

On Unbounded p -Summable Fredholm Modules¹

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We prove that odd unbounded p -summable Fredholm modules are also bounded p -summable Fredholm modules (this is the odd counterpart of a result of A. Connes for the case of even Fredholm modules). The approach we use is via estimates of the form $\|\phi(D) - \phi(D_0)\|_{L_p(\mathcal{M}, \tau)} \leq C \cdot \|D - D_0\|^{1/2}$, where $\phi(t) = t(1 + t^2)^{-1/2}$, $D_0 = D_0^*$ is an unbounded linear operator affiliated with a semifinite von Neumann algebra \mathcal{M} , $D - D_0$ is a bounded self-adjoint linear operator from \mathcal{M} and $(\mathbf{1} + D_0^2)^{-1/2} \in L_p(\mathcal{M}, \tau)$, where $L_p(\mathcal{M}, \tau)$ is a non-commutative L_p -space associated with \mathcal{M} . It follows from our results that if $p \in (1, \infty)$, then $\phi(D) - \phi(D_0)$ belongs to the space $L_p(\mathcal{M}, \tau)$. © 2000 Academic Press

0. INTRODUCTION

This paper concerns the question arising in the quantised calculus of Alain Connes [Co1, Co2] outlined in the abstract. To explain our results we need some further notation. Let \mathcal{M} be a semifinite von Neumann algebra on a separable Hilbert space \mathcal{H} and let $L_p(\mathcal{M}, \tau)$ be a non-commutative

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L_p -space associated with (\mathcal{M}, τ) , where τ is a faithful, normal semifinite trace on \mathcal{M} . The identity in \mathcal{M} is denoted by $\mathbf{1}$. Let \mathcal{A} be a unital Banach $*$ -algebra which is represented in \mathcal{M} via a continuous $*$ -homomorphism π which, without loss of generality, we may assume to be faithful. Where no confusion arises we suppress π in the notation. The fundamental objects of our analysis are explained in the following definition.

DEFINITION 0.1 ([Co1, Co2, CP, S]). *An odd unbounded p -summable (respectively, bounded pre-) Breuer–Fredholm module for \mathcal{A} , is a pair (\mathcal{M}, D_0) (respectively, (\mathcal{M}, F_0)) where D_0 (respectively, F_0) is an unbounded (respectively, bounded) self-adjoint operator affiliated with \mathcal{M} (respectively, in \mathcal{M}) satisfying:*

- (1) $(\mathbf{1} + D_0^2)^{-1/2}$ (respectively, $|\mathbf{1} - F_0^2|^{1/2}$) belongs to $L_p(\mathcal{M}, \tau)$; and
- (2) $\mathcal{A}_0 := \{a \in \mathcal{A} \mid a(\text{dom } D_0) \subset \text{dom } D_0, [D_0, a] \in \mathcal{M}\}$ (respectively, $\mathcal{A}_p := \{a \in \mathcal{A} \mid [F_0, a] \in L_p(\mathcal{M}, \tau)\}$) is a dense $*$ -subalgebra of \mathcal{A} .

When $F_0^2 = \mathbf{1}$ we drop the prefix “pre-.”

In the special case when $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and τ is the standard trace Tr , we shall omit the word “Breuer” from the definition and speak about *unbounded (respectively, bounded) p -summable Fredholm modules* (\mathcal{H}, D_0) (respectively, (\mathcal{H}, F_0)). In this case, the non-commutative L_p -space coincides with the Schatten–von Neumann ideal \mathcal{C}_p of compact operators and the “bounded” part of Definition 0.1 is a slight extension of Definition 3 from [Co2, p. 290] (where $\mathcal{A}_0 = \mathcal{A}$ and $F_0^2 = \mathbf{1}$, see also [Co1, Appendix 2, Co2, p. 298]). In the special case when \mathcal{M} is a semifinite factor, the “unbounded” part of Definition 0.1 coincides with [CP, Definition 2.1]; and, in the case $\mathcal{M} = \mathcal{B}(\mathcal{H})$, it may be considered as an odd counterpart of the notion of an unbounded even p -summable pre-Fredholm module from [Co1, Section 6, Corollary 3, and the remarks thereafter]. Definition 0.1 is adapted from [S] where the notion of Fredholm module is studied in the more general setting of symmetric operator spaces.

The approach of this paper to the study of these Breuer–Fredholm modules is motivated by one of the basic problems of perturbation theory which may be formulated as follows.

- I. If F and G are continuous functions on $(-\infty, \infty)$ under what conditions does the smallness of $G(D - D_0)$ imply that of $F(D) - F(D_0)$?

We will present a study of this problem when the function $F(t) = t(1 + t^2)^{-1/2}$, $G(t) = \sqrt{t}$ and D_0 (respectively, $D - D_0$) is some self-adjoint (respectively, bounded self-adjoint) operator affiliated with a semifinite von Neumann algebra \mathcal{M} (respectively in \mathcal{M}). We measure the “smallness” of

$G(D - D_0)$ (respectively, $F(D) - F(D_0)$) in the uniform operator norm (respectively, in the norm of $L_p(\mathcal{M}, \tau)$). This setting appeared first in [CP] and further extensive considerations were presented in [S]. The choice of F and G is suggested by the theory of unbounded Fredholm modules [Co1, Co2] and work on spectral flow [P1, P2, CP].

Following Connes' results for the even case (see [Co1, I.6]), the difficulties associated with the mapping $(\mathcal{H}, D_0) \rightarrow (\mathcal{H}, \text{sgn}(D_0))$ in the odd case were outlined in [CP] (see also [S]). Introduce the map ϕ defined by

$$\phi(D) = D(\mathbf{1} + D^2)^{-1/2}$$

which is a smooth approximation of the function sgn and hence explains our interest in the difference $\phi(D) - \phi(D_0)$. The results presented in this article contribute also to the study of the mapping $(\mathcal{M}, D_0) \rightarrow (\mathcal{M}, \text{sgn}(D_0))$ which was initiated in [CP] for odd p -summable Breuer–Fredholm modules and continued in [S]. The choice $F = \phi$ is also dictated by the following problem suggested from [Co, CP, S].

II. Does it follow from (\mathcal{M}, D_0) being an odd unbounded p -summable Breuer–Fredholm module that $(\mathcal{M}, \text{sgn}(D_0))$ is an odd bounded p -summable Breuer–Fredholm module?

The major technical problem in our setting for question I lies in the difference between norms on the right and left hand sides. In particular, it makes it virtually impossible to apply well-known double operator integral techniques from [BSo] and therefore a new technique is required even in the situation when \mathcal{M} coincides with the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on \mathcal{H} . Variants of this new technique are given in [CP, S]. In our present approach to Problems I and II we follow the direction outlined in [S, Sect. 6] where questions concerning Hölder estimates of the type

$$\|\phi(D) - \phi(D_0)\|_{L_p(\mathcal{M}, \tau)} \leq C \cdot \|D - D_0\|^{1/2} \quad (0.1)$$

were shown to be relevant to the Lipschitz continuity of the absolute value in the setting of non-commutative L_p -spaces. The result of [S, Corollary 6.3] (see also Proposition 2.3 in the present paper) asserts that if $(\mathbf{1} + D_0^2)^{-1/2} \in L_p(\mathcal{M}, \tau)$ then the functions

$$\frac{\|D(\mathbf{1} + D^2)^{-1/2} - D_0(\mathbf{1} + D_0^2)^{-1/2}\|_{L_p(\mathcal{M}, \tau)}}{\max\{\|D - D_0\|^{1/2}, \|D - D_0\|\}}$$

and

$$\frac{\||D| - |D_0|\} \cdot (\mathbf{1} + D_0^2)^{-1/2}\|_{L_p(\mathcal{M}, \tau)}}{\max\{\|D - D_0\|^{1/2}, \|D - D_0\|\}}$$

are bounded or unbounded simultaneously for each self-adjoint operator $D - D_0 \in \mathcal{M}$. In other words the question (0.1) is reduced to the study of the “weighted” difference

$$(|D| - |D_0|) \cdot (\mathbf{1} + D_0^2)^{-1/2}. \tag{0.2}$$

While it is possible to obtain results in the case when $D - D_0$ is D_0 -bounded (see [CP]) we will not consider this approach here. We use instead the approach applied in [S] based on the following result obtained jointly with Yu. B. Farforovskaya.

PROPOSITION 0.2 (cf. [S, Proposition 6.5]). *Let f be a Lipschitz function with constant 1 and let $p \in [1, 2]$. If $T \in \mathcal{C}_p$ commutes with D_0 , then $(f(D) - f(D_0)) T \in \mathcal{C}_p$ and, moreover*

$$\|(f(D) - f(D_0)) T\|_{\mathcal{C}_p} \leq \|D - D_0\| \cdot \|T\|_{\mathcal{C}_p}.$$

The proof of Proposition 0.2 relies strongly on the matrix representation of operators from \mathcal{C}_p and is not applicable in the case of L_p -spaces affiliated with an arbitrary von Neumann algebra. In the present article we shall work with the weighted difference (0.2) motivated by the methods from [DDPS] where it was established that the absolute value is Lipschitz continuous in any reflexive L_p -space associated with an arbitrary von Neumann algebra. The following theorem is our main result. It extends Proposition 0.2 in the special case that f is the absolute value function and $p \neq 1$ and contributes further to the solution of Problem I.

THEOREM 0.3. (i) *Let x, y be self-adjoint operators affiliated with \mathcal{M} with $x = y + a$ with $a \in \mathcal{M}$ and let $z = (\mathbf{1} + x^2)^{-1/2} \in L_p(\mathcal{M}, \tau) \cap \mathcal{M}$ for some fixed $p \in (1, \infty)$. We then have*

$$(|x| - |y|) z \in L_p(\mathcal{M}, \tau) \quad \text{and} \quad \frac{y}{(\mathbf{1} + y^2)^{1/2}} - \frac{x}{(\mathbf{1} + x^2)^{1/2}} \in L_p(\mathcal{M}, \tau).$$

Moreover

$$\|(|x| - |y|) z\|_{L_p(\mathcal{M}, \tau)} \leq \mathcal{L}_p \max\{\|x - y\|^{1/2}, \|x - y\|\} \cdot \|z\|_{L_p(\mathcal{M}, \tau)} \tag{0.3}$$

and

$$\left\| \frac{y}{(\mathbf{1} + y^2)^{1/2}} - \frac{x}{(\mathbf{1} + x^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \leq \mathcal{L}'_p \max\{\|x - y\|^{1/2}, \|x - y\|\} \cdot \|z\|_{L_p(\mathcal{M}, \tau)}, \tag{0.3'}$$

for some positive constants \mathcal{L}_p and \mathcal{L}'_p which depend on p only.

(ii) Let x, y be self-adjoint, τ -measurable operators affiliated with \mathcal{M} . Let $x = y + a$ with $a \in \mathcal{M}$ and suppose that $z \geq 0$ belongs to $L_p(\mathcal{M}, \tau) \cap \mathcal{M}$, for some fixed $p \in (1, \infty)$, commutes with x and has support projection **1**. Then $(|x| - |y|)z \in L_p(\mathcal{M}, \tau)$ and, moreover

$$\|(|x| - |y|)z\|_{L_p(\mathcal{M}, \tau)} \leq \mathcal{K}_p \|x - y\| \cdot \|z\|_{L_p(\mathcal{M}, \tau)}$$

for some positive constant \mathcal{K}_p which depends on p only.

The proofs of Theorem 0.3(i) and (ii) we present here are independent of each other (although initially we used (ii) to prove (i)). Notice the differences in the technical assumptions, these are important and force us to use rather different arguments. For the purposes of this paper Theorem 0.3(i) is the main result because from it we can deduce the following corollary which answers Question II in the affirmative and extends earlier results of the third named author [S] for the case $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $1 \leq p \leq 2$.

COROLLARY 0.4. *If $1 < p < \infty$ and (\mathcal{M}, D_0) is an odd unbounded p -summable Breuer–Fredholm module for the Banach $*$ -algebra \mathcal{A} then $(\mathcal{M}, \text{sgn}(D_0))$ is an odd bounded p -summable Breuer–Fredholm module for \mathcal{A} .*

The organisation of the paper is straightforward. In the next section we shall present a few facts and definitions which are necessary for the proof of Theorem 0.3. Our presentation of the proof of Theorem 0.3 in Section 2 requires us to develop further some ideas from [DDPS] although our considerations here are largely independent of that paper with the exception of one technical lemma. We have deliberately made our discussion independent of [S] including the needed results in the Appendix. Section 3 contains our applications to non-commutative geometry.

1. PRELIMINARIES

We denote by \mathcal{M} a semifinite von Neumann algebra on the Hilbert space \mathcal{H} , with a fixed faithful, normal semifinite trace τ . A linear operator $x: \text{dom}(x) \rightarrow \mathcal{H}$, with domain $\text{dom}(x) \subseteq \mathcal{H}$, is said to be *affiliated with* \mathcal{M} if $ux = xu$ for all unitaries u in the commutant \mathcal{M}' of \mathcal{M} (our basic references for facts about von Neumann algebras are [D, SZ]). Given a positive self-adjoint operator x in \mathcal{H} , we denote by E_t^x (or just E_t if there is no danger of confusion) the spectral projection of x corresponding to the interval $(-\infty, t)$. If x is a positive self-adjoint operator in \mathcal{H} affiliated with \mathcal{M} , then $E_{[0, t)}^x \equiv E_t^x \in \mathcal{M}$ and $x E_t^x \in \mathcal{M}$ for all $t > 0$ [SZ, E.9.10, E.9.25]. If x is a closed linear operator in \mathcal{H} with polar decomposition $x = v|x|$, then $v^*v = s(|x|)$, where $s(|x|)$ is the support projection of $|x|$ [SZ, 9.4]. If x is affiliated with

\mathcal{M} , then $v \in \mathcal{M}$ and $|x|$ is affiliated with \mathcal{M} [SZ, 9.29]. The set of all closed, densely defined operators affiliated with \mathcal{M} will be denoted by $\tilde{\mathcal{M}}$.

An operator $x \in \tilde{\mathcal{M}}$ is called τ -measurable (affiliated with \mathcal{M}) if and only if there exists $s > 0$ such that $\tau(1 - E_s^{|x|}) < \infty$. The set of all τ -measurable operators forms a $*$ -algebra $\tilde{\mathcal{M}}$ with the sum and product defined as the respective closures of the algebraic sum and product. For $\varepsilon, \delta > 0$ we denote by $N(\varepsilon, \delta)$ the set of all $x \in \tilde{\mathcal{M}}$ for which there exists an orthogonal projection $p \in \mathcal{M}$ such that $p(\mathcal{H}) \subseteq \text{dom}(x)$, $\|xp\| \leq \varepsilon$ and $\tau(1 - p) \leq \delta$. The sets $\{N(\varepsilon, \delta) : \varepsilon, \delta > 0\}$ form a base at 0 for a metrizable Hausdorff topology in $\tilde{\mathcal{M}}$, which is called the *measure topology*. Equipped with this measure topology, $\tilde{\mathcal{M}}$ is a complete topological $*$ -algebra. These facts and their proofs can be found in the papers [Ne, Te, and FK]. It is known (see [Ti, DDPS, Theorem 1.1]) that if $x \in \tilde{\mathcal{M}}$, $\{x_n\}_{n=1}^\infty \subset \tilde{\mathcal{M}}$ and if $x_n \rightarrow x$ for the measure topology, then also $|x_n| \rightarrow |x|$ for the measure topology.

The space $L_p(\mathcal{M}, \tau)$, $1 \leq p < \infty$ is the Banach space of all operators $A \in \tilde{\mathcal{M}}$ such that $\tau(|A|^p) < \infty$ with the norm $\|A\|_{L_p(\mathcal{M}, \tau)} := (\tau(|A|^p))^{1/p}$, where $|A| = (A^*A)^{1/2}$, $i = 1, 2$. If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and τ is the standard trace Tr , then $\tilde{\mathcal{M}} = \mathcal{M}$ and, then $L_p(\mathcal{M}, \tau)$ is precisely the *Schatten class* \mathcal{C}_p , $1 \leq p < \infty$.

If $\{x_\alpha\}_{\alpha \in \mathcal{A}} \subseteq \mathcal{M}$ is a net and if $x \in \mathcal{M}$, then we will write $x_\alpha \xrightarrow{(s)} x$ to denote convergence in the σ -strong (operator) topology (see [Ta, p. 68; SZ, p. 132]). If we consider the left regular representation of \mathcal{M} on $\mathcal{H} = L_2(\mathcal{M}, \tau)$, then it is straightforward that the convergence in the σ -strong topology coincides with the convergence in the strong operator topology. It is well known (see [Da1, p. 115; Si, p. 40; DDPS, Corollary 1.4]) that if $x_\alpha = x_\alpha^*$, $\forall \alpha$, $\sup_\alpha \|x_\alpha\|_\infty < \infty$ and if $x_\alpha \rightarrow x$ in the strong operator topology, then $|x_\alpha| \rightarrow |x|$ in the strong operator topology. In particular, if $\{e_n\}_{n=1}^\infty$ is a sequence of projections from \mathcal{M} such that $e_n \uparrow_n \mathbf{1}$ and $x = x^* \in \mathcal{M}$, then $e_n x e_n \rightarrow x$ and $|e_n x e_n| \rightarrow |x|$ in the strong operator topology, whence

$$e_n x e_n \xrightarrow{(s)} x \quad \text{and} \quad |e_n x e_n| \xrightarrow{(s)} |x|. \tag{1.1}$$

In the proof of Theorem 0.3 we shall use the following easily verified fact. If $x_\alpha \in L_p(\mathcal{M}, \tau)$ for $1 < p < \infty$ with $\|x_\alpha\|_{L_p(\mathcal{M}, \tau)} \leq C < \infty$ for all α and either $x_\alpha \rightarrow x$ in the measure topology or we have $x_\alpha = x_\alpha^*$ for all α , $\sup_\alpha \|x_\alpha\|_\infty < \infty$ and $x_\alpha \xrightarrow{(s)} x$, then

$$x \in L_p(\mathcal{M}, \tau) \quad \text{and} \quad \|x\|_{L_p(\mathcal{M}, \tau)} \leq C. \tag{1.2}$$

The rigorous proof of the latter fact in a slightly more general situation may be found in [DDPS, Proposition 1.6; FK, Theorems 3.5, 3.6].

An important fact from the geometry of non-commutative L_p -spaces used in [DDPS] is that any reflexive L_p -space associated with an arbitrary

semifinite von Neumann algebra (\mathcal{M}, τ) is a UMD-space (see [BGM1]). An equivalent form of the latter fact is that the $L_p(\mathcal{M}, \tau)$ -valued generalization of the Riesz projection is bounded in any Bochner space $L_p(G, L_p(\mathcal{M}, \tau))$, where G is an arbitrary connected compact Abelian group, the Riesz projection is defined with respect to a positive cone of a linear ordering of the dual group \hat{G} and $p \in (1, \infty)$. This fact together with the so-called transference method (see [BGM2]) was used in [DDPS] to establish the following result (which in the special case $\mathcal{M} = \mathcal{B}(\mathcal{H})$ was first established by E. B. Davies in [Da2]).

LEMMA 1.1 (cf. [DDPS, Lemma 3.2]). *If $1 < p < \infty$, then there exists a constant $K_p > 0$, which depends only on p such that*

$$\left\| \sum_{m,n=1}^N \frac{\lambda_m - \mu_n}{\lambda_m + \mu_n} p_m a p_n \right\|_{L_p(\mathcal{M}, \tau)} \leq K_p \|a\|_{L_p(\mathcal{M}, \tau)},$$

for all semifinite von Neumann algebras (\mathcal{M}, τ) , for all finite sequences p_1, p_2, \dots, p_N of mutually orthogonal projections in \mathcal{M} , for all $a \in L_p(\mathcal{M}, \tau)$ and all choices $0 \leq \lambda_1, \lambda_2, \dots, \lambda_N; \mu_1, \mu_2, \dots, \mu_N \in \mathbb{R}$ with $\lambda_m + \mu_n > 0$ for all $m, n = 1, 2, \dots, N$.

2. LIPSCHITZ AND COMMUTATOR ESTIMATES

This section contains the main proofs. We begin with three technical propositions. The first gives an estimate of the “weighted” commutator $[x, y]z$ which generalizes similar considerations of [DDPS].

PROPOSITION 2.1. *If $x = x^* \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}$, if $y \in \mathcal{M}$ and if $z \in L_p(\mathcal{M}, \tau) \cap \mathcal{M}$ commutes with x , then*

$$\|[x, y]z\|_{L_p(\mathcal{M}, \tau)} \leq 2(1 + K_p) \|[x, y]z\|_{L_p(\mathcal{M}, \tau)}.$$

Proof. Let $x \in \mathcal{M} \cap L_1(\mathcal{M}, \tau)$ be a self-adjoint element of the form

$$x = (\lambda_1 p_1 + \lambda_2 p_2 + \dots + \lambda_N p_N) - (\mu_1 q_1 + \mu_2 q_2 + \dots + \mu_N q_N),$$

where $p_1, p_2, \dots, p_N, q_1, q_2, \dots, q_N$ are mutually orthogonal projections in \mathcal{M} and $\{\lambda_i\}_{i=1}^N, \{\mu_j\}_{j=1}^N \subset [0, \infty)$. Note that there is no loss of generality in having the same number of p_i 's and q_j 's as we can allow some of them

to be zero. Let $z \in L_p(\mathcal{M}, \tau) \cap \mathcal{M}$ commute with these projections. It follows immediately from Lemma 1.1 that

$$\left\| \sum_{m,n=1}^N \frac{\lambda_m - \mu_n}{\lambda_m + \mu_n} p_m y p_n z \right\|_p \leq K_p \|yz\|_p. \tag{2.1}$$

Letting

$$y' := \left(\sum_{m=1}^N \lambda_m p_m \right) y + y \left(\sum_{n=1}^N \mu_n p_n \right), \quad y'' = \left(\sum_{m=1}^N p_m \right) y' \left(\sum_{n=1}^N p_n \right)$$

we see that

$$p_m y'' p_n = p_m y' p_n = (\lambda_m + \mu_n) p_m y p_n, \quad m, n = 1, 2, \dots, N$$

and we infer from (2.1) applied to y'' and z that

$$\begin{aligned} \left\| \sum_{m,n=1}^N (\lambda_m - \mu_n) p_m y p_n z \right\|_p &= \left\| \sum_{m,n=1}^N \frac{\lambda_m - \mu_n}{\lambda_m + \mu_n} p_m y'' p_n z \right\|_p \\ &\leq K_p \|y''z\|_p \\ &= \left\| \sum_{m,n=1}^N (\lambda_m + \mu_n) p_m y p_n z \right\|_p. \end{aligned} \tag{2.2}$$

We set

$$p = \sum_{i=1}^N p_i, \quad q = \sum_{j=1}^N q_j$$

and note that without loss of generality we may assume that $p + q = \mathbf{1}$. Following [DDPS, Proposition 2.4(vii) \Rightarrow (viii)] we have, as z commutes with p and q ,

$$\begin{aligned} p[|x|, y] z p &= \sum_{i=1}^N \sum_{j=1}^N (\lambda_i - \lambda_j) p_i y p_j z = p[x, y] z p \\ p[|x|, y] z q &= \sum_{i=1}^N \sum_{j=1}^N (\lambda_i - \mu_j) p_i y q_j z, \\ p[x, y] z q &= \sum_{i=1}^N \sum_{j=1}^N (\lambda_i + \mu_j) p_i y q_j z, \\ q[|x|, y] z p &= \sum_{i=1}^N \sum_{j=1}^N (\mu_j - \lambda_i) q_j y p_i z, \end{aligned}$$

$$q[x, y] zp = - \sum_{i=1}^N \sum_{j=1}^N (\mu_j + \lambda_i) q_j y p_i z,$$

$$q[|x|, y] zq = \sum_{i=1}^N \sum_{j=1}^N (\mu_i - \mu_j) q_i y q_j z = -q[x, y] zq.$$

Using (2.2) we now have

$$\begin{aligned} \|p[|x|, y] zp\|_{L_p(\mathcal{M}, \tau)} &= \|p[x, y] zp\|_{L_p(\mathcal{M}, \tau)}, \\ \|p[|x|, y] zq\|_{L_p(\mathcal{M}, \tau)} &\leq K_p \|p[x, y] zq\|_{L_p(\mathcal{M}, \tau)}, \\ \|q[|x|, y] zp\|_{L_p(\mathcal{M}, \tau)} &\leq K_p \|q[x, y] p\|_{L_p(\mathcal{M}, \tau)}, \\ \|q[|x|, y] zq\|_{L_p(\mathcal{M}, \tau)} &= \|q[x, y] zq\|_{L_p(\mathcal{M}, \tau)}. \end{aligned}$$

It now follows that

$$\begin{aligned} &\| [|x|, y] z \|_{L_p(\mathcal{M}, \tau)} \\ &= \|(p+q)[|x|, y] z(p+q)\|_{L_p(\mathcal{M}, \tau)} \leq 2(1+K_p) \| [x, y] z \|_{L_p(\mathcal{M}, \tau)}. \end{aligned} \tag{2.3}$$

We suppose now that x is an arbitrary self-adjoint element from $\mathcal{M} \cap L_1(\mathcal{M}, \tau)$. There exists a sequence $\{x_n\} \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ such that each x_n , $n \geq 1$ is a finite linear combination of spectral projections of x , such that $x_n \rightarrow x$, $|x_n| \rightarrow |x|$ in $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$. It follows that $[x_n, y] \rightarrow [x, y]$, $[|x_n|, y] \rightarrow [|x|, y]$ in $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ and hence $[x_n, y]z \rightarrow [x, y]z$, $[|x_n|, y]z \rightarrow [|x|, y]z$ in $L_p(\mathcal{M}, \tau)$ by the continuity of the embedding of $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ into $L_p(\mathcal{M}, \tau)$ (here we have adopted the argument used in [DDPS, Proposition 2.4(viii) \Rightarrow (ii)]). Noting that x_n commutes with z for every $n \geq 1$ we have via (2.3) that

$$\| [|x_n|, y] z \|_{L_p(\mathcal{M}, \tau)} \leq 2(1+K_p) \| [x_n, y] z \|_{L_p(\mathcal{M}, \tau)}$$

for all $n \geq 1$ (note that the assumptions $x_n \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}$, $n = 1, 2, \dots$ and $y \in \mathcal{M}$ guarantee $[x_n, y] z \in L_p(\mathcal{M}, \tau)$), and so also

$$\| [|x|, y] z \|_{L_p(\mathcal{M}, \tau)} \leq 2(1+K_p) \| [x, y] z \|_{L_p(\mathcal{M}, \tau)}.$$

This completes the proof of Proposition 2.1. \blacksquare

We shall now modify a matrix argument from the proof of the implication (ii) \Rightarrow (i) in [DDPS, Theorem 2.2]. It should be noted that the assumptions imposed on the element y in the next proposition are more stringent than in Proposition 2.1.

PROPOSITION 2.2. *If $x = x^*$, $y = y^* \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ and $z \in L_p(\mathcal{M}, \tau) \cap \mathcal{M}$ commutes with x , then*

$$\left\| (|x| - |y|) z \right\|_{L_p(\mathcal{M}, \tau)} \leq 2(1 + K_p) \left\| (x - y) z \right\|_{L_p(\mathcal{M}, \tau)}.$$

Proof. It should be noted now that the assertion of Proposition 2.1 holds for an arbitrary semifinite von Neumann algebra, in particular it holds for the von Neumann algebra $\mathcal{M}_1 := \mathcal{M} \otimes M_2(\mathbb{C})$ of all 2×2 matrices

$$[x_{ij}] = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

with $x_{ij} \in \mathcal{M}$, $i, j = 1, 2$, acting on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ with the trace τ_1 given by setting

$$\tau_1([x_{ij}]) = \tau(x_{11}) + \tau(x_{22}).$$

If

$$X^* = X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix},$$

then

$$[|X|, Y] Z = \begin{pmatrix} 0 & 0 \\ (|y| - |x|) z & 0 \end{pmatrix},$$

and

$$[X, Y] Z = \begin{pmatrix} 0 & 0 \\ (y - x) z & 0 \end{pmatrix}. \tag{2.4}$$

Since $X = X^* \in L_1(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1$, $Y \in \mathcal{M}_1$ and $ZX = XZ$, $Z \in L_p(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1$, it follows from Proposition 2.1 and (2.4) that

$$\left\| [|X|, Y] Z \right\|_{L_p(\mathcal{M}_1, \tau_1)} \leq 2(1 + K_p) \left\| [X, Y] Z \right\|_{L_p(\mathcal{M}_1, \tau_1)}.$$

From (2.4) it is clear that

$$\begin{aligned} \left\| [|X|, Y] Z \right\|_{L_p(\mathcal{M}_1, \tau_1)} &= \left\| (|y| - |x|) z \right\|_{L_p(\mathcal{M}, \tau)}, \\ \left\| [X, Y] Z \right\|_{L_p(\mathcal{M}_1, \tau_1)} &= \left\| (y - x) z \right\|_{L_p(\mathcal{M}, \tau)} \end{aligned}$$

and the assertion of the proposition follows. ■

Our proof of Theorem 0.3(i) rests on Proposition 2.3 below combined with a refinement of the approach from [S]. Crucial to our arguments is

the following inequality (Theorem 6.2 of [S]) whose proof we include in the Appendix so that this paper may be read independently of [S]. For any $x = x^*$, $y = y^* \in \tilde{\mathcal{M}}$ such that $x = y + a$ with $a \in \mathcal{M}$ and $z = (\mathbf{1} + x^2)^{-1/2} \in L_p(\mathcal{M}, \tau)$ we have

$$\left\| \frac{|y|}{(1 + y^2)^{1/2}} - \frac{|x|}{(1 + x^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \leq 2^{3/2} \|z\|_{L_p(\mathcal{M}, \tau)} \cdot \max\{\|x - y\|^{1/2}, \|x - y\|\}. \quad (2.5)$$

Our final technical proposition rests on (2.5) and is a slight refinement of [S, Corollary 6.3].

PROPOSITION 3.3. *Let $x = x^*$, $y = y^* \in \tilde{\mathcal{M}}$ and let $z = (\mathbf{1} + x^2)^{-1/2} \in L_p(\mathcal{M}, \tau)$. If $x = y + a$ with $a \in \mathcal{M}$ and the following inequality holds for some constant $c_p > 0$*

$$\|(|y| - |x|)z\|_{L_p(\mathcal{M}, \tau)} \leq c_p \max\{\|x - y\|^{1/2}, \|x - y\|\} \quad (2.6)$$

then we have

$$\|y(\mathbf{1} + y^2)^{-1/2} - x(\mathbf{1} + x^2)^{-1/2}\|_{L_p(\mathcal{M}, \tau)} \leq c'_p \max\{\|x - y\|^{1/2}, \|x - y\|\}, \quad (2.7)$$

for the constant $c'_p := c_p + (2^{3/2} + 1) \|z\|_{L_p(\mathcal{M}, \tau)}$

In its turn, if (2.7) holds for some constant c'_p and some (self-adjoint) $a = x - y \in \mathcal{M}$, then (2.6) holds with $c_p := c'_p + (2^{3/2} + 1) \|z\|_{L_p(\mathcal{M}, \tau)}$.

Proof. Let (2.6) be satisfied for some constant c_p and some $a = x - y \in \mathcal{M}$. Then from the equality

$$\begin{aligned} |y| \left(\frac{\mathbf{1}}{(\mathbf{1} + y^2)^{1/2}} - \frac{\mathbf{1}}{(\mathbf{1} + x^2)^{1/2}} \right) \\ = -(|y| - |x|) \frac{\mathbf{1}}{(\mathbf{1} + x^2)^{1/2}} + \frac{|y|}{(\mathbf{1} + y^2)^{1/2}} - \frac{|x|}{(\mathbf{1} + x^2)^{1/2}} \end{aligned}$$

and (2.5) we infer that

$$\begin{aligned} & \left\| |y| \left(\frac{\mathbf{1}}{(\mathbf{1} + y^2)^{1/2}} - \frac{\mathbf{1}}{(\mathbf{1} + x^2)^{1/2}} \right) \right\|_{L_p(\mathcal{M}, \tau)} \\ & \leq \|(|y| - |x|)z\|_{L_p(\mathcal{M}, \tau)} + \left\| \frac{|y|}{(\mathbf{1} + y^2)^{1/2}} - \frac{|x|}{(\mathbf{1} + x^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \\ & \leq (c_p + 2^{3/2} \|z\|_{L_p(\mathcal{M}, \tau)}) \max\{\|x - y\|^{1/2}, \|x - y\|\}. \end{aligned}$$

It follows immediately that

$$\begin{aligned} & \left\| y \left(\frac{\mathbf{1}}{(\mathbf{1} + y^2)^{1/2}} - \frac{\mathbf{1}}{(\mathbf{1} + x^2)^{1/2}} \right) \right\|_{L_p(\mathcal{M}, \tau)} \\ & \leq (c_p + 2^{3/2} \|z\|_{L_p(\mathcal{M}, \tau)}) \max\{\|x - y\|^{1/2}, \|x - y\|\}. \end{aligned} \quad (2.8)$$

Now from the equality

$$y \left(\frac{\mathbf{1}}{(\mathbf{1} + y^2)^{1/2}} - \frac{\mathbf{1}}{(\mathbf{1} + x^2)^{1/2}} \right) = \frac{y}{(\mathbf{1} + y^2)^{1/2}} - \frac{x}{(\mathbf{1} + x^2)^{1/2}} + (y - x) \frac{1}{(\mathbf{1} + x^2)^{1/2}}$$

combined with (2.8) we arrive at (2.7)

$$\begin{aligned} & \left\| \frac{y}{(\mathbf{1} + y^2)^{1/2}} - \frac{x}{(\mathbf{1} + x^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \\ & \leq (c_p + 2^{3/2} \|z\|_{L_p(\mathcal{M}, \tau)}) \max\{\|x - y\|^{1/2}, \|x - y\|\} + \|z\|_{L_p(\mathcal{M}, \tau)} \cdot \|x - y\| \\ & \leq (c_p + (2^{3/2} + 1) \|z\|_{L_p(\mathcal{M}, \tau)}) \max\{\|x - y\|^{1/2}, \|x - y\|\}. \end{aligned}$$

The second assertion is established similarly. \blacksquare

With these preliminary results established we are now in a position to prove Theorem 0.3.

Proof of Theorem 0.3(i). Recall that we assume that $x = x^*$, $y = y^* \in \tilde{\mathcal{M}}$ are such that $x = y + a$ with $a \in \mathcal{M}$ and that $z = (\mathbf{1} + x^2)^{-1/2} \in L_p(\mathcal{M}, \tau)$. Introduce the sequence $\{e_n = E_{[1/n, 1]}^z\}_{n=1}^\infty \subset \mathcal{M}$. Note that

$$e_n z = z e_n, \quad \forall n \geq 1, \quad e_n \uparrow_n \mathbf{1}.$$

It is straightforward to see from the definition of e_n , $n \geq 1$ and the fact $z = (\mathbf{1} + x^2)^{-1/2}$, that

$$e_n x = x e_n \in \mathcal{M}, \quad \forall n \geq 1$$

and that $e_n \leq n z \in L_p(\mathcal{M}, \tau)$. It is immediate that $e_n \in L_1(\mathcal{M}, \tau)$ for all $n \geq 1$ and further that

$$e_n x e_n, e_n y e_n \in L_1(\mathcal{M}, \tau) \cap \mathcal{M}$$

for all $n \geq 1$.

Appealing to Proposition 2.2 we have

$$\begin{aligned} & \left\| (|e_n x e_n| - |e_n y e_n|) e_n z e_n \right\|_{L_p(\mathcal{M}, \tau)} \\ & \leq \mathcal{K}_p \|e_n(x - y) e_n z e_n\|_{L_p(\mathcal{M}, \tau)} \\ & \leq \mathcal{K}_p \|z\|_{L_p(\mathcal{M}, \tau)} \max\{\|e_n(x - y) e_n\|^{1/2}, \|e_n(x - y) e_n\|\}, \end{aligned} \quad (2.9)$$

for all $n \geq 1$ and all (self-adjoint) $x - y \in \mathcal{M}$. Noting that

$$\begin{aligned} (|e_n x e_n| - |e_n y e_n|) e_n z e_n &= (|e_n x e_n| - |e_n y e_n|) e_n (\mathbf{1} + x^2)^{-1/2} e_n \\ &= (|e_n x e_n| - |e_n y e_n|) (e_n + (e_n x e_n)^2)^{-1/2} \\ &= (|e_n x e_n| - |e_n y e_n|) e_n (\mathbf{1} + (e_n x e_n)^2)^{-1/2} \\ &= (|e_n x e_n| - |e_n y e_n|) (\mathbf{1} + (e_n x e_n)^2)^{-1/2} \end{aligned}$$

we may now combine (2.9) with Proposition 2.3 to obtain

$$\begin{aligned} &\left\| \frac{e_n y e_n}{(\mathbf{1} + (e_n y e_n)^2)^{1/2}} - \frac{e_n x e_n}{(\mathbf{1} + (e_n x e_n)^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \\ &\leq c'_p \max\{\|e_n x e_n - e_n y e_n\|^{1/2}, \|e_n x e_n - e_n y e_n\|\} \\ &\leq c'_p \max\{\|x - y\|^{1/2}, \|x - y\|\} \end{aligned} \quad (2.10)$$

for all $n \geq 1$ and all (self-adjoint) $x - y \in \mathcal{M}$ with

$$c'_p = (\mathcal{K}_p + 2^{3/2} + 1) \|z\|_{L_p(\mathcal{M}, \tau)}. \quad (2.11)$$

It should be noted that since $e_n \uparrow_n \mathbf{1}$ and $y = x - a$, $a \in \mathcal{M}$ we have

$$e_n x e_n(\xi) \rightarrow x(\xi), \quad e_n y e_n(\xi) \rightarrow y(\xi),$$

as $n \rightarrow \infty$ for any $\xi \in \text{dom } x = \text{dom } y$.

Combining [RS1, Theorems VIII.25(a) and VIII.20(b)] we have

$$\frac{e_n y e_n}{(\mathbf{1} + (e_n y e_n)^2)^{1/2}} - \frac{e_n x e_n}{(\mathbf{1} + (e_n x e_n)^2)^{1/2}} \rightarrow \frac{y}{(\mathbf{1} + y^2)^{1/2}} - \frac{x}{(\mathbf{1} + x^2)^{1/2}}, \quad n \rightarrow \infty \quad (2.12)$$

in the strong operator topology as $n \rightarrow \infty$.

We may now proceed to the final part of the proof. All the operators in (2.12) are uniformly bounded. Hence we may apply (1.2) to deduce that

$$\begin{aligned} &\left\| \frac{y}{(\mathbf{1} + y^2)^{1/2}} - \frac{x}{(\mathbf{1} + x^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \\ &\leq \liminf_n \left\| \frac{e_n y e_n}{(\mathbf{1} + (e_n y e_n)^2)^{1/2}} - \frac{e_n x e_n}{(\mathbf{1} + (e_n x e_n)^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \end{aligned}$$

and, using (2.10),

$$\left\| \frac{y}{(\mathbf{1} + y^2)^{1/2}} - \frac{x}{(\mathbf{1} + x^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \leq c'_p \max\{\|x - y\|^{1/2}, \|x - y\|\}.$$

Letting (see equality (2.11))

$$\mathcal{L}'_p := \mathcal{K}_p + 2^{3/2} + 1$$

we arrive at the inequality (0.3)'. The inequality (0.3) of Theorem 0.3(i) follows from the inequality (0.3)' via Proposition 2.3. ■

Proof of Theorem 0.3(ii). We suppose first that $x = x^*$, $y = y^* \in \mathcal{M}$. Let

$$e_n := \mathbf{1} - E_{1/n}^z, \quad n = 1, 2, \dots$$

Then, using $0 \leq z \in L_p(\mathcal{M}, \tau) \cap \mathcal{M}$ and $s(z) = \mathbf{1}$, we have that $e_n \uparrow_n \mathbf{1}$ and that

$$e_n z = z e_n, \quad e_n x = x e_n, \quad \tau(e_n) < \infty, \quad \forall n \geq 1.$$

Since $e_n \uparrow_n \mathbf{1}$ we have (see (1.1))

$$e_n x e_n \xrightarrow{(s)} x, \quad e_n y e_n \xrightarrow{(s)} y, \quad |e_n x e_n| \xrightarrow{(s)} |x|, \quad |e_n y e_n| \xrightarrow{(s)} |y|. \tag{2.13}$$

It follows from the assumptions $x, y \in \mathcal{M}$ and from the inequality $\tau(e_n) < \infty$ that

$$e_n x e_n = (e_n x e_n)^*, \quad e_n y e_n = (e_n y e_n)^* \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}, \quad \forall n \geq 1.$$

Since

$$e_n(x - y) z e_n = e_n(x - y) e_n z \quad \text{and} \\ \|e_n(x - y) z e_n\|_{L_p(\mathcal{M}, \tau)} \leq \|(x - y) z\|_{L_p(\mathcal{M}, \tau)},$$

it follows from Proposition 2.2 that

$$\begin{aligned} \|(|e_n x e_n| - |e_n y e_n|) z\|_{L_p(\mathcal{M}, \tau)} &\leq 2(1 + K_p) \|e_n(x - y) e_n z\|_{L_p(\mathcal{M}, \tau)} \\ &\leq 2(1 + K_p) \|(x - y) z\|_{L_p(\mathcal{M}, \tau)}, \quad \forall n \geq 1. \end{aligned} \tag{2.14}$$

The inequality

$$\||x| - |y|\| z\|_{L_p(\mathcal{M}, \tau)} \leq 2(1 + K_p) \|(x - y) z\|_{L_p(\mathcal{M}, \tau)} \tag{2.15}$$

is now clear from (2.13), (2.14) combined with (1.2).

We shall assume now that $x = x^*$, $y = y^* \in \tilde{\mathcal{M}}$, $x - y \in \mathcal{M}$.

There exist self-adjoint projections $\{p_n\} \subseteq \mathcal{M}$ such that $p_n \uparrow_n \mathbf{1}$, $\tau(\mathbf{1} - p_n) \rightarrow 0$ and such that $x p_n \in \mathcal{M}$, $n \geq 1$. It follows immediately from the assumption

$x - y \in \mathcal{M}$ that $yp_n \in \mathcal{M}$, $n \geq 1$. Since it is possible to choose the sequence $\{p_n\}_{n=1}^\infty$ from the set of spectral projections of x we also have

$$p_n z = z e_n, \quad p_n x = x p_n, \quad n \geq 1.$$

Thus, appealing to the preceding part of the proof and applying (2.15) we get

$$\begin{aligned} \|(|e_n x e_n| - |e_n y e_n|) z\|_{L_p(\mathcal{M}, \tau)} &\leq 2(1 + K_p) \|(e_n x e_n - e_n y e_n) z\|_{L_p(\mathcal{M}, \tau)} \\ &= 2(1 + K_p) \|e_n(x - y) z e_n\|_{L_p(\mathcal{M}, \tau)} \\ &\leq 2(1 + K_p) \|(x - y) z\|_{L_p(\mathcal{M}, \tau)}. \end{aligned} \quad (2.16)$$

It is easily seen that $p_n x p_n \rightarrow x$, $p_n y p_n \rightarrow y$ for the measure topology and therefore (see Section 1) we have

$$|p_n x p_n| - |p_n y p_n| \rightarrow |x| - |y|$$

for the measure topology. It follows immediately that

$$(|p_n x p_n| - |p_n y p_n|) z \rightarrow (|x| - |y|) z$$

for the measure topology. This fact, combined with (2.16) and (1.2) implies

$$\|(|x| - |y|) z\|_{L_p(\mathcal{M}, \tau)} \leq 2(1 + K_p) \|(x - y) z\|_{L_p(\mathcal{M}, \tau)}.$$

The proof of Theorem 0.3(ii) is completed with $\mathcal{H}_p = 2(1 + K_p)$. \blacksquare

3. APPLICATIONS

Before we move to the applications we need a preliminary result. Recall that \mathcal{A} is a Banach $*$ -algebra with a bounded $*$ -representation $\pi: \mathcal{A} \rightarrow \mathcal{M}$. As the kernel of this $*$ -representation is a closed two-sided $*$ -ideal in \mathcal{A} , we can (and do) assume for our purposes that π is faithful. We let $\|\cdot\|_{\mathcal{A}}$ denote the Banach $*$ -algebra norm on \mathcal{A} and by renorming \mathcal{A} if necessary we can (and do) assume that $\|\pi(a)\| \leq \|a\|_{\mathcal{A}}$. From now on we suppress the notation π , but not the distinct norm $\|\cdot\|_{\mathcal{A}}$ on \mathcal{A} .

LEMMA 3.1. *The set*

$$\mathcal{A}_0 := \{a \in \mathcal{A} \mid a(\text{dom } D_0) \subset \text{dom } D_0, [D_0, a] \in \mathcal{M}\}$$

is a Banach $$ -algebra in the norm $\|a\|_0 = \|a\|_{\mathcal{A}} + \|[D_0, a]\|$.*

Proof. This result appears to be well known (cf. [CM]) and in any case is a good exercise in careful applications of the definition of the adjoint of an unbounded operator; see, [RN, pp. 299–300]. ■

We are now in a position to present the proof of Corollary 0.4. We follow the argument in [S, Corollary 6.8].

Proof of Corollary 0.4. We first show that $(\mathcal{M}, \phi(D_0))$ is an odd bounded p -summable pre-Breuer–Fredholm module for \mathcal{A} . Recall that by assumption (see the part (1) of Definition 0.1 for unbounded odd Breuer–Fredholm modules) the element $(\mathbf{1}/(\mathbf{1} + D_0^2))^{1/2}$ belongs to $L_p(\mathcal{M}, \tau)$ and therefore

$$(\mathbf{1} - \phi(D_0)^2)^{1/2} = \left(\mathbf{1} - \frac{D_0^2}{\mathbf{1} + D_0^2} \right)^{1/2} = \left(\frac{\mathbf{1}}{\mathbf{1} + D_0^2} \right)^{1/2}$$

belongs to $L_p(\mathcal{M}, \tau)$ too. Thus part (1) of Definition 0.1 for bounded odd pre-Breuer–Fredholm modules is satisfied. Thus, we need to check only the second part of Definition 0.1. It suffices to show that

$$\mathcal{A}_0 \subseteq \mathcal{A}_p. \tag{3.1}$$

Using Lemma 3.1 we may now apply a result of [Pa, Theorem 7], to see that the linear span of the set of all unitary elements $U(\mathcal{A}_0)$ coincides with \mathcal{A}_0 . Hence in order to establish (3.1) we need to show only that

$$[\phi(D_0), u] \in L_p(\mathcal{M}, \tau), \quad \forall u \in U(\mathcal{A}_0). \tag{3.2}$$

To establish (3.2), we note that for an arbitrary $u \in U(\mathcal{A}_0)$, we have

$$\begin{aligned} [\phi(D_0), u] &= \phi(D_0) u - u \phi(D_0) = u(u^* \phi(D_0) u - \phi(D_0)) \\ &= u(\phi(u^* D_0 u) - \phi(D_0)). \end{aligned}$$

From our assumptions (see the part (2) of Definition 0.1 for unbounded odd Breuer–Fredholm modules) we have that

$$u^* D_0 u - D_0 = u^* [D_0, u] \in \mathcal{M}, \quad \forall u \in U(\mathcal{A}_0).$$

Therefore, letting $D = u^* D_0 u$, we have by Theorem 0.3(i), that

$$\phi(D) - \phi(D_0) \in L_p(\mathcal{M}, \tau)$$

and this shows immediately that (3.2) holds.

It is now easy to verify that $(\mathcal{M}, \text{sgn}(D_0))$ is an odd bounded p -summable Breuer–Fredholm module for \mathcal{A} . Indeed, condition (1) from Definition 0.1 obviously holds. To verify that condition (2) holds, we note that

$$\begin{aligned}
(\operatorname{sgn}(D_0) - \phi(D_0))(\operatorname{sgn}(D_0) + \phi(D_0)) &= \mathbf{1} - \phi(D_0)^2 \\
&= \mathbf{1} - D_0^2(1 + D_0^2)^{-1} \\
&= (1 + D_0^2)^{-1} \\
&\leq (1 + D_0^2)^{-1/2} \in L_p(\mathcal{M}, \tau),
\end{aligned}$$

and since

$$(\operatorname{sgn}(D_0) + \phi(D_0))^{-1} \in \mathcal{M}$$

it follows that

$$\operatorname{sgn}(D_0) - \phi(D_0) = (\mathbf{1} + D_0^2)^{-1} ((\operatorname{sgn}(D_0) + \phi(D_0))^{-1}) \in L_p(\mathcal{M}, \tau),$$

whence (via the first part of the proof)

$$[\operatorname{sgn}(D_0), u] = [\operatorname{sgn}(D_0) - \phi(D_0), u] + [\phi(D_0), u] \in L_p(\mathcal{M}, \tau)$$

for any $u \in U(\mathcal{A}_0)$. ■

The significance of this result for Connes' quantised calculus is that it fills a lacuna in [Co1]. There the relationship between bounded and unbounded Fredholm modules is presented in the even case but not in the odd case. This is rectified by taking a different viewpoint in [Co2] utilising the Dixmier trace, a device which is clearly natural from the viewpoint of the geometric examples described there. However, using Corollary 0.4 we can fill this lacuna by a different method which we now explain.

Given an odd p -summable unbounded Breuer–Fredholm module (\mathcal{M}, D_0) for the algebra \mathcal{A} , we have by Corollary 0.4, that $(\mathcal{M}, F_0 = \operatorname{sgn}(D_0))$ is an odd p -summable bounded Breuer–Fredholm module. Now, each p -summable bounded Breuer–Fredholm module for \mathcal{A} has associated with it a cyclic $(p - 1)$ -dimensional cycle over \mathcal{A} (see [Co2, p. 292]). We may therefore utilise the standard formula for the character of this cycle which is, with $p = 2n + 2$,

$$\tau_{2n+1}(a^0, a^1, \dots, a^{2n+1}) = \tau(F_0[F_0, a^0][F_0, a^1] \cdots [F_0, a^{2n+1}]),$$

where $a^j \in \mathcal{A}$. Thus τ_{2n+1} may be regarded as the cyclic cocycle associated with both the odd unbounded Breuer–Fredholm module and the bounded one.

Note that in [Co1] in the case of even p -summable Fredholm modules an expression is given directly in terms of the unbounded operator D_0 for an associated cyclic cocycle. We have not investigated the existence of such an expression in the odd case but presumably one exists.

It is worth mentioning at this point the motivating example of spectral flow in [CP]. Given an odd p -summable unbounded Breuer–Fredholm module (\mathcal{M}, D_0) [CP] (following [G]) introduce an affine space $\Phi_p = \{D = D_0 + A \mid A \in \mathcal{M}_{sa}\}$. Then it is shown in [CP] that the map $D \mapsto \phi(D) = D(\mathbf{1} + D^2)^{-1/2}$ takes, for all $q > p$, the space Φ_p continuously into the affine space

$$\mathcal{M}_q = \{F = F_0 + X \mid X \in L_{q, q/2}(\mathcal{M}, \tau)_{sa}\},$$

where $F_0 = \phi(D_0)$ and $L_{q, q/2}(\mathcal{M}, \tau)_{sa}$ are the bounded self adjoint elements of $L_q(\mathcal{M}, \tau)$ that satisfy the additional condition

$$XF_0 + F_0X \in L_{q/2}(\mathcal{M}, \tau).$$

This second constraint plays a key role in the analytic formulae for spectral flow in [CP]. In fact the import of Corollary 0.4 is that Φ_p actually maps continuously into \mathcal{M}_p by the following proposition.

PROPOSITION 3.2. *If $1 < p < \infty$ and (\mathcal{M}, D_0) is an odd unbounded p -summable Breuer–Fredholm module then the map*

$$\phi: \Phi_p \rightarrow \phi(D_0) + L_{p, p/2}(\mathcal{M}, \tau)_{sa}$$

is well defined and continuous.

Proof. By Theorem 0.3(i), $\phi(D) = F$ lies in $\phi(D_0) + L_p(\mathcal{M}, \tau)_{sa}$ and the mapping is continuous in that space. To take account of the additional condition set $F_0 = \phi(D_0)$ and define $X_D = F - F_0$ so that the map $D \mapsto X_D \in L_p(\mathcal{M}, \tau)_{sa}$ is continuous on Φ_p . Now the map

$$D \mapsto 1 - \phi(D)^2 = (1 + D^2)^{-1} \in L_{p/2}(\mathcal{M}, \tau)_{sa}$$

is continuous by Corollary A.2 (or [CP, Proposition 10 of Appendix B]). But

$$D \mapsto X_D \mapsto X_D^2 \in L_{p/2}(\mathcal{M}, \tau)_{sa}$$

is also continuous. We observe that

$$\begin{aligned} 1 - \phi(D)^2 &= 1 - F_0^2 - (X_D^2 + F_0X_D + X_DF_0) \\ &= (1 + D_0^2)^{-1} - X_D^2 - (F_0X_D + X_DF_0) \end{aligned}$$

and so $D \mapsto (F_0 X_D + X_D F_0) \in L_{p/2}(\mathcal{M}, \tau)_{sa}$ is continuous. That is, $X_D \in L_{p, p/2}(\mathcal{M}, \tau)_{sa}$ and continuity in the norm $\|\cdot\|_{p, p/2}$ on the latter space is clear from the definition of this norm,

$$\begin{aligned} & \|X_{D_1} - X_{D_2}\|_{p, p/2} \\ &= \|X_{D_1} - X_{D_2}\|_p + \|(X_{D_1} - X_{D_2})F_0 + F_0(X_{D_1} - X_{D_2})\|_{p/2}. \quad \blacksquare \end{aligned}$$

In particular continuous paths

$$\{D_t = D_0 + A_t\} \tag{3.3}$$

in Φ_p map to continuous paths of Breuer–Fredholm operators in \mathcal{M}_p under ϕ . Thus using [P2] we define the spectral flow along the path $\{D_t\}$ as the spectral flow along $\{\phi(D_t)\}$ and this will be independent of the path in \mathcal{M}_p joining the endpoints as \mathcal{M}_p is simply connected.

Remark 3.3. Generalizing the notions of p -summable and θ -summable Fredholm modules (see [Co1, Co2]) one of us introduced in [S] the notion of an odd (un)bounded Breuer–Fredholm module associated with an arbitrary symmetric operator space $E(\mathcal{M}, \tau)$ (which we now abbreviate to $E(\mathcal{M}, \tau)$ -summable Breuer–Fredholm module). For the definitions and additional information concerning these spaces we refer to [DDP, DDPS, SC]. An inspection of the proofs presented in Section 2 shows that the assertions given in Proposition 2.1, 2.2, 2.3 and Theorem 0.3 would also hold if

(i) $L_p(\mathcal{M}, \tau)$ is replaced by an arbitrary symmetric operator space $E(\mathcal{M}, \tau)$ which is an interpolation space for any couple $(L_{p_1}(\mathcal{M}, \tau), L_{p_2}(\mathcal{M}, \tau))$ with $1 < p_1 \leq p_2 < \infty$ and

(ii) which has the Fatou property (see, e.g., [DDPS]).

In particular, these conditions are satisfied when the corresponding symmetric function space E has non-trivial Boyd indices (see, e.g., [LT]) and the Fatou property (see [DDP, DDPS]). The latter assumption about the Fatou property is automatically satisfied whenever E is an Orlicz, Lorentz or Marcinkiewicz function space (see [LT, BS, KPS]).

By way of an example, it follows from [FG, Theorem 4.1] that any reflexive Orlicz space L_ϕ has non-trivial Boyd indices. It is also well-known that the latter property holds also for the family of spaces $L_{p, q}$, $1 < p < \infty$, $1 \leq q \leq \infty$ (for the definition of the latter spaces we refer to [LT]; the spaces $L_{p, \infty}$ are known as *weak L_p -spaces*, see, e.g., [BS]). Thus we obtain the following strengthening of Corollary 0.4.

COROLLARY 0.5. *If $E(\mathcal{M}, \tau)$ is either a reflexive Orlicz operator space $L_\Phi(\mathcal{M}, \tau)$, or $L_{p,q}(\mathcal{M}, \tau)$, $1 < p < \infty$, $1 \leq q \leq \infty$ and if (\mathcal{M}, D_0) is an odd unbounded $E(\mathcal{M}, \tau)$ -summable Breuer–Fredholm module for \mathcal{A} , then $(\mathcal{M}, \text{sgn}(D_0))$ is an odd bounded $E(\mathcal{M}, \tau)$ -summable Breuer–Fredholm module for \mathcal{A} .*

It is also worth mentioning specifically that this last corollary holds when $E(\mathcal{M}, \tau)$ is the space $L_{p,\infty}(\mathcal{M}, \tau)$, $1 < p < \infty$. This latter fact may also be deduced from [CP, CM]. It has implications for analytic formulae for spectral flow.

APPENDIX

The result we need in Theorem 0.3(i) is the following.

THEOREM. *Let (\mathcal{M}, τ) be an arbitrary semifinite von Neumann algebra, let $x = x^*$ be affiliated with \mathcal{M} such that $z = (1 + x^2)^{-1/2} \in L_p(\mathcal{M}, \tau)$. Then there exists a constant $C > 0$ (depending on $L_p(\mathcal{M}, \tau)$ and x) such that for all bounded self-adjoint $y - x \in \mathcal{M}$ we have*

$$\left\| \frac{|y|}{(1 + y^2)^{1/2}} - \frac{|x|}{(1 + x^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \leq C \max\{\|y - x\|^{1/2}, \|y - x\|\}. \quad (\text{A.1})$$

We introduce some notation. We let $\mu(x)$ denote the generalised singular value function for $x \in \tilde{\mathcal{M}}$ (see [FK] for details). If $x, y \in \tilde{\mathcal{M}}$, then we say that x is submajorized by y and write $x \ll y$ if and only if

$$\int_0^t \mu_s(x) ds \leq \int_0^t \mu_s(y) ds, \quad t \geq 0.$$

The proof of the theorem rests on the following:

LEMMA A.1. *Let $x = x^* \in \tilde{\mathcal{M}}$ and $0 \leq y \in \tilde{\mathcal{M}}$ and let $-y \leq x \leq y$. Then $\mu_s(x) \leq \mu_{s/2}(y)$ for all $s > 0$.*

Proof. We let p_\pm denote the spectral projections corresponding to the positive and negative parts of the spectrum of x . We have $x_+ \leq p_+ y p_+$ and $x_- \leq p_- y p_-$. So $\mu_s(x_+) \leq \mu_s(y)$ and $\mu_s(x_-) \leq \mu_s(y)$, whence $\mu_s(x_+^n) \leq \mu_s(y)^n$ and $\mu_s(x_-^n) \leq \mu_s(y)^n$ for all $n = 1, 2, \dots$ and all $s > 0$. Thus

$$\begin{aligned} \mu_s(x)^n &= \mu_s(|x|^n) = \mu_s(x_+^n + x_-^n) \leq \mu_{s/2}(x_+^n) + \mu_{s/2}(x_-^n) \\ &\leq 2\mu_{s/2}(y)^n, \quad n = 1, 2, \dots \end{aligned}$$

It follows that $\mu_s(x) \leq \mu_{s/2}(y)$, $s > 0$. ■

COROLLARY A.1. *If f is any continuous increasing function on $[0, \infty)$ with $f(0) \geq 0$ and $-y \leq x \leq y$ then $\mu_s(f(|x|)) \leq \mu_{s/2}(f(y))$ and*

$$\int_0^r \mu_s(f(|x|)) ds \leq 2 \int_0^r \mu_s(f(y)) ds.$$

That is, $f(|x|) \ll 2f(y)$ so that in particular $|x|^{1/2} \ll 2y^{1/2}$.

Proof. The first claim follows from [FK, Lemma 2.5(iv)]. The second claim holds because

$$\int_0^r \mu_s(f(|x|)) ds \leq \int_0^r \mu_{s/2}(f(y)) ds = 2 \int_0^{r/2} \mu_t(f(y)) dt \leq 2 \int_0^r \mu_t(f(y)) dt. \quad \blacksquare$$

COROLLARY A.2. *Given a fully symmetric function space $E(\mathcal{M}, \tau)$ and any continuous increasing function f on $[0, \infty)$ with $f(0) \geq 0$ we have, for $x = x^*$, $y \geq 0$, $-y \leq x \leq y$ and $f(y) \in E(\mathcal{M}, \tau)$, that $f(|x|) \in E(\mathcal{M}, \tau)$ and $\|f(|x|)\|_{E(\mathcal{M}, \tau)} \leq 2 \|f(y)\|_{E(\mathcal{M}, \tau)}$.*

We now move to the proof of the theorem.

Proof. It follows from the proof in [CP, Appendix B, Proposition 10] that

$$\begin{aligned} -2 \max\{\|y-x\|^2, \|y-x\|\} \cdot \frac{1}{(1+x^2)} &\leq \frac{1}{(1+y^2)} - \frac{1}{(1+x^2)} \\ &\leq 2 \max\{\|y-x\|^2, \|y-x\|\} \cdot \frac{1}{(1+x^2)} \end{aligned}$$

and, by Corollary A.1, it follows that

$$\left| \frac{1}{(1+y^2)} - \frac{1}{(1+x^2)} \right|^{1/2} \ll 2(2 \max\{\|y-x\|^2, \|y-x\|\})^{1/2} \left(\frac{1}{(1+x^2)^{1/2}} \right). \quad (\text{A.2})$$

The latter inequality implies immediately that

$$\begin{aligned} &\left\| \left| \frac{1}{(1+y^2)} - \frac{1}{(1+x^2)} \right|^{1/2} \right\|_{L_p(\mathcal{M}, \tau)} \\ &\leq 2^{3/2} \max\{\|y-x\|^{1/2}, \|y-x\|\} \left\| \frac{1}{(1+x^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} \end{aligned}$$

or, equivalently,

$$\left\| \left| \frac{y^2}{(1+y^2)} - \frac{x^2}{(1+x^2)} \right|^{1/2} \right\|_{L_p(\mathcal{M}, \tau)} \leq C \max\{|y-x|^{1/2}, \|y-x\|\}, \quad (\text{A.3})$$

where $C := 2^{3/2} \|1/(1+x^2)^{1/2}\|_{L_p(\mathcal{M}, \tau)}$.

The following inequality is developed in [BKS] for the case of symmetrically normed ideals of compact operators, extended to measurable operators affiliated with an arbitrary semifinite von Neumann algebra \mathcal{M} by H. Kosaki (it is given in the appendix to [HN]) with an alternative version of the proof given in [DD]. By Theorem 1.1 from [DD] (see also [BKS, Theorem 1]) we have

$$a^{1/2} - b^{1/2} \ll\ll |a-b|^{1/2} \quad (\text{A.4})$$

for any $0 \leq a, b \in \tilde{\mathcal{M}}$. Combining (A.4) with (A.3) and using the fact that the space $L_p(\mathcal{M}, \tau)$ is fully symmetric we have

$$\begin{aligned} \left\| \frac{|y|}{(1+y^2)^{1/2}} - \frac{|x|}{(1+x^2)^{1/2}} \right\|_{L_p(\mathcal{M}, \tau)} &= \left\| \left(\frac{y^2}{(1+y^2)} \right)^{1/2} - \left(\frac{x^2}{(1+x^2)} \right)^{1/2} \right\|_{L_p(\mathcal{M}, \tau)} \\ &\leq \left\| \left| \frac{y^2}{(1+y^2)} - \frac{x^2}{(1+x^2)} \right|^{1/2} \right\|_{L_p(\mathcal{M}, \tau)} \\ &\leq C \max\{\|y-x\|^{1/2}, \|y-x\|\} \end{aligned}$$

which is the desired inequality. ■

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