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On the perturbations of spectra of upper triangular operator matrices

Mohamed Barraa [a], Mohamed Boumazgour [b,*]

[a] Département de Mathématiques, Faculté des Sciences Semlalia, Université Cadi Ayyad, B.P. 2390 Marrakech, Morocco
[b] Faculté des Sciences Juridiques, Economiques et Sociales, B.P. 8658 Cité Dakhla, Agadir 80000, Morocco

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ABSTRACT

In this paper we investigate perturbations of the regular spectrum of an upper triangular operator matrix such as $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ acting on a Hilbert space $H \oplus K$.

1. Introduction

Let H and K be Hilbert spaces, let $\mathcal{L}(H, K)$ denote the space of all bounded linear operators from H to K, and write $\mathcal{L}(H) = \mathcal{L}(H, H)$. If $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$ and $C \in \mathcal{L}(K, H)$ are given, we denote by M_C the operator acting on $H \oplus K$ of the form

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

For $T \in \mathcal{L}(H, K)$, let T^*, $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$ denote the adjoint, the null space, the range, the spectrum and the point spectrum of T, respectively. The nullity and the deficiency of T are defined respectively by $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim K/R(T)$. The reduced minimum modulus $\gamma(T)$ of T is defined by

$$\gamma(T) = \begin{cases} \inf\{\|Tx\|:\ \mathrm{dist}(x, N(T)) = 1\} & \text{if } T \neq 0, \\ 0 & \text{if } T = 0. \end{cases}$$

It is well known that $\gamma(T) > 0$ if and only if $R(T)$ is closed. Moreover $\gamma(T) = \gamma(T^*)$ and $\|Tx\| \geqslant \gamma(T)\|x\|$ for every $x \in N(T)^{\perp}$, where $N(T)^{\perp}$ stands for the orthogonal complement of $N(T)$ (see [1]).

Recall that an operator $T \in \mathcal{L}(H)$ is said to be generalized invertible (g-invertible for short) if there exists an operator $T^+ \in \mathcal{L}(H)$ such that

$$TT^+T = T \quad \text{and} \quad T^+TT^+ = T^+.$$

The operator T^+ is known as a generalized inverse of T.

It is well known that $T \in \mathcal{L}(H)$ has a generalized inverse if and only if its range $R(T)$ is closed (see [4]). If there is an operator $S \in \mathcal{L}(H)$ such that $TST = T$, then T is g-invertible. Indeed, let $S_1 = STS$. Then clearly S_1 satisfies $TS_1T = T$ and $S_1TS_1 = S_1$.

* Corresponding author.
E-mail addresses: barraa@ucam.ac.ma (M. Barraa), boumazgour@hotmail.com (M. Boumazgour).

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We shall call $T \in \mathcal{L}(H)$ Kato non-singular if it is g-invertible and satisfies the following condition:

$$N(T) \subseteq R(T^n) \quad \text{for all } n \geqslant 0. \tag{1.1}$$

Note that the inequality in (1.1) is equivalent to

$$N(T^n) \subseteq R(T) \quad \text{for all } n \geqslant 0. \tag{1.2}$$

Given an arbitrary operator $T \in \mathcal{L}(H)$, the regular region $\text{reg}(T)$ of T is defined by

$$\text{reg}(T) = \{\lambda \in \mathbb{C}\colon\ T - \lambda I \text{ is Kato non-singular}\}.$$

The regular spectrum $\sigma_g(T)$ of T is defined to be the set

$$\sigma_g(T) := \mathbb{C} \setminus \text{reg}(T).$$

The set $\sigma_g(T)$ is compact and nonempty. Moreover, $\partial\sigma(T) \subseteq \sigma_g(T)$ (see [17]), where we write ∂K for the topological boundary of a subset $K \subseteq \mathbb{C}$. We also have from [17], $\sigma_g(T^*) = \overline{\sigma_g(T)}$ (the bar stands for the complex conjugates points), and $\sigma_g(T) = \sigma(T)$ whenever T is a normal operator. More other properties of the regular spectrum can be found in [1,12,16,17].

Perturbations of different spectra of operator matrices have been studied by numerous authors, see for example [2,3,5,6,8,13,14,19] and the references cited therein. This paper is concerned with the regular spectrum of 2×2 upper triangular operator matrices.

Recall that a hole in a compact subset $\Delta \subseteq \mathbb{C}$ is a bounded component. The polynomially convex hull of Δ is the topological object obtained by filling in holes. We denote it by $\eta(\Delta)$. We also denote $\text{int}(\Delta)$ for the interior points of Δ.

2. Kato non-singularity of M_C

In this section we investigate the Kato non-singularity of the matrix M_C. We begin with the following theorem.

Theorem 2.1. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ be given operators. Suppose that A is Kato non-singular, $R(B)$ is closed and $\alpha(B) \leqslant \beta(A)$. Then, there exists an operator $C \in \mathcal{L}(K, H)$ such that M_C is Kato non-singular.*

Proof. Since $\alpha(B) \leqslant \beta(A)$, there exists an isometry $J : N(B) \to R(A)^\perp$. Define an operator $C \in \mathcal{L}(K, H)$ by

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N(B) \\ N(B)^\perp \end{bmatrix} \to \begin{bmatrix} R(A)^\perp \\ R(A) \end{bmatrix}. \tag{2.1}$$

We claim that M_C is Kato non-singular. Let $\binom{x}{y} \in N(M_C)$. Then

$$M_C \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} Ax + Cy = 0, \\ By = 0, \end{cases} \quad \Rightarrow \quad \begin{cases} x \in N(A), \\ y = 0, \end{cases}$$

where the second implication follows from the fact that $R(C) \cap R(A) = \{0\}$ and $N(C) \subseteq N(B)^\perp$. Hence

$$N(M_C) = N(A) \subseteq R(A^n) \subseteq R(M_C^n),$$

for all n because A is Kato non-singular.

Next we prove that $R(M_C)$ is closed. To do this, let $\binom{x}{y} \in N(M_C)^\perp$. Then $x \in N(A)^\perp$ and

$$\left\| M_C\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right\|^2 = \|Ax + Cy\|^2 + \|By\|^2 = \|Ax\|^2 + \|Cy\|^2 + \|By\|^2.$$

Write $y := u + v$, where $u \in N(B)$ and $v \in N(B)^\perp$. Then $\|Cy\| = \|u\|$ and $\|By\| = \|Bv\| \geqslant \gamma(B)\|v\|$ since $R(B)$ is closed. Hence

$$\begin{aligned} \left\| M_C\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right\|^2 &\geqslant \|Ax\|^2 + \|u\|^2 + \gamma^2(B)\|v\|^2 \\ &\geqslant \gamma^2(A)\|x\|^2 + \|u\|^2 + \gamma^2(B)\|v\|^2 \\ &\geqslant \min\bigl(\gamma^2(A), \gamma^2(B), 1\bigr)\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^2. \end{aligned}$$

Thus $\gamma(M_C) > 0$. Consequently M_C is Kato non-singular. This ends the proof. □

Corollary 2.1. *If A is Kato non-singular with $\beta(A) = \infty$ and $R(B)$ is closed, then there exists an operator C such that M_C is Kato non-singular.*

Corollary 2.2. *For a given pair (A, B) of operators, we have*

$$\bigcap_{C\in\mathcal{L}(K,H)} \sigma_g(M_C) \subseteq \sigma_g(A) \cup \sigma_f(B) \cup \{\lambda \in \mathbb{C}\colon\ \beta(A-\lambda I) < \alpha(B-\lambda I)\},$$

where $\sigma_f(B) = \{\lambda \in \mathbb{C}\colon\ R(B-\lambda I)$ is not closed$\}$.

In the case where the range $R(B)$ is not closed we have the following results:

Theorem 2.2. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ be given operators such that $R(B)$ is not closed. If A is Kato non-singular and $\beta(A) = \infty$, then there exists $C \in \mathcal{L}(K, H)$ such that M_C is Kato non-singular.*

Proof. Since $R(B)$ is not closed and $\beta(A) = \infty$, there exists an isomorphism $J : K \to R(A)^{\perp}$. Define an operator $C : K \to H$ in the following way:

$$C := (J\ \ 0) : K \to \begin{bmatrix} R(A)^{\perp} \\ R(A) \end{bmatrix}. \tag{2.2}$$

We claim that M_C is Kato non-singular. As in the proof of Theorem 2.1, we check easily that $N(M_C) = N(A)$ and $\gamma(M_C) > 0$. Next, we claim that $N(M_C) \subseteq R(M_C^n)$ for all n. Indeed, let $\binom{x}{y} \in N(M_C^2)$; then we have

$$\begin{cases} A^2x + ACy + CBy = 0, \\ B^2y = 0. \end{cases}$$

Since $R(C)$ is orthogonal to $R(A)$, we derive that

$$A^2x + ACy = CBy = 0.$$

Hence

$$Ax + Cy \in N(A) \subseteq R(A),$$

because A is Kato non-singular. It follows then that $Cy \in R(A)$, so $y = 0$. Therefore

$$N(M_C^2) = N(A^2) \oplus \{0\} \subseteq R(A) \oplus \{0\} \subseteq R(M_C).$$

Then, using an induction argument, we deduce that

$$N(M_C^n) = N(A^n) \oplus \{0\} \subseteq R(M_C),$$

for all $n \geqslant 1$. That is, M_C is Kato non-singular. □

To prove the next theorem we need a lemma.

Lemma 2.1. *Let S, U and $T \in \mathcal{L}(H)$ be given such that U is invertible. If $\dim N(S)$ is finite and $R(SUT)$ is closed, then $R(T)$ is also closed.*

Proof. Since $R(SUT)$ is closed, it follows from [10, Theorem 1] that $N(S) + R(UT)$ is closed. But $\dim N(S) < \infty$, hence we deduce that $R(UT)$ is closed. Using again [10, Theorem 1] and the fact that U is invertible, we conclude that $R(T)$ is closed. □

Theorem 2.3. *Suppose that B is injective and $R(B)$ is not closed. Then there exists $C \in \mathcal{L}(K, H)$ such that M_C is Kato non-singular if and only if A is Kato non-singular and $\beta(A) = \infty$.*

Proof. Assume that M_C is Kato non-singular for some $C \in \mathcal{L}(K, H)$. Since B is injective, we easily check that $N(M_C^n) = N(A^n) \oplus \{0\} \subseteq R(M_C)$ for every $n \geqslant 1$. From this we deduce that $N(A^n) \subseteq R(A)$ for all n. On the other hand, we have

$$M_C = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix},$$

hence, by virtue of Lemma 2.1, we conclude that $R(\left[\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right])$ is closed, that is, $R(A)$ is closed. Therefore A is Kato non-singular. Next, suppose in the contrary that $\beta(A) < \infty$. Then

$$\dim N\left(\begin{bmatrix} A^* & 0 \\ 0 & I \end{bmatrix}\right) = \dim N(A^*) = \beta(A) < \infty.$$

Since $R(M_C^*)$ is closed and $\left[\begin{smallmatrix} I & 0 \\ C^* & I \end{smallmatrix}\right]$ is invertible, we have by Lemma 2.1 that $R(\left[\begin{smallmatrix} I & 0 \\ 0 & B^* \end{smallmatrix}\right])$ is closed, that is, $R(B^*)$ is closed. This contradicts our assumption. Therefore we must have $\beta(A) = \infty$.

The reverse implication is proved in Theorem 2.2. □

As a consequence of Theorem 2.3, we have

Corollary 2.3. *Suppose that $R(B)$ is not closed. If $\sigma_p(B)$ is empty, then*

$$\bigcap_{C \in \mathcal{L}(K,H)} \sigma_g(M_C) = \sigma_g(A) \cup \{\lambda \in \mathbb{C}: \beta(A - \lambda) < \infty\}.$$

Remark 2.1. One might guess that if M_C is Kato non-singular, then $R(B)$ is closed. But this is not the case. By [11, Example 3], there are Hilbert space operators A, B and C such that M_C is bounded below and such that $R(B)$ is not closed.

We known that $\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{L}(K,H)$, however this inclusion fails to be true for the regular spectrum in a general setting. To see this, consider the following example:

Example 2.1. Let H be a Hilbert space with an orthonormal basis $\{e_{i,j}\}$ where i and j are integers such that $ij \leqslant 0$. Define operators A and $B \in \mathcal{L}(H)$ by

$$Ae_{i,j} = \begin{cases} 0 & \text{if } j = 0,\ i > 0, \\ e_{i,j+1} & \text{otherwise} \end{cases}$$

and

$$Be_{i,j} = \begin{cases} 0 & \text{if } i = 0,\ j > 0, \\ e_{i+1,j} & \text{otherwise.} \end{cases}$$

Then $N(A) = \bigvee\{e_{i,0},\ i > 0\} \subseteq R^n(A)$ and $N(B) = \bigvee\{e_{0,j},\ j > 0\} \subseteq R^n(B)$ for all n, and both $R(A)$ and $R(B)$ are closed (see [18]). Thus A and B are Kato non-singular.

Define an operator $C \in \mathcal{L}(H)$ by

$$C := -e_{0,1} \otimes e_{0,1} + e_{0,0} \otimes e_{-1,1} - e_{-1,0} \otimes e_{-1,1}.$$

We have $\binom{e_{0,0}}{e_{0,1}} \in N(M_C)$, but $\binom{e_{0,0}}{e_{0,1}} \notin R(M_C)$. Indeed, suppose that there exists a vector $\binom{x}{y}$ such that $\binom{e_{0,0}}{e_{0,1}} = M_C\binom{x}{y}$. Then a straightforward computation shows that $y = e_{-1,1}$ and $Ax = e_{-1,0}$, which is a contradiction since $e_{-1,0} \notin R(A)$. Therefore $N(M_C) \nsubseteq R(M_C)$, so that M_C is not Kato non-singular.

In [1, Lemma 1.4], it was shown that, for given operators A, B and C, the operator M_C is Kato non-singular whenever A is surjective and B is bounded below. In the sequel we give a generalization of this result.

Lemma 2.2. *Suppose that $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ have generalized inverses A^+ and B^+ respectively. If an operator $C \in \mathcal{L}(K,H)$ satisfies the equation $C = AA^+C - AA^+CB^+B + CB^+B$ then, the operator M_C is g-invertible. If, in particular, either A is right invertible and B is g-invertible or A is g-invertible and B is left invertible, then M_C is g-invertible for every $C \in \mathcal{L}(K,H)$.*

Proof. A simple computation shows that

$$M_C \begin{bmatrix} A^+ & -A^+CB^+ \\ 0 & B^+ \end{bmatrix} M_C = M_C.$$

So it follows that M_C is g-invertible. □

Theorem 2.4. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$ be given Kato non-singular operators. If either A is surjective or B is injective, then M_C is Kato non-singular for every $C \in \mathcal{L}(K,H)$.*

Proof. We claim that M_C is Kato non-singular. We consider two cases.

Case 1. Assume that B is injective. If $\binom{x}{y} \in N(M_C)$, then $x \in N(A)$ and $y = 0$. Thus $N(M_C) \subseteq N(A)$. It follows then from the Kato non-singularity of A that

$$N(M_C) \subseteq R(A^n) \subseteq R(M_C^n) \quad \text{for all } n.$$

Next, we will show that $R(M_C)$ is closed. Since B is injective and $R(B)$ is closed, we conclude that B is left invertible. Hence Lemma 2.2 implies that M_C is g-invertible. Consequently, we have that M_C is Kato non-singular.

Case 2. Assume that A is surjective. Hence A^* is injective. Since

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} B^* & C^* \\ 0 & A^* \end{bmatrix}, \tag{2.3}$$

we conclude that

$$\sigma_g\left(\begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix}\right) = \sigma_g\left(\begin{bmatrix} B^* & C^* \\ 0 & A^* \end{bmatrix}\right). \tag{2.4}$$

Since $0 \notin \sigma_g(A) \cup \sigma_g(B)$, it follows from the first case that

$$0 \notin \sigma_g\left(\begin{bmatrix} B^* & C^* \\ 0 & A^* \end{bmatrix}\right).$$

Using (2.4), we conclude that $0 \notin \sigma_g(M_C^*)$. Thus M_C is Kato non-singular, which completes the proof. □

From the above theorem we obtain the following consequence.

Corollary 2.4. *If A, B and C are given operators, then*

$$\sigma_g(M_C) \subseteq \sigma_g(A) \cup \sigma_g(B) \cup \left(\overline{\sigma_p(A^*)} \cap \sigma_p(B)\right). \tag{2.5}$$

The inclusion in (2.5) may be proper. To see this, consider the following example.

Example 2.2. Let $\{e_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ be orthonormal basis for H and K respectively. Define the operators A and B by $Ae_i = e_{i+1}$, $i = 1, 2, \ldots,$ and

$$\begin{cases} Bf_1 = 0, \\ Bf_i = f_{i-1}, \quad i = 1, 2, \ldots. \end{cases}$$

It is well known that

$$\sigma_p(A^*) = \sigma_p(B) = \{\lambda \in \mathbb{C}\colon\ |\lambda| < 1\}$$

and that

$$\sigma_g(A) = \sigma_g(B) = \{\lambda \in \mathbb{C}\colon\ |\lambda| = 1\}.$$

Define an operator C from K into H by

$$C := e_1 \otimes f_1\colon\ x \in K \mapsto \langle x, f_1\rangle e_1.$$

It is not hard to show that M_C is a unitary operator. So, by [17, Theorem 1.5], we have

$$\sigma_g(M_C) = \sigma(M_C) \subseteq \{\lambda \in \mathbb{C}\colon\ |\lambda| = 1\}.$$

Hence, we see that the inclusion

$$\sigma_g(M_C) \subseteq \sigma_g(A) \cup \sigma_g(B) \cup \left(\overline{\sigma_p(A^*)} \cap \sigma_p(B)\right)$$

is proper.

Theorem 2.5. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$. For every $C \in \mathcal{L}(K, H)$, we have*

$$\left(\sigma_g(A) \setminus \sigma_p(B)\right) \cup \left(\sigma_g(B) \setminus \overline{\sigma_p(A^*)}\right) \subseteq \sigma_g(M_C).$$

Proof. It suffices to claim that $\sigma_g(A) \setminus \sigma_p(B)) \subseteq \sigma_g(M_C)$, then as in the above we deduce from (2.3) that $\sigma_g(B) \setminus \overline{\sigma_p(A^*)}) \subseteq \sigma_g(M_C)$.

Suppose $\lambda \in \sigma_g(A) \setminus \sigma_p(B)$ and $\lambda \notin \sigma_g(M_C)$. Without loss of generality we may take $\lambda = 0$. Since $0 \notin \sigma_p(B)$, we must have $N(M_C) = N(A)$. Thus, it follows from the Kato non-singularity of M_C that $N(A) \subseteq R(A^n)$ for all n. On the other hand, since $0 \notin \sigma_g(M_C)$, there is an operator $M = \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right] \in \mathcal{L}(H \oplus K)$ such that $M_C M M_C = M_C$. Hence, we obtain

$$\begin{cases} AX_1A + CX_3A = A, \\ BX_3A = 0. \end{cases}$$

Since B is injective, we get $X_3A = 0$. Hence $AX_1A = A$, and then A is g-invertible. Consequently, we deduce $0 \notin \sigma_g(A)$, which contradicts our assumption. This completes the proof. □

3. The passage from $\sigma_g(A) \cup \sigma_g(B) \cup (\overline{\sigma_p(A^*)} \cap \sigma_p(B))$ to $\sigma_g(M_C)$

In this section we give a description of the passage from $\sigma_g(A) \cup \sigma_g(B) \cup (\overline{\sigma_p(A^*)} \cap \sigma_p(B))$ to $\sigma_g(M_C)$ for a given operators A, B and C. We shall prove that this passage is accomplished by removing certain subsets of $\overline{\sigma_p(A^*)} \cup \sigma_p(B)$ from the former.

Theorem 3.1. *For a given pair (A, B) of operators there is equality, for every $C \in \mathcal{L}(K, H)$,*

$$\eta\big(\sigma(A) \cup \sigma(B)\big) = \eta\big(\sigma_g(A) \cup \sigma_g(B)\big) = \eta\big(\sigma_g(M_C)\big).$$

More precisely,

$$\sigma_g(A) \cup \sigma_g(B) \cup \big(\overline{\sigma_p(A^*)} \cap \sigma_p(B)\big) = \sigma_g(M_C) \cup W,$$

where W is the union of certain of the holes in $\sigma_g(M_C)$ which happen to be subsets of $\overline{\sigma_p(A^*)} \cup \sigma_p(B)$.

Proof. We first claim that, for every $C \in \mathcal{L}(K, H)$,

$$\eta\big(\sigma(A) \cup \sigma(B)\big) = \eta\big(\sigma_g(M_C)\big). \tag{3.1}$$

We know by [17] that $\partial\sigma(M_C) \subseteq \sigma_g(M_C)$ and $\sigma_g(M_C) \subseteq \sigma(M_C)$, hence it follows that

$$\eta\big(\sigma(M_C)\big) = \eta\big(\sigma_g(M_C)\big). \tag{3.2}$$

By [9, Theorem 6] and (3.2), we deduce that

$$\eta\big(\sigma(A) \cup \sigma(B)\big) = \eta\big(\sigma_g(M_C)\big). \tag{3.3}$$

On the other hand, we have by virtue of [17] and Theorem 2.4

$$\begin{aligned}\partial(\sigma(A) \cup \sigma(B)) &\subseteq \partial\big(\sigma(A)\big) \cup \partial\big(\sigma(B)\big)\\ &\subseteq \sigma_g(A) \cup \sigma_g(B)\\ &\subseteq \sigma_g(A) \cup \sigma_g(B) \cup \big(\overline{\sigma_p(A^*)} \cap \sigma_p(B)\big)\\ &\subseteq \sigma(A) \cup \sigma(B).\end{aligned}$$

Hence using (3.3), we conclude that

$$\begin{aligned}\eta\big(\sigma(A) \cup \sigma(B)\big) &= \eta\big(\sigma_g(A) \cup \sigma_g(B)\big)\\ &= \eta\big(\sigma_g(A) \cup \sigma_g(B) \cup \big(\overline{\sigma_p(A^*)} \cap \sigma_p(B)\big)\big)\\ &= \eta\big(\sigma_g(M_C)\big).\end{aligned} \tag{3.4}$$

Eq. (3.4) says that the passage from $\sigma_g(M_C)$ to $\sigma_g(A) \cup \sigma_g(B) \cup (\overline{\sigma_p(A^*)} \cap \sigma_p(B))$ is filling in certain of the holes in $\sigma_g(M_C)$. But by Theorem 2.5, we have for every $C \in \mathcal{L}(K, H)$,

$$\big(\sigma_g(A) \cup \sigma_g(B)\big) \setminus \big(\overline{\sigma_p(A^*)} \cup \sigma_p(B)\big) \subseteq \sigma_g(M_C).$$

Therefore, the filling in certain of the holes in $\sigma_g(M_C)$ should occur in $\overline{\sigma_p(A^*)} \cup \sigma_p(B)$. This ends the proof. □

Corollary 3.1. *If $\overline{\sigma_p(A^*)} \cup \sigma_p(B)$ has no interior points (if in particular A and B are compact), then*

$$\sigma_g(M_C) = \sigma_g(A) \cup \sigma_g(B) \cup \big(\overline{\sigma_p(A^*)} \cap \sigma_p(B)\big) \quad \text{for every } C \in \mathcal{L}(K, H).$$

Proof. This follows at once from Theorem 3.1. □

Recall that an operator $A \in \mathcal{L}(H)$ is said to be hyponormal if the commutator $A^*A - AA^* \geqslant 0$. The operator A is completely non-normal hyponormal if it has no reducing subspace on which it is normal.

Corollary 3.2. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$. If A^* and B are completely non-normal hyponormal, then*

$$\sigma_g(M_C) = \sigma_g(A) \cup \sigma_g(B), \quad \text{for every } C \in \mathcal{L}(K, H).$$

Proof. This follows from Theorem 3.1 and the general fact that $\sigma_p(T) = \emptyset$ for every completely non-normal hyponormal operator T on a Hilbert space, see for instance [15, p. 70]. □

In the remainder of this section consider operators $C \in \mathcal{L}(K, H)$ for which there is equality $\sigma_g(M_C) = \sigma_g(A) \cup \sigma_g(B)$.
For $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ let $\delta_{A,B}$ denote the generalized derivation defined by $\delta_{A,B}(X) = AX - XB$ $(X \in \mathcal{L}(K, H))$.

Theorem 3.2. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$, and let $C \in N(\delta_{AB}) + R(\delta_{AB})$. Then*

$$\sigma_g(M_C) = \sigma_g(A) \cup \sigma_g(B).$$

Proof. Since $C \in N(\delta_{AB}) + R(\delta_{AB})$, there exist $X, Y \in \mathcal{L}(K, H)$ such that $C = Y + AX - XB$ and $AY = YB$. Hence

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & Y \\ 0 & B \end{bmatrix}.$$

Therefore

$$\sigma_g\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = \sigma_g\left(\begin{bmatrix} A & Y \\ 0 & B \end{bmatrix}\right), \tag{3.5}$$

where the equality in (3.5) follows from the well-known fact that, if T is a bounded operator on a Hilbert space E, then $\sigma_g(STS^{-1}) = \sigma_g(T)$ for every invertible operator S on E.

Write

$$\begin{bmatrix} A & Y \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}.$$

Since the operator $\left[\begin{smallmatrix} 0 & Y \\ 0 & 0 \end{smallmatrix}\right]$ is nilpotent and commutes with $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$, we derive from (3.5) and [17, Theorem 4.8] that

$$\sigma_g(M_C) = \sigma_g\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right).$$

But it is easy to see that

$$\sigma_g\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \sigma_g(A) \cup \sigma_g(B).$$

Hence

$$\sigma_g(M_C) = \sigma_g(A) \cup \sigma_g(B),$$

which completes the proof. □

Remark 3.1. If $\sigma_g(M_C) = \sigma_g(A) \cup \sigma_g(B)$, then $\sigma_g(M_{C+D}) = \sigma_g(A) \cup \sigma_g(B)$ for every operator $D \in N(\delta_{AB})$. Indeed, we have $M_{C+D} = M_C + \left[\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right]$ and the operator $\left[\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right]$ is nilpotent and commutes with M_C. Thus [17, Theorem 4.8] applies.

As a corollary of Theorem 3.2, we have

Corollary 3.3. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ such that $\sigma_r(A) \cap \sigma_l(B) = \emptyset$, then*

$$\sigma_g(M_C) = \sigma_g(A) \cup \sigma_g(B) \quad \textit{for every } C \in \mathcal{L}(K, H).$$

Proof. Since $\delta_{A,B}$ is surjective (see [7]), the result follows from Theorem 3.2. □

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