamma_2) \cap \mathfrak{z}(\gamma_3).$$

Continuing in this way, we find $(V^{(1)})^\perp = \mathfrak{z}(\gamma_2) \cap \cdots \cap \mathfrak{z}(\gamma_\ell) = \mathfrak{z}(\rho)$. We have shown that $\mathrm{Im}(D\pi_\rho^{(1)}) = \mathfrak{z}(\rho)^\perp$. Hence, if the representation is irreducible (i.e. if $\mathfrak{z}(\rho) = \mathfrak{z}(U(n)) \simeq i\mathbb{R}$), then by transversality we conclude that $\mathrm{Hom}_{\mathfrak{a}}(\Gamma_\ell, U(n))$ is a manifold at an irreducible. Transversality applied to the product over s of $U(n, m^s)$ (or $U(n, m^s, 0)$ if $s \in z$) also gives the statement about local surjectivity and the fibration structure over the multiplicity space (see Lemma 4.1). For the dimension, we observe that

- $\dim \mathfrak{z}(\rho(\gamma_1)) = \sum_{j=1}^{l_1} (\mu_j^1)^2$;
- since $Z(\rho) = Z(U(n)) = U(1)$ by irreducibility, $n^2 - 1$ is the dimension of $U(n)$-orbit through ρ.

The dimension of $\pi^{-1}(\mathfrak{a})$ is computed by subtracting these from (9). Since this dimension depends only on the multiplicity structure, it is constant over the fixed multiplicity space; hence, the smoothness. This completes the proof. □

Remark 4.1. The surjectivity in Proposition 4.2 also follows from the Mehta–Seshadri Theorem [17] which describes irreducible representations with fixed conjugacy classes in terms of stable parabolic vector bundles. In the next section we will see that a similar result holds even if we restrict π to the Lagrangian representations, where we apparently have no such holomorphic description.

4.2. Twisting and bending deformations of Lagrangian representations

We approach the deformation theory of Lagrangian representations by introducing two special families: twist deformations and real bendings. Twist deformations are rather simple and apply equally well to unitary representations, while the bending deformations are particular to Lagrangian representations.

Definition 4.1. Let $\lambda = (L_1, \ldots, L_\ell) \in \Lambda^\ell(n)$, and $\rho = \tilde{\varphi}(\lambda)$. A *twist deformation* of the Lagrangian representation ρ is a Lagrangian representation of the form: $\rho_\tau = \tilde{\varphi}(\lambda_\tau)$, where $\lambda_\tau = (\tau_1 L_1, \ldots, \tau_\ell L_\ell)$ for some $\tau = (\tau_1, \ldots, \tau_\ell) \in U^\ell(1)$.

Remark 4.2. Since $\tilde{\varphi}$ always has the center of $U(n)$ as a fiber, the twist deformations naturally depend on $\ell - 1$ parameters in $U(1)$.

The following result is a calculation using the method in the proof of Lemma 3.4.

Lemma 4.2. *Let $\mathscr{T}_\rho \subset T_\rho\mathscr{L}\ \mathrm{Hom}(\Gamma_\ell, U(n))$ denote the subspace tangent to the twist deformations of ρ. Then $\mathscr{T}_\rho = [\mathfrak{u}(1) \times \cdots \times \mathfrak{u}(1)]_0$, where $\mathfrak{u}(1)$ is the Lie algebra of the center $U(1) \subset U(n)$, and the subscript 0 indicates that the sum of the entries vanishes.*

Definition 4.2. Let λ, ρ be as in Definition 4.1. A *real bending* of the Lagrangian representation ρ is a Lagrangian representation of the form $\rho_b = \tilde{\varphi}(\lambda_b)$, where

$$\lambda_b = (L_1, \ldots, L_s, bL_{s+1}, \ldots, bL_{s+r}, L_{s+r+1}, \ldots, L_\ell)$$

for some $s, r = 1, \ldots, \ell$, and $b \in O_{L_s}$ (as usual, we reduce mod ℓ any index greater than ℓ). Given s, r we shall say the bending is about L_s and has length r.

The twist deformations considered above are special cases of the action of the group $\mathrm{Hom}(\Gamma, Z(G))$ on $\mathrm{Hom}(\Gamma, G)$, and they were considered in [16]. Bending deformations are inspired by generalizations of Fenchel–Nielsen twists defined by Thurston (see [9,12]). An important difference is that the bending deformations defined in these references *fix* the conjugacy classes of $\rho(\gamma_s)$, whereas those in Definition 4.2 change certain conjugacy classes in a controlled way.

Indeed, from the definition we see that a real bending of length r about L_s has the form

$$\rho_b(\gamma_{s'}) = \begin{cases} \rho(\gamma_{s'}) & \text{if } s' = 1, \ldots, s-1, s+r+1, \ldots, \ell, \\ b\rho(\gamma_{s'})b^{-1} & \text{if } s' = s, \ldots, s+r-1. \end{cases} \tag{10}$$

Hence, the only conjugacy class which is potentially changed is that of $\rho(\gamma_{s+r})$. One can easily show that any deformation of a Lagrangian representation of the form (10) with $b \in O_{L_s}$ is necessarily Lagrangian and coincides with $\tilde{\varphi}(\lambda_b)$.

Lemma 4.3. *Let $\mathscr{B}_\rho(s, r) \subset T_\rho \mathscr{L}\ \mathrm{Hom}(\Gamma_\ell, U(n))$ denote the subspace tangent to the bending deformations of ρ of length r about L_s. Then the $(s+r)$th component $[\mathscr{B}_\rho(s, r)]_{s+r}$ is the orthogonal projection of* $\mathfrak{o}_s$ to $\mathfrak{o}_{s+r}^\perp$.

Proof. Using (10) and the calculation in the proof of Lemma 3.4, we see that for an infinitesimal bending $\dot{b} = B \in \mathfrak{o}_s$,

$$X_{s'} = \begin{cases} 0 & \text{if } s' = 1, \ldots, s-1, s+r+1, \ldots, \ell, \\ (\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_{s'})})B & \text{if } s' = s, \ldots, s+r-1, \\ (\mathbf{I} - \mathrm{Ad}_{\sigma_{s+r}})B & \text{if } s' = s+r. \end{cases}$$

Hence, $[\mathscr{B}_\rho(s, r)]_{s+r} = \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\sigma_{s+r}})|_{\mathfrak{o}_s}$, and the result follows from Lemma 3.1. $\square$

Our goal is to show that twistings and real bendings sweep out the full space of deformations of conjugacy classes in a neighborhood of an irreducible Lagrangian representation. We now prove

Proposition 4.3 (*cf. Proposition 4.2*). *Let $\rho : \Gamma_\ell \to U(n)$ be an irreducible Lagrangian representation with $\pi(\rho) = \mathfrak{a} \in \mathscr{L}_{I,\ell}(n, \mathfrak{m}, z)$. Then near ρ, $\mathscr{L}\ \mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$ is a smooth manifold of dimension*

$$\dim(\mathscr{L}\ \mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n))) = \frac{(\ell-2)}{2} n^2 + 1 - \frac{1}{2} \sum_{s=1}^{\ell} \sum_{j=1}^{l_s} (\mu_j^s)^2.$$

Moreover, the spectral projection

$$\pi : \mathscr{L}\ \mathrm{Rep}^{\mathrm{irr.}}(\Gamma_\ell, U(n)) \cap \pi^{-1}(\mathscr{L}_{I,\ell}(n, \mathfrak{m}, z)) \longrightarrow \mathscr{P}_{I,\ell}(n, \mathfrak{m}, z),$$

is locally surjective and is a fibration near ρ.

Proof. As in the proof of Proposition 4.2, we will first concentrate on deformations of $\rho(\gamma_1)$ up to conjugation. Thus, we consider bending deformations of L_s and length $r = \ell - s + 1$, for $s = 2, \ldots, \ell$. We also add twist deformations. It follows from Lemmas 4.2 and 4.3 that for $\rho = \tilde{\varphi}(\lambda)$,

$$P_1^{\perp}\mathfrak{v}_2 + \cdots + P_1^{\perp}\mathfrak{v}_\ell + \mathrm{i}\mathbb{R} + \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_1)}) \subset [\mathrm{Im} D\tilde{\varphi}_\lambda]_1,$$

where $P_1^{\perp}$ is the orthogonal projection to $\mathfrak{v}_1^{\perp}$. Since we are assuming ρ is irreducible, it follows as in the proof of Lemma 3.4 that

$$(\mathfrak{v}_1 + \cdots + \mathfrak{v}_\ell)^{\perp} = \mathrm{Ker}(D\tilde{\varphi}_\lambda) = \mathrm{i}\mathbb{R}.$$

Hence, denoting the traceless part with a subscript 0,

$$\begin{aligned}
&\{P_1^{\perp}\mathfrak{v}_2 + \cdots + P_1^{\perp}\mathfrak{v}_\ell + \mathrm{i}\mathbb{R} + \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_1)})\}^{\perp} \\
&\quad = \{P_1^{\perp}(\mathfrak{v}_1 + \cdots + \mathfrak{v}_\ell) + \mathrm{i}\mathbb{R} + \mathrm{Im}(\mathbf{I} - \mathrm{Ad}_{\rho(\gamma_1)})\}^{\perp} \\
&\quad = [\{\mathfrak{v}_1 + (\mathfrak{v}_1 + \cdots + \mathfrak{v}_\ell)^{\perp}\} \cap \mathfrak{z}(\rho(\gamma_1))]_0 \\
&\quad = [\mathfrak{v}_1 \cap \mathfrak{z}(\rho(\gamma_1))]_0 = \mathfrak{v}_1 \cap \mathfrak{v}_2
\end{aligned}$$

by Proposition 3.1 (2). Since we may do this calculation for any $\rho(\gamma_s)$, and since the variation preserves the conjugacy classes of $\rho(\gamma_{s'})$, $s' \neq s$ up to the twist deformations, we have shown that $D\tilde{\varphi}_\lambda$ is surjective onto

$$[(\mathfrak{v}_1 \cap \mathfrak{v}_2)^{\perp} \times (\mathfrak{v}_2 \cap \mathfrak{v}_3)^{\perp} \times \cdots \times (\mathfrak{v}_\ell \cap \mathfrak{v}_1)^{\perp}]_0 \subset [\mathfrak{z}(\rho(\gamma_1)) \times \cdots \times \mathfrak{z}(\rho(\gamma_\ell))]_0, \tag{11}$$

where now the subscript indicates that the sum of the traces vanishes. By Lemma 3.5, this must be exactly the image. Notice that

$$\mathfrak{z}^{ab.}(\rho(\gamma_s)) \oplus \mathfrak{z}^{\perp}(\rho(\gamma_s)) \subset (\mathfrak{v}_s \cap \mathfrak{v}_{s+1})^{\perp},$$

for all s (cf. Proposition 3.1 and Lemma 4.1). Hence, by transversality we deduce the local surjectivity and fiber structure onto the multiplicity space. We count dimensions:

- $\ell(n/2)(n+1)$ is dimension of $\Lambda^{\ell}(n)$;
- By Proposition 3.1 (4),

  $$\dim\, \mathfrak{z}(\sigma_s\sigma_{s+1}) - (\dim\, \mathfrak{v}_s \cap \mathfrak{v}_{s+1}) = (1/2)\dim\, \mathfrak{z}(\rho(\gamma_s)) + n/2.$$

  Hence, the dimension of the subspace in (11) is $(1/2)\sum_{s=1}^{\ell}\sum_{j=1}^{l_s}(\mu_j^s)^2 + \ell(n/2) - 1$.
- Finally, n^2 is the dimension of $U(n)$-orbit through ρ (notice that the action is free; see also Lemma 3.4).

The dimension follows by subtracting the last two items from the first. This completes the proof. □

Proposition 4.3 implies that, near irreducible representations, the allowed holonomies for unitary and Lagrangian representations coincide. In particular, a chamber either has no Lagrangian representations or is entirely populated by Lagrangians.

Corollary 4.1. *Let $\Delta \subset \mathscr{U}_{I,\ell}(n, \mathfrak{m}, z)$ be a chamber. Then $\Delta \cap \mathscr{L}_{I,\ell}(n, \mathfrak{m}, z) \neq \emptyset \iff \Delta \subset \mathscr{L}_{I,\ell}(n, \mathfrak{m}, z)$.*

Proof. By Remark 2.1 (2) and Proposition 4.3 it follows that $\Delta \cap \mathscr{L}_{I,\ell}(n, \mathfrak{m}, z)$ is open. On the other hand this set is also clearly closed in Δ; hence, the result. $\square$

We also have the

Proof of Theorem 2. Assume $\mathscr{L}\ \mathrm{Rep}^{\mathrm{irr.}}_{\mathfrak{a}}(\Gamma_\ell, U(n))$ is not empty. Then by Propositions 4.2 and 4.3, it is a smoothly embedded half-dimensional submanifold of $\mathrm{Rep}^{\mathrm{irr.}}_{\mathfrak{a}}(\Gamma_\ell, U(n))$. By Proposition 3.7, its tangent space is everywhere isotropic. The theorem follows. $\square$

4.3. Codimension of the reducibles

In this section, we use Proposition 4.3 to estimate the size of the set of reducible representations. Since we will only require the result for $\ell = 3$, we restrict to this case. We begin with the following simple observation.

Lemma 4.4. *Let $\rho : \Gamma_3 \to U(n)$ be irreducible with $\pi(\rho) = \mathfrak{a} \in \mathscr{P}_{I,3}(n, \mathfrak{m}, z)$. Then for at least two values of $s = 1, 2, 3$, all multiplicities $\mu^s_j = m^s_j - m^s_{j-1} \leqslant n/2$.*

Proof. Suppose not. Then there are two values of s, say $s = 1, 2$, and j_1, j_2, such that $\mu^1_{j_1} > n/2$ and $\mu^2_{j_2} > n/2$. If E_1 is the $\hat{\alpha}^1_{j_1}$ eigenspace of $\rho(\gamma_1)$ and E_2 is the $\hat{\alpha}^2_{j_2}$ eigenspace of $\rho(\gamma_2)$, then both $\rho(\gamma_1)$ and $\rho(\gamma_2)$, and hence also $\rho(\Gamma_3)$, leave invariant the intersection $E_1 \cap E_2$, which is positive dimensional. This contradicts the assumption of irreducibility. $\square$

Proposition 4.4. *Let $\Omega \subset \mathscr{L}\ \mathrm{Rep}_{\mathfrak{a}}(\Gamma_3, U(n))$ be an open connected subset containing an irreducible representation. Then the set of reducibles $\Omega \cap \mathscr{L}\ \mathrm{Rep}^{\mathrm{red.}}_{\mathfrak{a}}(\Gamma_3, U(n))$ has codimension $\geqslant n$.*

Proof. Suppose $\mathfrak{a} \in \mathscr{P}_{I,3}(n, \mathfrak{m}, z)$. If $\tilde{\rho} \in \mathscr{L}\ \mathrm{Rep}_{\mathfrak{a}}(\Gamma_3, U(n))$ is reducible, then we can decompose it into its irreducible components ρ_i, $i = 1, \ldots, k$, $k \geqslant 2$. Without loss of generality, we may assume ρ_i and ρ_j are non-isomorphic for $i \neq j$. Write: $\pi(\rho_i) = {}_i\mathfrak{a} = ({}_i\alpha^s_j) \in \mathscr{P}_{I_i,3}(n_i, {}_i\mathfrak{m}, z_i)$. Conversely, given a decomposition of $\mathfrak{a}$ into ${}_1\mathfrak{a}, \ldots, {}_k\mathfrak{a}$, it suffices to compute the codimension of the set of all reducibles with $\pi_i(\rho) = {}_i\mathfrak{a}$. We therefore assume this fixed decomposition, and let cod be the codimension of all reducibles compatible with the decomposition.

For each s let μ^s_j, $j = 1, \ldots, l_s$ denote the multiplicities from the partition m^s, and let $\hat{\alpha}^s_j$ denote the distinct entries of α^s. We define ${}_i\mu^s_j$ to be the multiplicity of $\hat{\alpha}^s_j$ if it appears in ${}_i\alpha^s$, and we set it to zero otherwise. The following are easy consequences of this definition:

$$\mu^s_j = \sum_{i=1}^{k} {}_i\mu^s_j, \tag{12}$$

$$n_i = \sum_{j=1}^{l_s} {}_i\mu^s_j, \tag{13}$$

$$n=\sum_{i=1}^{k} n_i=\sum_{j=1}^{l_s} \mu_j^s. \tag{14}$$

Counting dimensions as in the proof of Proposition 4.3 we find

$$\begin{aligned}
\mathrm{cod} &= (3/2)n^2-\left((1/2)\sum_{s=1}^{3}\sum_{j=1}^{l_s}(\mu_j^s)^2-1\right)-n^2 & (15)\\
&\quad -\left\{\sum_{i=1}^{k}\left[(3/2)n_i^2-\left((1/2)\sum_{s=1}^{3}\sum_{j=1}^{l_s}({}_i\mu_j^s)^2-1\right)-n_i^2\right]-\left(n^2-\sum_{i=1}^{k}n_i^2\right)\right\} & (16)\\
&= (1/2)\sum_{s=1}^{3}\left\{n^2-\sum_{j=1}^{l_s}(\mu_j^s)^2-\sum_{i=1}^{k}\left(n_i^2-\sum_{j=1}^{l_s}({}_i\mu_j^s)^2\right)\right\}+1-k. & (17)
\end{aligned}$$

The line (15) is the dimension count for the irreducibles. In line (16), we take this dimension for each irreducible factor, and then divide out by the part of the $U(n)$ which changes the splitting. It follows that for each s we need to estimate

$$C_s=n^2-\sum_{i=1}^{k}n_i^2-\sum_{j=1}^{l_s}(\mu_j^s)^2+\sum_{i=1}^{k}\sum_{j=1}^{l_s}({}_i\mu_j^s)^2.$$

Using (14) we have

$$n^2=\left(\sum_{i=1}^{k}n_i\right)^2=\sum_{i=1}^{k}n_i^2+\sum_{i\neq i'}n_in_{i'}.$$

Applying (13) to the second term on the right hand side above,

$$n^2-\sum_{i=1}^{k}n_i^2=\sum_{i\neq i'}\sum_{j,j'}({}_i\mu_j^s)({}_{i'}\mu_{j'}^s)=\sum_{i\neq i'}\sum_{j}({}_i\mu_j^s)({}_{i'}\mu_j^s)+\sum_{i\neq i'}\sum_{j\neq j'}({}_i\mu_j^s)({}_{i'}\mu_{j'}^s). \tag{18}$$

On the other hand, from (12) we have

$$\sum_{j=1}^{l_s}(\mu_j^s)^2=\sum_{i,i'}\sum_{j}({}_i\mu_j^s)({}_{i'}\mu_j^s)=\sum_{i,j}({}_i\mu_j^s)^2+\sum_{i\neq i'}\sum_{j}({}_i\mu_j^s)({}_{i'}\mu_j^s). \tag{19}$$

Combining (18) and (19), we find that

$$C_s=\sum_{i\neq i'}\sum_{j\neq j'}({}_i\mu_j^s)({}_{i'}\mu_{j'}^s).$$

We wish to estimate this quantity from below. Since there are at least two distinct eigenvalues, it follows that: $C_s \geqslant 2$. By Lemma 4.4, for at least two values of s we may assume that $\mu_j^s \leqslant n/2$ for all $j = 1, \ldots, l_s$. We estimate C_s in this case.

Case 1. Assume that for each i, j where ${}_i\mu_j^s \neq 0$ there are $i' \neq i$ and $j' \neq j$ such that ${}_{i'}\mu_{j'}^s \neq 0$. In this case we have

$$C_s \geqslant 2 \sum_{i,j} ({}_i\mu_j^s) \geqslant 2n, \tag{20}$$

by (13) and (14).

Case 2. If the condition in Case 1 is not satisfied, then there are i_0, j_0 such that ${}_{i_0}\mu_{j_0}^s \neq 0$ and for all $i \neq i_0$, $n_i = 1$ and ${}_i\mu_j^s = 1$ if $j = j_0$ and zero otherwise. This is true because if $n_i \geqslant 2$, then the ith block must have at least two distinct eigenvalues; in particular, one different from ${}_{i_0}\mu_{j_0}^s$. We also have $n_{i_0} - {}_{i_0}\mu_{j_0}^s = n - \mu_{j_0}^s$, and $n - n_{i_0} = k - 1$. Now

$$\begin{aligned}
\sum_{i \neq i'} \sum_{j \neq j'} ({}_i\mu_j^s)({}_{i'}\mu_{j'}^s) &= 2\left(\sum_{j \neq j_0} {}_{i_0}\mu_j^s\right)(n - n_{i_0}) + \sum_{i_0 \neq i \neq i' \neq i_0} ({}_i\mu_{j_0}^s)({}_{i'}\mu_{j_0}^s) \\
&= 2\left(\sum_j {}_{i_0}\mu_j^s - {}_{i_0}\mu_{j_0}^s\right)(n - n_{i_0}) + (1/2)(n - n_{i_0})(n - n_{i_0} - 1) \\
&= 2(n_{i_0} - {}_{i_0}\mu_{j_0}^s)(n - n_{i_0}) + (1/2)(n - n_{i_0})(n - n_{i_0} - 1) \\
&= 2(n - \mu_{j_0}^s)(n - n_{i_0}) + (1/2)(n - n_{i_0})(n - n_{i_0} - 1),
\end{aligned}$$

where in the third line we have used (14). Using the assumption that $\mu_j^s \leqslant n/2$, we have

$$C_s \geqslant n(k-1) + (1/2)(k-1)(k-2). \tag{21}$$

Hence, we have bounds on C_s from Cases 1 and 2 at two of the three values of s, and $C_s \geqslant 2$ at the third value. Putting (20) and (21) into expression (17) we find three possibilities:

$$\mathrm{cod} \geqslant \begin{cases} 2n + 2 - k; \\ n + (1/2)\{n(k-1) + (1/2)(k-1)(k-2)\} + 2 - k; \\ n(k-1) + (1/2)(k-1)(k-2) + 2 - k. \end{cases}$$

It is easily verified that the quantities on the right are all $\geqslant n$, with equality in the last case at $k = 2$. Since this is true for all of the finitely many possible types of reduction, the proof is complete. □

5. Proof of the main theorem

We have shown in Proposition 4.3 that $\mathscr{L}\ \mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$, if not empty, is a smoothly embedded submanifold of $\mathrm{Rep}_{\mathfrak{a}}^{\mathrm{irr.}}(\Gamma_\ell, U(n))$. In this section, we prove the existence of a Lagrangian representation with given holonomy whenever a unitary representation with the same holonomy exists. We first reduce the problem to the case of triples.

Proposition 5.1. *Suppose Theorem* 1 *holds for* $\ell = 3$. *Then it holds for all* ℓ.

Proof. By induction. Assume Theorem 1 holds for some $\ell \geqslant 3$, and also for $\ell = 3$. We show that it also holds for $\ell + 1$. Let $A_1, \ldots, A_{\ell+1}$ be unitary matrices satisfying $A_1 \cdots A_{\ell+1} = \mathbf{I}$ with given spectra. By induction, we may find Lagrangians $L_1, \ldots, L_{\ell-1}$ such that $\mathrm{spec}(A_i) = \mathrm{spec}(\sigma_{L_{i-1}}\sigma_{L_i})$, $i = 1, \ldots, \ell - 1$, and $\mathrm{spec}(A_\ell A_{\ell+1}) = \mathrm{spec}(\sigma_{L_{\ell-1}}\sigma_{L_0})$, where L_0 is as in Section 3.1. Write: $B_1 B_2 B_3 = \mathbf{I}$, where $B_1 \sim A_{\ell+1}^{-1}$, $B_2 \sim A_\ell^{-1}$, and $B_3 = \sigma_{L_{\ell-1}}\sigma_{L_0}$. Using the result for $\ell = 3$ we may find Lagrangians L', L'' such that $B_1 \sim \sigma_{L_0}\sigma_{L'}$, $B_2 \sim \sigma_{L'}\sigma_{L''}$, and $B_3 \sim \sigma_{L''}\sigma_{L_0}$.By Lemma 3.3, both $\sigma_{L_{\ell-1}}\sigma_{L_0}$ and $\sigma_{L''}\sigma_{L_0}$ are conjugate by elements in $O(n)$ to diagonal matrices. Since they furthermore have the same spectrum, it follows from Proposition 3.2 that there is some $g \in O(n)$ with $gL'' = L_{\ell-1}$. Set $L_\ell = gL'$. Then $A_\ell \sim \sigma_{L_{\ell-1}}\sigma_{L_\ell}$, and $A_{\ell+1} \sim \sigma_{L_\ell}\sigma_{L_0}$, and the result follows. □

By Proposition 5.1, it suffices to prove Theorem 1 for triples of Lagrangians. For the rest of this section, we consider the problem of specifying three conjugacy classes. To simplify notation, we will omit the subscript "$\ell = 3$", and write Γ for Γ_3, and $\overline{\mathscr{U}_I}^*(n)$ for $\overline{\mathscr{U}}^*_{I,3}(n)$, for example.

Definition 5.1. A reducible representation $\rho : \Gamma \to U(n_1) \times \cdots \times U(n_k) \hookrightarrow U(n)$, $\sum_{i=1}^k n_i = n$, will be called *relatively irreducible* with respect to $U(n_1) \times \cdots \times U(n_k)$ if the induced representations $\rho_i : \Gamma \to U(n_i)$ are irreducible for each $i = 1, \ldots, k$.

Our goal is to show that $\overline{\mathscr{L}}^*_I(n) = \overline{\mathscr{U}}^*_I(n)$, for all I and n. Using the stratification of $\overline{\mathscr{P}}^*_I(n)$ described in Section 2.1, the argument proceeds by induction on the four parameters available:

- Fix the rank n. We assume that we have shown $\mathscr{L}_I(\tilde{n}, \mathfrak{m}, z) = \mathscr{U}_I(\tilde{n}, \mathfrak{m}, z)$ for all $\tilde{n} < n$ and all $(\mathfrak{m}, z)$. The result for $U(1)$ or $U(2)$ representations holds, as has already been mentioned.
- Next, fix a multiplicity structure $\mathfrak{m}$. Assume we have proven that $\mathscr{L}_I(\tilde{n}, \mathfrak{p}, z) = \mathscr{U}_I(\tilde{n}, \mathfrak{p}, z)$ for all $\mathfrak{p} < \mathfrak{m}$ and all z. We may clearly do this, since a partition giving multiplicity n for each s corresponds to $U(1)$ representations.
- Fix a subset $z \subset \{1, 2, 3\}$ and assume that $\mathscr{L}_I(n, \mathfrak{m}, \tilde{z}) = \mathscr{U}_I(n, \mathfrak{m}, \tilde{z})$ for all $z \subsetneq \tilde{z}$. We will justify this assumption below.
- Finally, the last part of the inductive scheme is to assume that $\mathscr{L}_{\overline{I}}(n, \bar{\mathfrak{m}}, \bar{z}) = \mathscr{U}_I(n, \bar{\mathfrak{m}}, \bar{z})$ for all $\overline{I} < I$, and all $\bar{\mathfrak{m}}$ and $\bar{z}$. Notice that $\overline{I} = 0$ involves only the trivial representation.

If the stratum $\mathscr{P}_I(n, \mathfrak{m}, z)$ is degenerate, then either $\mathscr{U}_I(n, \mathfrak{m}, z) = \emptyset$, in which case there is nothing to prove, or each ρ with $\pi(\rho) \in \mathscr{U}_I(n, \mathfrak{m}, z)$ is reducible by Proposition 4.2. Hence, by induction on the rank n, $\mathscr{L}_I(n, \mathfrak{m}, z) = \mathscr{U}_I(n, \mathfrak{m}, z)$ if $\mathscr{P}_I(n, \mathfrak{m}, z)$ is degenerate. Thus, we assume that $\mathscr{P}_I(n, \mathfrak{m}, z)$ is nondegenerate. If $\mathscr{L}_I(n, \mathfrak{m}, z) \neq \mathscr{U}_I(n, \mathfrak{m}, z)$ then there is a connected component Δ of $\mathscr{U}_I(n, \mathfrak{m}, z) \setminus \mathscr{L}_I(n, \mathfrak{m}, z)$ which by Corollary 4.1 is a union of chambers. By Remark 2.1 (1), $\partial\Delta$ consists of a union of convex subsets of affine planes. By Proposition 4.3, it follows that any $\rho \in \mathscr{L}$ $\mathrm{Hom}(\Gamma, U(n))$ for which $\pi(\rho) \in \partial\Delta$ is reducible. Finally, we claim that $\partial\Delta \cap \mathring{\mathscr{U}}_I(n, \mathfrak{m}, z)$ is unbounded. To see this, choose $\rho \in \partial\Delta \cap \mathring{\mathscr{U}}_I(n, \mathfrak{m}, z)$ contained in a cell of minimal dimension. Then ρ is relatively irreducible with respect to some reduction $U(n_1) \times \cdots \times U(n_k)$ (see Definition 5.1). Among the induced representations $\Gamma \to U(n_j)$ there mustbe one, say ρ_j, that is nontrivial, since the total index is positive. Hence, $\pi(\rho_j) \in \mathring{\mathscr{U}}_{I_j}(n_j, {}_j\mathfrak{m}, z_j)$ for some induced multiplicity structure. Since $\overset{\circ}{\mathscr{U}}_{I_j}(n_j, {}_j\mathfrak{m}, z_j)$ is positive dimensional, the claim follows from this fact.

From the discussion above and the description of the stratification and wall structure in Sections 2.1 and 2.2 we see that there are four (not necessarily exclusive) possibilities:

(1) $\partial\Delta$ intersects an outer wall in $\mathscr{P}_I(n, \mathfrak{m}, z)$;
(2) $\partial\Delta$ intersects a stratum $\mathscr{P}_I(n, \mathfrak{p}, z)$, $\mathfrak{p} < \mathfrak{m}$;
(3) $\partial\Delta$ intersects a stratum $\mathscr{P}_I(n, \mathfrak{p}, \tilde{z})$, $\mathfrak{p} \leqslant \mathfrak{m}$, $z \subsetneq \tilde{z}$;
(4) $\partial\Delta$ intersects a stratum $\mathscr{P}_{\bar{I}}(n, \bar{\mathfrak{p}}, \bar{z})$, for some $\bar{I} < I$, $z \subset \bar{z}$.

In each case, our inductive hypothesis assumes the result for the lower dimensional stratum, and we will use this below to derive a contradiction. Here we remark that possibility (3) does not occur if $z = \{1, 2, 3\}$. The derivation of a contradiction for this case therefore justifies the inductive hypothesis on z. The structure of the argument deriving a contradiction is actually identical for each of the four possibilities above, *mutatis mutandis*. We will give a detailed account of how this works in case (1), the modifications necessary for the other cases being straightforward.

Consider then the case where $\partial\Delta$ intersects the outer walls $\mathscr{W}_I(n, \mathfrak{m}, z)$ at a point in $\mathscr{P}_I(n, \mathfrak{m}, z)$. To simplify notation, for the following discussion we set $\mathscr{U}_I = \mathscr{U}_I(n, \mathfrak{m}, z)$, $\mathscr{W}_I = \mathscr{W}_I(n, \mathfrak{m}, z)$, $\mathscr{P}_I = \mathscr{P}_I(n, \mathfrak{m}, z)$, and $\Lambda_I = \Lambda_I^3(n, \mathfrak{m}, z)$. Also, let l_s be the lengths of the partitions m^s, $s = 1, 2, 3$. The intersection $H = \partial\Delta \cap \mathscr{W}_I$ is a union of convex subsets of intersections of affine planes corresponding to reductions of Lagrangian representations. We claim that H must have positive codimension in $\mathscr{W}_I$. For if not, we could find an outer wall W and point $\mathfrak{a} \in \partial\Delta \cap W$ such that $\mathfrak{a} \notin W'$ for any outer *or inner* wall W'. In particular, if N is a sufficiently small neighborhood of $\mathfrak{a}$, then $N \cap \overset{\circ}{\mathscr{U}}_I = N \cap \Delta$. By the induction hypothesis, we may find (a reducible) $\rho \in \mathscr{L}_I$ such that $\pi(\rho) = \mathfrak{a}$. Now any Lagrangian may be perturbed slightly to give an irreducible Lagrangian representation $\tilde{\rho}$. It follows from Proposition 2.1 (1) that for sufficiently small perturbations, $\pi(\tilde{\rho}) \in N \cap \overset{\circ}{\mathscr{U}}_I \subset \Delta$; contradiction.

Hence, we may assume H has positive codimension. To illustrate the basic idea of the proof, suppose first that H has codimension one inside $\mathscr{W}_I$, so that H locally disconnects $\mathscr{W}_I$. We choose $\mathfrak{a} \in H$ with minimal valency with respect to the outer wall structure along H. By this we mean that there are outer walls $W_1, \ldots, W_p$ meeting at $\mathfrak{a}$, and $p \geqslant 1$ is the minimal number of such intersections among all points in H. With this choice, and using the convexity of $\mathscr{U}_I$, we see that the number p of outer walls meeting at $\mathfrak{a}$ is 1 or 2. Let us first assume that $p = 1$, and let W denote the outer wall in question. Choose a neighborhood U of $\mathfrak{a}$ in the wall W such that $H \cap U$ is a cell. Since W is the only outer wall at $\mathfrak{a}$, we may also assume that the neighborhood U is contained in $\mathscr{U}_I$. Let N be a neighborhood of $\mathfrak{a}$ in $\mathscr{P}_I$ such that the following hold:

(1) $U = N \cap W$;
(2) $N \backslash W$ consists precisely of two components N^+, N^-;
(3) $N^- \cap \mathscr{U}_I = \emptyset$ and $N^+ \subset \mathscr{U}_I$ is homomorphic to a ball;
(4) $N^+ \backslash \Delta$ has the topology of $U \backslash H$.

Choose a point $\rho \in \pi^{-1}(W)$ as follows: W corresponds to a reduction $U(k) \times U(n-k)$. We may find a point ρ, $\pi(\rho) = \mathfrak{a}$, such that ρ is relatively irreducible with respect to $U(k) \times U(n-k)$. By

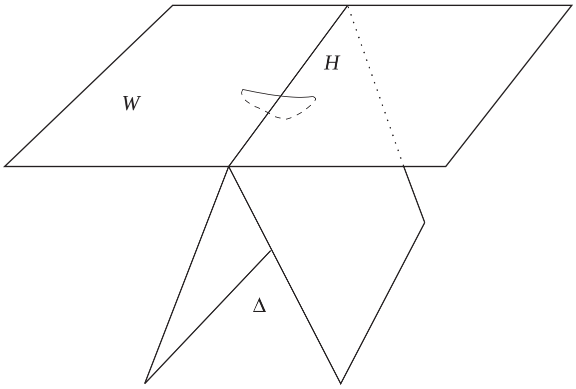

Fig. 1. Intersection H of a chamber Δ with an outer wall W.

Proposition 4.3, we may assume that Λ_I is a manifold near ρ. With this understood, let $\widetilde{B} \subset \Lambda_I$ be a ball about ρ such that $\pi(\widetilde{B}) \cap W \subset U$. By our choice of ρ it follows, again by Proposition 4.3, that $\pi(\widetilde{B})$ intersects *both* components of $U \backslash H$. By Proposition 4.4, $\widetilde{B} \cap \Lambda_I^{\text{irr.}}$ is connected; hence, so is $\pi(\widetilde{B} \cap \Lambda_I^{\text{irr.}})$. On the other hand, by the previous remark, $\pi(\widetilde{B} \cap \Lambda_I^{\text{irr.}}) \subset N^+ \backslash \Delta$ must intersect both components of $N^+ \backslash \Delta$. This contradicts the connectedness of $\pi(\widetilde{B} \cap \Lambda_I^{\text{irr.}})$ (see Fig. 1).

The case $p = 2$ requires only a small modification of the above argument: Let W_1 and W_2 be outer walls meeting along H at $\mathfrak{a}$. We choose the set $U \subset W_1 \cup W_2$ to consist of two pieces: $U_1 = U \cap W_1 \subset W_1 \cap \mathscr{U}_I(n)$, and $U_2 = U \cap W_2 \subset W_2 \cap \mathscr{U}_I(n)$. Since $\mathfrak{a}$ is at the intersection of precisely two outer walls, it corresponds to a reduction of the form $U(n_1) \times U(n_2) \times U(n_3)$; the wall W_1 corresponds to a $U(n_1 + n_2) \times U(n_3)$ reduction, say, and the wall W_1 corresponds to a $U(n_1) \times U(n_2 + n_3)$ reduction. Now since deformations along the wall W_1 can only take values on one side of W_2, and vice versa, it follows that the image by π of a neighborhood of any ρ, $\pi(\rho) = \mathfrak{a}$, intersects *both* components of $U \backslash H$. In the choice of the neighborhood N we modify the first two criteria so that

1'. $U = N \cap (W_1 \cup W_2) \cap \mathscr{U}_I(n)$;
2'. $N \backslash (W_1 \cup W_2) \cap \mathscr{U}_I(n)$ consists precisely of two components N^+, N^-,

and keep items (3) and (4) as above. The rest of the argument then proceeds exactly as before.

Next, let us consider the case where H has higher codimension d, $d \geqslant 2$, in $\mathscr{W}_I(n)$. If we again choose $\mathfrak{a} \in H$ with minimal valency with respect to the outer wall structure along H, then we see that at most $d + 1$ outer walls meet at $\mathfrak{a}$. As before, we first consider the case where there is just one outer wall W. Choose a neighborhood U of $\mathfrak{a}$ in W as above. We also choose N satisfying conditions (1–4) above. Let $D \subset U$ be a cell in U of dimension equal to the codimension d of H in W and intersecting H precisely in $\mathfrak{a}$. Hence, the boundary ∂D is the link of H in W. We regard D as the image of a continuous map, $f : B^d \longrightarrow U$. We may further assume that $f = \pi \circ \tilde{f}$ for a map $\tilde{f} : B^d \longrightarrow \Lambda_I$, taking the origin to ρ. Indeed, choosing a relatively irreducible ρ and using Proposition 4.3, $\pi : \pi^{-1}(W) \cap \Lambda_I \to W$ is a fibration in a neighborhood of ρ and $\pi(\rho) = \mathfrak{a}$. Hence, we may define $\tilde{f}$ by taking a section of this fibration.

Claim. $\dim H \geqslant \sum l_s - n - |z|$.

Proof. Assume that ρ is relatively irreducible with respect to a reduction $U(n_1)\times\cdots\times U(n_k)$. Then restricted to representations near ρ which are relatively irreducible of this type, the map π is locally surjective onto $\overline{\mathscr{P}}_{I_1}(n_1)\times\cdots\times\overline{\mathscr{P}}_{I_k}(n_k)$ (cf. Proposition 4.2). Assume first that $|z|\neq 3$. Then all $I_j>0$. In particular,

$$\dim(\overline{\mathscr{P}}_{I_1}(n_1)\times\cdots\times\overline{\mathscr{P}}_{I_k}(n_k))=\sum_{j=1}^{k}(3n_j-1)=3n-k.$$

Since l_s is the number of distinct eigenvalues of $\rho(\gamma_s)$, it follows that

$$\dim\ H=3n-k-\sum_{s=1}^{3}(n-l_s)-|z|=\sum_{s=1}^{3}l_s-k-|z|\geqslant\sum_{s=1}^{3}l_s-n-|z|.$$

Now suppose that $I_1=\cdots=I_q=0$ for some $1\leqslant q<k$, and $I_j\neq 0$ for $j=q+1,\ldots,k$. Since we are assuming $\pi(\rho)\in\mathscr{P}_I(n,\mathfrak{m},z)$, this can only happen if $z=\{1,2,3\}$, i.e. $|z|=3$. Also, $n_1=\cdots=n_q=1$. It follows that

$$\begin{aligned}\dim(\overline{\mathscr{P}}_{I_1}(n_1)\times\cdots\times\overline{\mathscr{P}}_{I_k}(n_k)) &= \dim(\overline{\mathscr{P}}_{I_{q+1}}(n_{q+1})\times\cdots\times\overline{\mathscr{P}}_{I_k}(n_k))\\ &=\sum_{j=1}^{k-q}(3n_{q+j}-1)=3(n-q)-(k-q).\end{aligned}$$

Now for each $s=1,2,3$, either $q=m_1^s$, in which case there are precisely l_s-1 distinct nonzero eigenvalues among the remaining $n-q$; or, $q<m_1^s$, in which case there are l_s distinct eigenvalues, but one of them is zero. In both cases, this imposes: $n-q-(l_s-1)$ conditions on the eigenvalues. Hence, we have

$$\dim\ H=3(n-q)-(k-q)-\sum_{s=1}^{3}(n-q-(l_s-1))=\sum_{s=1}^{3}l_s-(k-q)-3.$$

Since $k-q\leqslant n-1$, and $|z|=3$, the claim follows in this case as well. $\square$

Now $d=\dim W-\dim H\leqslant\sum_{s=1}^{3}l_s-2-|z|-(\sum_{s=1}^{3}l_s-n-|z|)=n-2$. Notice that this computation is still valid even if $\sum_{s=1}^{3}l_s-n-|z|\leqslant 0$. By Proposition 4.4, $\Lambda_I^{\text{red.}}$ has codimension at least: $n>n-2$ in Λ_I. Hence, we may find a perturbed map $\tilde{f}_\varepsilon: B^d\to\Lambda_I^{\text{irr.}}$. For sufficiently small perturbations we clearly may assume that $f_\varepsilon=\pi\circ\tilde{f}_\varepsilon$ has image in N. It follows that in fact $f_\varepsilon: B^d\to N^+\backslash\Delta$. Now $N^+\backslash\Delta$ has the topology of $U\backslash H$, and under this equivalence $f_\varepsilon(\partial B^d)$ is the link of $N\cap\Delta$. The continuous extension of f_ε to B^d is therefore a contradiction.

When the number p of outer walls meeting at $\mathfrak{a}$ is greater than one, the configuration of outer walls at $\mathfrak{a}$ forms a "corner" in $\mathscr{W}_I$ (see Fig. 2). As in the case $p=2$ above, we want to choose the set U to mimic this configuration. The technical result we will require is the following:

Lemma 5.1. *Suppose that $\rho\in\Lambda_I$ is such that $\pi(\rho)$ lies in the intersection $W_1\cap\cdots\cap W_p$ of p distinct outer walls, where p is the minimal such number, and that ρ is relatively irreducible with respect to the reduction corresponding to $W_1\cap\cdots\cap W_p$. Then for any small neighborhood $\Omega\subset\Lambda_I$ of ρ*

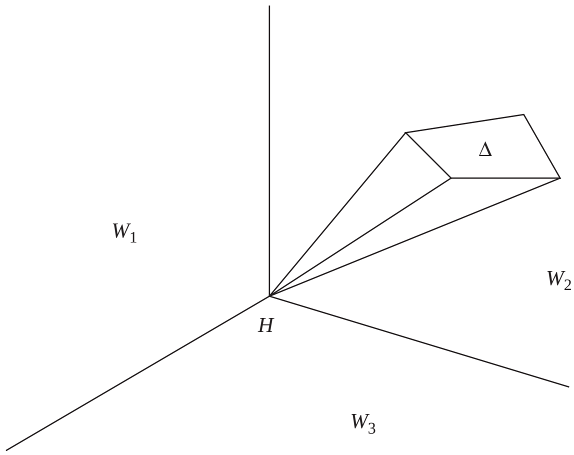

Fig. 2. Intersection H of a chamber Δ with three outer walls.

there is a continuous map $\tilde{f} : B^{p-1} \to \Omega$ *satisfying the following*:

(1) $\tilde{f}(0) = \rho$;
(2) $\pi \circ \tilde{f}(\partial B^{p-1}) \subset W_1 \cup \cdots \cup W_p$;
(3) $\pi \circ \tilde{f}(\partial B^{p-1}) \cap W_1 \cap \cdots \cap W_p = \emptyset$.

Moreover, $\tilde{f}$ *may be chosen to vary continuously with* ρ *satisfying the hypothesis*.

Given the lemma, the rest of the argument proceeds as in the previous paragraph. Indeed, choose $\mathfrak{a} \in H$ with minimal valency with respect to the outer wall structure along $H \subset W_1 \cap \cdots \cap W_p$, and choose a neighborhood N of $\mathfrak{a}$ such that $N \setminus (W_1 \cup \cdots \cup W_p) \cap \mathscr{U}_I$ consists precisely of two components N^+, N^-, and which also satisfies items (3) and (4) above. Let ρ, $\pi(\rho) = \mathfrak{a}$, be relatively irreducible, and choose a neighborhood Ω of ρ such that $\pi(\Omega) \subset N$. Choose a continuous map $\tilde{g} : B^{d+1-p} \to \Omega$ such that $\pi \circ \tilde{g} : B^{d+1-p} \to W_1 \cap \cdots \cap W_p$ is transverse to H at $\mathfrak{a}$. As before, we can do this because ρ is relatively irreducible. Now use Lemma 5.1 to extend $\tilde{g}$ to a continuous map: $\tilde{f} : B^d \simeq B^{d+1-p} \times B^{p-1} \to \Omega$. By the construction, we can easily arrange that

$$f = \pi \circ \tilde{f}(\partial B^{d+1-p} \times \{y\}) \cap H = \emptyset, \tag{22}$$

for all $y \in B^{p-1}$. By Lemma 5.1 (3) we also have

$$f(\{x\} \times \partial B^{p-1}) \cap W_1 \cap \cdots \cap W_p = \emptyset, \tag{23}$$

for all $x \in B^{d+1-p}$. It follows from (22) and (23) that $f : S^{d-1} \to W_1 \cup \cdots \cup W_p$ is a link of H in $W_1 \cup \cdots \cup W_p$. We may now perturb the map $\tilde{f}$ as above so that $f_\varepsilon(S^{d-1}) \subset N^+ \setminus \Delta$ is a link of $N^+ \cap \Delta$. The extension $f_\varepsilon(B^d) \subset N^+ \setminus \Delta$ gives a contradiction as before.

Proof of Lemma 5.1. Suppose ρ is of type $U(n_1) \times \cdots \times U(n_p) \times U(n_{p+1})$, where each wall W_i corresponds to a reduction $U(n) \to U(n_i) \times U(n - n_i)$, $i = 1, \ldots, p$. Let $\rho = (\rho_1, \ldots, \rho_p, \rho_{p+1})$ be the irreducible factors. Notice that the assumption of minimal valency of $\mathfrak{a}$ implies that $n_{p+1}=n-\sum_{i=1}^{p} n_i \neq 0$. Let $e_1 \cdots e_p$ be a $(p-1)$-simplex in $\mathbb{R}^{p-1}$ with the origin e_0 as barycenter. For each i, we may find a path $\tilde{g}_i'(t)$ of Lagrangian representations into $U(n_{p+1}+n_i)$ such that $\tilde{g}_i'(0)=(\rho_i,\rho_0)$ and $\tilde{g}_i'(t)$ is irreducible for $t \neq 0$. Keeping the other factors fixed, these define paths

$$\tilde{g}_i : [0,1] \longrightarrow \mathscr{L}\ \mathrm{Hom}(\Gamma, U(n_1) \times \cdots \times \widehat{U}(n_i) \times \cdots \times U(n_p) \times U(n_{p+1}+n_i)),$$

where $\widehat{\ }$ means that factor is deleted. Combining these paths defines a continuous map $\tilde{f} : \cup_{i=1}^{p} e_0 e_i \to \Omega$. Suppose inductively that we have defined $\tilde{f}$ on all simplices of the form $e_{i_1} \cdots e_{i_k}$, $2 \leqslant k < p-1$, $1 \leqslant i_1 < \cdots < i_k$. For each such simplex, let $\{j_1, \ldots, j_{p-k}\}$ be the complimentary set to $\{i_1, \ldots, i_k\}$ in $\{1, \ldots, p\}$. We will assume $\tilde{f}$ has been defined such that the following hold:

(1) $\pi \circ \tilde{f}(e_{i_1} \cdots e_{i_k}) \subset W_{j_1} \cap \cdots \cap W_{j_{p-k}}$;
(2) For each $x \in e_{i_1} \cdots e_{i_k}$, $\tilde{f}(x)$ is relatively irreducible with respect to the decomposition $U(n_{j_1}) \times U(n_{j_{p-k}}) \times U(n - \sum_{\mu=1}^{k} n_{j_\mu})$;
(3) $\pi \circ \tilde{f}(e_{i_1} \cdots e_{i_k}) \cap W_1 \cap \cdots \cap W_p = \emptyset$.

We now extend $\tilde{f}$ to a simplex of the form $e_{i_1} \cdots e_{i_{k+1}}$ as follows. By assumption (1), for the complimentary set of indices $\{j_1, \ldots, j_{p-k-1}\}$ we have $\pi \circ f(\partial(e_{i_1} \cdots e_{i_{k+1}})) \subset W_{j_1} \cap \cdots \cap W_{j_{p-k-1}}$. Assuming Ω has been chosen sufficiently small so that $\Omega \cap \pi^{-1}(W_{j_1} \cap \cdots \cap W_{j_{p-k-1}})$ is contractible, we may extend $\tilde{f}$ to a map $e_{i_1} \cdots e_{i_{k+1}} \to W_{j_1} \cap \cdots \cap W_{j_{p-k-1}}$. Applying the same codimension argument we have used several times already, we can further assume that this extended map satisfies conditions (2) and (3) as well. Continuing in this way, we have defined $\tilde{f}$ on the boundary of $e_1 \cdots e_p$. Recall that f is also defined on the one simplices $e_0 e_i$, $i = 1, \ldots, p$. Again using contractibility of Ω, we extend $\tilde{f}$ inductively and arbitrarily to simplices of the form $e_0 e_{i_1} \cdots e_{i_k}$, $k = 1, \ldots, p$. This completes the definition of $\tilde{f}$. □

6. Examples

In this last section, we illustrate some of the ideas in the paper by explicity giving the wall structure for the cases: $\ell = 3$, $n = 2, 3$. For convenience, we will only consider distinct eigenvalues different from unity. The case of $U(2)$ representations was first proven [11], and more generally [4]. The inequalities were later derived from spherical triangles in [6].

Let us first introduce some useful notation. For integers i_s, $1 \leqslant i_s \leqslant n$, $s = 1, \ldots, \ell$, define the collection of subsets as in Section 2.2 $\wp_{(1)} = (\wp_{(1)}^s)$, $\wp_{(1)}^s = \{i_s\}$. For $\mathfrak{a} = (\alpha_j^s) \in \overline{\mathscr{A}}_\ell(n)$, we will use the notation (cf. (3))

$$[i_1, \ldots, i_s]_{\mathfrak{a}} = I(\mathfrak{a}, \wp_{(1)}) = \sum_{s=1}^{\ell} \alpha_{i_s}^s.$$

By a permutation of $[i_1, \ldots, i_s]_{\mathfrak{a}}$, we mean a quantity of the form: $[i_{\tau(1)}, \ldots, i_{\tau(s)}]_{\mathfrak{a}}$, for some τ in the group of permutations of $\{1, \ldots, \ell\}$. With this understood, we may write the $U(2)$ inequalities as

Theorem 6.1 (*cf. Biswas [4], Falbel et al. [6]*). *There exist representations* $\rho : \Gamma_3 \to U(2)$ *with* $\mathfrak{a} = \pi(\rho) \in \mathscr{U}_{I,3}(2)$, *if and only if*

- $I = 2$, *and*: $[2, 1, 1]_{\mathfrak{a}} \leqslant 1$, *plus all permutations*;
- $I = 3$, *and*: $[2, 2, 1]_{\mathfrak{a}} \leqslant 2 \leqslant [2, 2, 2]_{\mathfrak{a}}$, *plus all permutations*; *or*,
- $I = 4$, *and*: $[2, 1, 1]_{\mathfrak{a}} \leqslant 2$, *plus all permutations*;

The bounds on the index come from Proposition 2.2. Notice that for each index there are no inner walls. Indeed, any equality of the form: $[i_1, i_2, i_3]_{\mathfrak{a}} = K$ implies

$$I = [i_1, i_2, i_3] + [\bar{i}_1, \bar{i}_2, \bar{i}_3] = K + [\bar{i}_1, \bar{i}_2, \bar{i}_3] \geqslant K + 1,$$

where $\bar{i}_s = \{1, 2\} \backslash \{i_s\}$. Now if $I = 2$, for example, then $K = 1$, and it is easy to see that the outer walls are the only possible solutions for distinct nonzero eigenvalues.

We have used a duality in the wall structure. In general, if $\wp_{(k)} = (\wp^s_{(k)})$ is a collection of subsets of $\{1, \ldots, n\}$ of cardinality k, then let $\wp^*_{(k)}$ denote the collection of subsets of cardinality $n - k$ defined by $(\wp^*_{(k)})^s = (\wp^s_{(k)})^c$. It follows that $I(\mathfrak{a}, \wp_{(k)}) + I(\mathfrak{a}, \wp^*_{(k)}) = I(\mathfrak{a})$. So an inequality of the form $I(\mathfrak{a}, \wp_{(k)}) \leqslant K$ may be written $I(\mathfrak{a}, \wp^*_{(k)}) \geqslant I(\mathfrak{a}) - K$. In particular, this means that for $n = 3$ we may express all the inequalities in terms of the $[i_1, \ldots, i_\ell]_{\mathfrak{a}}$'s.

Theorem 6.2. *There exist representations* $\rho : \Gamma_3 \to U(3)$ *with* $\mathfrak{a} = \pi(\rho) \in \mathscr{U}_{I,3}(3)$, *if and only if*

- $I = 3$, *and*

$$[3,1,1]_{\mathfrak{a}},\ [2,2,1]_{\mathfrak{a}} \leqslant 1 \leqslant [3,3,1]_{\mathfrak{a}},\ [3,2,2]_{\mathfrak{a}},$$
$$2 \leqslant [3,3,2]_{\mathfrak{a}},$$

  plus all permutations;
- $I = 4$, *and*

$$[2,1,1]_{\mathfrak{a}} \leqslant 1 \leqslant [3,2,1]_{\mathfrak{a}},\ [2,2,2]_{\mathfrak{a}},$$
$$[3,3,1]_{\mathfrak{a}},\ [3,2,2]_{\mathfrak{a}} \leqslant 2 \leqslant [3,3,3]_{\mathfrak{a}},$$

  plus all permutations;
- $I = 5$, *and*

$$[1,1,1]_{\mathfrak{a}} \leqslant 1 \leqslant [2,2,1]_{\mathfrak{a}},\ [3,1,1]_{\mathfrak{a}},$$
$$[3,2,1]_{\mathfrak{a}},\ [2,2,2]_{\mathfrak{a}} \leqslant 2 \leqslant [3,3,2]_{\mathfrak{a}},$$

  plus all permutations; *or*,
- $I = 6$, *and*

$$1 \leqslant [2,1,1]_{\mathfrak{a}}$$
$$[3,1,1]_{\mathfrak{a}},\ [2,2,1]_{\mathfrak{a}} \leqslant 2 \leqslant [3,3,1]_{\mathfrak{a}},\ [3,2,2]_{\mathfrak{a}}$$

  plus all permutations.

The result is proven using the procedure given in [5]. Since this is straightforward, we will not give the details. It turns out that there are no inner walls for this case either, though this is certainly tedious to check by hand. For example, take $[1, 2, 3]_{\mathfrak{a}} = 1$ for the $I = 3$ case. This is compatible with the first set of inequalities. However, since the total index is 3, we have $[3, 3, 2]_{\mathfrak{a}} + [2, 1, 1]_{\mathfrak{a}} = 2$, and this violates the inequality $[3, 3, 2]_{\mathfrak{a}} \geqslant 2$.

Indeed, by combining Propositions 2.1 (3) and 4.2, and using the connectivity of the moduli of parabolic bundles, one can show that the smallest $U(n)$ for which inner walls can appear is $n = 5$ (still assuming $\ell = 3$).

Acknowledgments

Proposition 3.7 above has been independently proven by Florent Schaffhauser in [19]. His method is to realize the Lagrangian representations as fixed points of an antisymplectic involution acting on the space of all unitary representations. The authors would like to thank him for many discussions about this problem. They are also grateful to the mathematics departments at Johns Hopkins University and the Université Paris VI for their generous hospitality during the course of this research. Funding for this work was provided by a US/France Cooperative Research Grant: NSF OISE-0232724, CNRS 14551. R.W. received additional support from NSF DMS-0204496.

References

[1] A. Alekseev, A. Malkin, E. Meinrenken, Lie group valued moment maps, J. Diff. Geom. 48 (3) (1998) 445–495.
[2] S. Agnihotri, C. Woodward, Eigenvalues of products of unitary matrices and quantum Schubert calculus, Math. Res. Lett. 5 (6) (1998) 817–836.
[3] P. Belkale, Local systems on $\mathbb{P}^1 - S$ for S a finite set, Compositio Math. 129 (1) (2001) 67–86.
[4] I. Biswas, A criterion for the existence of a parabolic stable bundle of rank two over the projective line, Int. J. Math. 9 (5) (1998) 523–533.
[5] I. Biswas, On the existence of unitary flat connections over the punctured sphere with given local monodromy around the punctures, Asian J. Math. 3 (2) (1999) 333–344.
[6] E. Falbel, J.-P. Marco, F. Schaffhauser, Classifying triples of Lagrangians in a Hermitian vector space, Topol. Appl. 144 (2004) 1–27.
[7] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, Bull. Amer. Math. Soc. 37 (3) (2000) 209–249.
[9] W. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, Invent. Math. 85 (1986) 1–40.
[11] L. Jeffrey, J. Weitsman, Bohr–Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula, Comm. Math. Phys. 150 (1992) 593–630.
[12] D. Johnson, J. Millson, Deformation spaces associated to compact hyperbolic manifolds, Discrete Groups in Geometry and Analysis, New Haven, Conn., 1984, pp. 48–106 (Progr. Math., vol. 67, Birkhauser, Boston, MA, 1987).
[13] M. Kapovich, J. Millson, The symplectic geometry of polygons in Euclidean space, J. Diff. Geom. 44 (3) (1996) 479–513.
[14] M. Kashiwara, P. Schapira, Sheaves on Manifolds, Grundlehren der Mathematischen Wissenshaften 292, Springer, Berlin, 1990.
[15] A. Klyachko, Stable bundles, representation theory and Hermitian operators, Selecta Math. (NS) 4 (3) (1998) 419–445.
[16] A. Lubotzky, A. Magid, Varieties of representations of finitely generated groups, Mem. Amer. Math. Soc. 58 (1985) 336.
[17] V. Mehta, C. Seshadri, Moduli of vector bundles on curves with parabolic structures, Math. Ann. 248 (3) (1980) 205–239.
[18] M. Narasimhan, C. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, Ann. Math. 82 (2) (1965) 540–567.
[19] F. Schaffhauser, Representations of the fundamental group of an *l*-punctured sphere generated by products of Lagrangian involutions, Can. Math. J., to appear.
[20] A. Weil, Remarks on the cohomology of groups, Ann. Math. 80 (2) (1964) 149–157.

Further reading

[8] W. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. Math. 54 (1984) 200–225.
[10] W. Goldman, J. Millson, Eichler–Shimura homology and the finite generation of cusp forms by hyperbolic Poincaré series, Duke Math. J. 53 (1986) 1081–1091.