Journal of Functional Analysis **183**, 451–471 (2001)
doi:10.1006/jfan.2001.3748, available online at http://www.idealibrary.com on IDEAL®

On the Computation of Spectra in Free Probability

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Communicated by A. Connes

Received September 19, 2000; accepted November 24, 2000

We use free probability techniques to compute borders of spectra of non-hermitian operators in finite von Neumann algebras which arise as "free sums" of "simple" operators. To this end, the resolvent is analyzed with the aid of the Haagerup inequality. Concrete examples coming from reduced C^*-algebras of free product groups and leading to systems of polynomial equations illustrate the approach. © 2001 Academic Press

Key Words: spectrum of convolution operator; free probability; free product group; random walk; Haagerup inequality.

1. INTRODUCTION

Computation of spectra of convolution operators on discrete groups was one of the motivations behind the development of free probability. It has some interest for harmonic analysis and analysis of random walks on free products of discrete groups, where under the name of the *transition operator* these operators carry much information about the random walks under consideration. *Free probability* is an abstract framework for harmonic analysis on the free group in the language of non-commutative probability. Let us introduce briefly the terminology of the latter, which is used throughout the paper.

Definition 1.1. A *non-commutative probability space* is a pair $(\mathscr{A}, \varphi)$ of a (complex) algebra $\mathscr{A}$ with unit I and a linear function φ on $\mathscr{A}$ which satisfies $\varphi(I) = 1$. We will usually work with C^*-algebras and faithful states, i.e. positive unital linear functionals. The elements of the algebra are called (*non-commutative*) *random variables*. The distribution of a random variable $a \in \mathscr{A}$ is given by the collection of its *moments* $\varphi(a^n)$, $n = 0, 1, 2, \ldots$. If a is self-adjoint, then this corresponds to a probability measure on the

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0022-1236/01 $35.00

spectrum of a. The distribution of a family of random variables is the collection of their mixed moments, which abstractly can be interpreted as a linear functional on the polynomials on noncommutative variables.

Given a noncommutative probability space $(\mathscr{A}, \varphi)$, the subalgebras $\mathscr{A}_i \subseteq \mathscr{A}$ are called *free independent* (or *free* for short) if

$$\varphi(a_1 a_2 \cdots a_n) = 0 \tag{1.1}$$

whenever $a_j \in \mathscr{A}_{i_j}$ with $\varphi(a_j) = 0$ and $i_j \neq i_{j+1}$ for $j = 1, ..., n-1$.

In the rest of this section we review some of the necessary facts from free probability theory and refer the reader to [25] and [19] for further information.

One of the basic problems is *free convolution*: Given free non-commutative random variables $a, b \in \mathscr{A}$ whose moments are known, compute the moments of their sum $a + b$. From (1.1) one sees immediately by induction that the mixed moments of a and b only depend on the individual distributions of a and b. Therefore the moments of $a + b$ only depend on the individual moments of a and b, and thus the term *free convolution* of distributions is justified. In the case of selfadjoint random variables, once the moments are known, one can proceed to compute the spectra, see Section 2. What to do in the non-selfadjoint case will be the subject of the rest of this paper.

The computational machinery of free convolution was found independently and about the same time by W. Woess [27], D. Cartwright and P. Soardi [6], J. C. McLaughlin [22] in the language of random walks, and in most generality by D. Voiculescu [26]. Using the conventions of the latter, the recipe goes as follows. The Cauchy transform of a random variable a is the function

$$G_a(\zeta) = \varphi((\zeta - a)^{-1}) = \frac{1}{\zeta} \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{\zeta^n} \tag{1.2}$$

which is defined and analytic at least for $|\zeta| > \|a\|$. It has an inverse (under composition) in some neighbourhood of infinity which has the form

$$K_a(z) = G_a^{-1}(z) = \frac{1}{z} (1 + R_a(z)), \tag{1.3}$$

where $R_a(z) = c_1 z + c_2 z^2 + \cdots$ is analytic and one has

$$R_{a+b}(z) = R_a(z) + R_b(z). \tag{1.4}$$

This allows in principle to compute the moments of the sum $a + b$. However, function inversion is difficult and for non-selfadjoint operators knowledge of moments is not sufficient to determine the spectrum. For the latter, a more detailed analysis of the resolvent is necessary.

The paper is organized as follows.

In Section 2 we review the case of self-adjoint operators, where the computation is reduced to the solution of a moment problem.

In Section 3 we use Haagerup inequality to estimate the norm of the resolvent of a sum of free operators.

In Section 4 we look at the sum of two free operators and give a somewhat easier proof for this case.

Finally in Section 5 we present some examples of computations.

2. SELF-ADJOINT OPERATORS

We collect here some well known techniques about computation of spectra of sums of self-adjoint free operators. The main tool to study spectra of selfadjoint operators is the spectral measure and its Cauchy–Stieltjes transform. Recall that to every selfadjoint operator T in a C^*-probability space with faithful state φ one can associate a probability measure μ which is characterized by the property that $\varphi(T^n)=\int t^n\,d\mu(t)$. By faithfulness of the state φ one has $\operatorname{supp}\mu=\sigma(T)$. Thus knowing this measure one knows a fortiori the spectrum of T. In order to compute this measure one has to solve a moment problem. This is usually done via the Cauchy transform (1.2), which can be written in this case

$$G(\zeta)=\int\frac{d\mu(t)}{\zeta-t}.$$

This is an analytic function on $\mathbf{C}\backslash\sigma(T)$ mapping the upper half plane to the lower half plane. The absolute continuous part of the measure can be found by the Stieltjes inversion formula (cf. [3])

$$\frac{d\mu}{dt}(t)=-\frac{1}{\pi}\lim_{\varepsilon\searrow 0}\operatorname{Im}G(t+i\varepsilon).$$

Atoms can be detected by studying the poles of $G(\zeta)$, namely

$$\mu(\{t\})=\lim_{\substack{\zeta\to t\\ \text{non-tangentially}}}(\zeta-t)\,G(\zeta).$$

When studying free sums of self-adjoint operators, by (1.4) one usually is given the inverse $K(z)=\frac{1}{z}(1+\sum R_i(z))$ of the Cauchy transform $G(\zeta)$ and the latter is not accessible directly, but it is still possible to compute the spectrum. We are grateful to U. Haagerup for showing the following lemma to us.

LEMMA 2.1. *Let $\mathcal{A}$ be a C^*-probability space with faithful state φ, and let $L^2(\varphi)$ be the associated L^2-space (the closure of $\mathcal{A}$ under the Hilbert norm $\|X\|_2 = \varphi(X^*X)^{1/2}$). Let T be a normal operator, then $\sigma(T) = \overline{\{\lambda\colon \|(\lambda - T)^{-1}\|_2 = \infty\}}$.*

Proof. Given $\lambda_0 \in \sigma(T)$ and $\varepsilon > 0$ we have to find $\lambda \in \sigma(T)$ such that $|\lambda - \lambda_0| < \varepsilon$ and $\|(\lambda - T)^{-1}\|_2 = \infty$. To this end consider the square Q_0 of diameter 2ε centered at λ_0. The spectral measure of this square is positive, say $\mu_T(Q_0) = \delta > 0$. It follows that one of the four quarters of Q_0 has mass at least $\delta/4$. Let Q_1 be such a quarter. Doing the same argument again, we find a subsquare of Q_1 of mass at least $\delta/16$. Repeating this construction gives rise to a sequence $Q_n \subseteq Q_0$ with $\operatorname{diam}(Q_n) = \varepsilon/2^{n-1}$ and mass $\mu_T(Q_n) \geqslant \delta/4^n$. Let λ be the limit point of the sequence Q_n, then every point $\eta \in Q_n$ is at distance $|\eta - \lambda| \leqslant \varepsilon/2^n$ from λ and thus

$$
\begin{aligned}
\int \frac{d\mu_T(t)}{|\lambda - t|^2} &\geqslant \frac{\mu_T(Q_0)}{\varepsilon^2} + \sum_{n=1}^{\infty} \mu_T(Q_n) \left(\frac{1}{(2^{-n}\varepsilon)^2} - \frac{1}{(2^{1-n}\varepsilon)^2} \right) \\
&\geqslant \frac{\delta}{\varepsilon^2} + \sum_{n=1}^{\infty} \frac{\delta}{4^n} \frac{4^n - 4^{n-1}}{\varepsilon^2} \\
&= \frac{\delta}{\varepsilon^2} \left(1 + \sum_{n=1}^{\infty} \frac{3}{4} \right) \\
&= \infty. \quad \blacksquare
\end{aligned}
$$

It is therefore enough to know the L^2-norm of the inverse, which in the case of a self-adjoint operator is easy to compute. For $\operatorname{Im} \zeta \neq 0$ it is

$$
\begin{aligned}
\|(\zeta - T)^{-1}\|_2^2 &= \int \frac{d\mu_T(t)}{|\zeta - t|^2} \\
&= \int \frac{1}{\bar{\zeta} - \zeta} \left(\frac{1}{\zeta - t} - \frac{1}{\bar{\zeta} - t} \right) d\mu_T(t) \\
&= -\frac{G(\zeta) - G(\bar{\zeta})}{\zeta - \bar{\zeta}}
\end{aligned}
$$

while for $\operatorname{Im} \zeta = 0$ we take the limit to get

$$
\|(\zeta - T)^{-1}\|_2^2 = -G'(\zeta).
$$

In terms of the inverse function $K(z)$ (1.3) this becomes

$$\|(K(z)-T)^{-1}\|_2^2=\begin{cases} -\dfrac{z-\bar{z}}{K(z)-\overline{K(z)}} & \operatorname{Im} z\neq 0 \\ -\dfrac{1}{K'(z)} & \operatorname{Im} z=0. \end{cases}$$

For concrete examples of the use of this see e.g. [1, 14].

3. THE NON-SELFADJOINT CASE

Spectral theory of non-selfadjoint operators is complicated by the absence of the spectral measure[2] and one has to analyze the resolvent directly. We will see that the L^2-norm of the resolvent of free sums is easily computable and Haagerup inequality will provide a replacement of Lemma 2.1.

3.1. *The Free Resolvent*

The resolvent for the sum of convolution operators on free products of discrete groups has been computed by several authors, among the first are [6, 23, 24, and 27]. On the free group itself it was also computed in [2] and [15]. The resolvent of a sum of two free operators has been computed in [21] and very elegantly in [17], see Section 4 below.

Another proof for the free resolvent formula and the R-transform can be based on the following lemma (cf. [14, 20]).

LEMMA 3.1. *Let $S_1, ..., S_N \in B(H)$ be arbitrary operators and assume that the sum of alternating products*

$$\sum_{n=1}^{\infty} \sum_{i_1\neq i_2\neq \cdots \neq i_n} S_{i_1}S_{i_2}\cdots S_{i_n}$$

(*the sum over all products where neighbouring factors are different*) *converges, then it equals*

$$\left(I-\sum_{i=1}^{N} S_i(I+S_i)^{-1}\right)^{-1}.$$

Convergence criteria will be presented below. The following is a simple reformulation of this summation formula.

[2] In some cases *Brown's spectral measure* [5] is computable and can serve a replacement—see [4, 18].

PROPOSITION 3.1. *Let $X_1, ..., X_N$ be operators on Hilbert space. Assume that $I-X_i$ is invertible for every $i \in \{1, ..., N\}$ with inverse $(I-X_i)^{-1} = I+S_i$. Then*

$$\left(I-\sum X_i\right)^{-1} = I + \sum_{n=1}^{\infty} \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} S_{i_1} S_{i_2} \cdots S_{i_n},$$

provided the sum on the right hand side converges.

Sums over alternating indices are well suited for free probability, as the very definition of freeness (1.1) suggests and we have the following theorem.

THEOREM 3.1. *Let $(\mathscr{A}, \varphi)$ be a non-commutative probability space and let $T_i \in \mathscr{A}$ be freely independent random variables with Cauchy transforms $G_i(\zeta) = \varphi((\zeta - T_i)^{-1})$. The Cauchy transform is invertible at infinity with an inverse of the form $K_i(z) = \frac{1}{z}(1+R_i(z))$, i.e. $\varphi((K_i(z)-T_i)^{-1}) = z$ and thus*

$$(K_i(z) - T_i)^{-1} = z(1+S_i(z))$$

with $\varphi(S_i(z)) = 0$. Then the resolvent of $T = \sum T_i$ at $K(z) = \frac{1}{z}(1+\sum R_i(z))$ can be written formally as an infinite sum

$$\left(K(z) - \sum T_i\right)^{-1} = z\left(I + \sum_{n=1}^{\infty} \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} S_{i_1} S_{i_2} \cdots S_{i_n}\right). \tag{3.1}$$

In particular, the expectation of the resolvent is $\varphi((K(z) - \sum T_i)^{-1}) = z$ and (1.4) holds.

If in addition

$$\sum \frac{\|S_i\|_2^2}{1+\|S_i\|_2^2} < 1 \tag{3.2}$$

then the sum (3.1) converges in $L^2(\varphi)$ and its norm is

$$\begin{aligned}
&\left\| \left(K(z) - \sum T_i\right)^{-1} \right\|_2^2 \\
&\quad = |z|^2 \left(1 + \sum_{n=1}^{\infty} \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} \|S_{i_1}\|_2^2 \, \|S_{i_2}\|_2^2 \cdots \|S_{i_n}\|_2^2\right) \\
&\quad = |z|^2 \left(1 - \sum \frac{\|S_i\|_2^2}{1+\|S_i\|_2^2}\right)^{-1} \\
&\quad = |z|^2 \left(1 - \sum \left(\frac{|z|^2}{\|(K_i(z)-T_i)^{-1}\|_2^2}\right)\right)^{-1}.
\end{aligned} \tag{3.3}$$

Moreover, boundedness of $\|(K(z)-\sum T_i)^{-1}\|_2$ *implies boundedness of* $\|(K(z)-\sum T_i)^{-1}\|$ *(operator norm).*

The reader easily verifies the following reformulation in terms of moment generating functions.

COROLLARY 3.1. *Let* a_i, $i=1, ..., n$ *be freely independent random variables in the noncommutative probability space* $(\mathscr{A}, \varphi)$. *Let* $f_i(s)=\frac{1}{s}G_i(\frac{1}{s})=\varphi((1-sa_i)^{-1})$ *the moment generating function and denote*

$$\mathring{a}_i(s)=(1-sa_i)^{-1}-f_i(s)$$

the centered part of the resolvent. Given $\lambda \in \mathbf{C}$, *assume that there exist numbers* $z \neq 0$ *and* s_i *such that* $s_i f_i(s_i)=z$ *for each* $i=1, ..., n$ *and such that* $\lambda=\frac{1}{z}+\sum(\frac{1}{s_i}-\frac{1}{z})$

1. *If*

$$\sum \frac{\|\mathring{a}_i\|_2^2}{1+\|\mathring{a}_i\|_2^2}<1 \tag{3.4}$$

then $\lambda \notin \sigma(\sum a_i)$.

2. *If*

$$\sum \frac{\|\mathring{a}_i\|_2^2}{1+\|\mathring{a}_i\|_2^2}=1$$

and there are λ' *arbitrary close to* λ *for which* (3.4) *holds, then* $\lambda \in \sigma(\sum a_i)$.

Theorem 3.1 provides an easy to check sufficient criterion for the boundedness in $L^2(\varphi)$. By virtue of the following variant of the Haagerup inequality we will see that it is also sufficient for boundedness in operator norm.

PROPOSITION 3.2 [12, Lemma 3.4]. *Let* $(\mathscr{A}, \tau)$ *be a tracial* C^**-probability space and let* $\mathscr{A}_i$ *be free subalgebras with orthonormal bases* $X_i=\{I, x_{i,1}, x_{i,2}, ...\}$. *Let* $\mathring{X}_i=X_i \setminus \{I\}$ *be the centered part and*

$$X=\{x_{1,k_1}x_{2,k_2}\cdots x_{n,k_n} : x_{j,k_j} \in \mathring{X}_{i_j}, i_j \neq i_{j+1}\} \cup \{I\}$$

the free product of these bases. This is an orthonormal basis of the free product $\bigstar \mathscr{A}_i$ *and can be decomposed* $X=\bigcup_n E_n$ *into the subsets* E_n *of words of length* n. *For a finitely supported operator* $a \in \operatorname{span} X$ *denote by* $F_i(a)$ *its* *i-support, i.e., the set of all* $x \in \mathring{X}_i$ *appearing in the words of the expansion of* a. *Then for any* $a \in \operatorname{span} E_n$ *we have*

$$\|a\| \leqslant (2n+1) \max_i \left(\sum_{x \in F_i(a)} \|x\|^2 \right)^{1/2} \|a\|_2.$$

Proof of Theorem 3.1. Formula (3.1) follows from Proposition 3.1. For the L^2-norm, note that the freeness condition implies that for different sets of indices $i_1, i_2, \ldots, i_m$ and $j_1, j_2, \ldots, j_n$ the summands $S_{i_1} S_{i_2} \cdots S_{i_m}$ and $S_{j_1} S_{j_2} \cdots S_{j_n}$ are orthogonal and that $\|S_{i_1} S_{i_2} \cdots S_{i_m}\|_2^2 = \|S_{i_1}\|_2^2 \, \|S_{i_2}\|_2^2 \cdots \|S_{i_m}\|_2^2$. Then formula (3.3) follows from Lemma 3.1. To show boundedness of the resolvent in operator norm, we resort to analytic functions, as in [14, Chapter 2, Lemma 1.6]. Assume that the L^2-norm is bounded, i.e., $\sum \|S_{i_1}\|_2^2 \cdots \|S_{i_n}\|_2^2 < \infty$. Then we can define an analytic function on the open disk $\mathbf{D}$

$$F(\xi) = 1 + \sum_{n=1}^{\infty} \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} \xi^n \|s_{i_1}\|_2^2 \cdots \|S_{i_n}\|_2^2$$

which by the summation formula from Lemma 3.1 equals

$$F(\xi) = \left(1 - \sum \frac{\xi \|S_i\|_2^2}{1 + \xi \|S_i\|_2^2}\right)^{-1}.$$

This is a rational function and by assumption it has no pole on the circle $\{\xi \colon |\xi| = 1\}$. As a rational function it has finitely many singularities and is therefore analytic on some disk of radius $1 + \varepsilon$ with $\varepsilon > 0$. It follows that there exists some constant C independent of n such that the Taylor coefficients satisfy

$$\sum_{i_1 \neq i_2 \neq \cdots \neq i_n} \|S_{i_1}\|_2^2 \cdots \|S_{i_n}\|_2^2 \leqslant C(1+\varepsilon)^{-n}.$$

Now with the help of the Haagerup inequality (Proposition 3.2) we obtain the estimate

$$\begin{aligned}
&\left\| I + \sum_{n=1}^{\infty} \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} S_{i_1} S_{i_1} \cdots S_{i_n} \right\| \\
&\quad \leqslant 1 + \sum_{n=1}^{\infty} (2n+1) \max \|S_i\| \left(\sum_{i_1 \neq i_2 \neq \cdots \neq i_n} \|S_{i_1}\|_2^2 \cdots \|S_{i_n}\|_2^2 \right)^{1/2} \\
&\quad \leqslant 1 + \sum_{n=1}^{\infty} (2n+1) \max \|S_i\| \sqrt{C}\, (1+\varepsilon)^{-n/2} \\
&\quad < \infty. \quad \blacksquare
\end{aligned}$$

The criterion for boundedness of the L^2-norm

$$\sum \frac{\|S_i\|_2^2}{1+\|S_i\|_2^2} < 1 \tag{3.5}$$

can be reformulated in terms of symmetric functions. Recall that the *elementary symmetric functions* in n variables $x_1, x_2, ..., x_n$ are defined as

$$E_k = E_k(x_1, x_2, ..., x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k};$$

their generating function is

$$\prod_{i=1}^{n} (1+tx_i) = 1 + \sum_{k=1}^{n} t^k E_k.$$

The left hand side of (3.5) is a rational symmetric function of $x_i = \|S_i\|_2^2$ and can be expressed in terms of the elementary symmetric functions as follows.

$$\begin{aligned}
\sum \frac{x_i}{1+tx_i} &= \frac{\sum\limits_i x_i \prod\limits_{j\neq i} (1+tx_j)}{\prod (1+tx_i)} \\
&= \frac{\dfrac{d}{dt} \prod (1+tx_j)}{\prod (1+tx_i)} \\
&= \frac{\sum\limits_{k=1}^{n} kE_k t^{k-1}}{1+\sum\limits_{k=1}^{n} E_k t^k}
\end{aligned}$$

at $t=1$ and $x_i = \|S_i\|_2^2$ we get therefore the condition

$$\frac{\sum_{k=1}^{n} kE_k(\|S_1\|_2^2, ..., \|S_n\|_2^2)}{1+\sum_{k=1}^{n} E_k(\|S_1\|_2^2, ..., \|S_n\|_2^2)} < 1$$

which is equivalent to

$$\sum_{k=1}^{n} (k-1)\, E_k(\|S_1\|_2^2, ..., \|S_n\|_2^2) < 1.$$

For $n=2$ this reduces to the simple condition $\|S_1\|_2^2 \, \|S_2\|_2^2 < 1$. We will investigate this case in Section 4 below.

4. CASE OF TWO OPERATORS

In the particular case of the sum of two operators the analysis can be somewhat simplified and there are shorter proofs. We can use the following proposition instead of the Haagerup inequality from Proposition 3.2. It is much easier to prove.

Proposition 4.1 [18, Prop. 4.1]. *Let $(\mathscr{M}, \tau)$ be a non-commutative von Neumann probability space with faithful trace state τ and let $a, b \in \mathscr{M}$ be arbitrary centered $*$-free random variables. Then the spectral radius of their product is*

$$\rho(ab) = \|ab\|_2 = \|a\|_2 \, \|b\|_2.$$

Corollary 4.1. *For a and b as in Proposition 4.1, the circle of radius $\rho(ab) = \|ab\|_2$ is part of the spectrum $\sigma(ab)$.*

Proof. We can rescale a and b and assume that $\|a\|_2 = \|b\|_2 = 1$. We have to show that $1 - tab$ is not invertible whenever $|t| = 1$. When $|t| < 1$, then $1 - tab$ is invertible and $\frac{1}{t}$ is in the complement of the spectrum. By orthogonality, the L^2-norm of the inverse is

$$\|(1 - tab)^{-1}\|_2^2 = \sum_{n=0}^{\infty} |t|^{2n} \, \|a\|_2^{2n} \, \|b\|_2^{2n} = \frac{1}{1 - |t|^2 \, \|a\|_2^2 \, \|b\|_2^2}$$

and this grows unboundedly, as $|t|$ tends to one. Consequently the operator norm of the inverse becomes unbounded as $|t| \to 1$. Since the resolvent is continuous (even analytic) on the complement of the spectrum, any number t of modulus $|t| = 1$ is in the spectrum of ab. ▮

Part (i) of the following proposition is taken from [17].

Proposition 4.2. *Let $a, b \in \mathscr{A}$ and $|s| < \frac{1}{\rho(a)}$, $|t| < \frac{1}{\rho(b)}$, $f(s) = \varphi((1 - sa)^{-1})$, $g(t) = \varphi((1 - tb)^{-1})$. Put $\mathring{a}(s) = (1 - sa)^{-1} - f(s)$, $\mathring{b}(t) = (1 - tb)^{-1} - g(t)$ and assume that*

$$sf(s) = tg(t) \neq 0. \tag{4.1}$$

Set

$$\lambda = \frac{f(s) + g(t) - 1}{sf(s)} = \frac{1}{s} + \frac{1}{t} - \frac{1}{sf(s)}, \tag{4.2}$$

then

(i)

$$\lambda - a - b = \frac{g(t)}{s}\,(1 - sa)\left(1 - \frac{\mathring{a}(s)\,\mathring{b}(t)}{f(s)\,g(t)}\right)(1 - tb).$$

(ii) *If* $\|\mathring{a}(s)\|_2^2\,\|\mathring{b}(t)\|_2^2 < |f(s)|^2\,|g(t)|^2$, *then* $\lambda \in \mathbf{C}\backslash\sigma(a+b)$.

(iii) *If* $\|\mathring{a}(s)\|_2^2\,\|\mathring{b}(t)\|_2^2 = |f(s)|^2\,|g(t)|^2$, *then* $\lambda \in \sigma(a+b)$.

Proof. Part (i) can be derived from Theorem 3.1 and can also be verified directly by expanding the right hand side. Parts (ii) and (iii) follow from Proposition 4.1 and Corollary 4.1 respectively and the fact that $1 - sa$ and $1 - tb$ are invertible. ∎

5. EXAMPLES

The most interesting examples are perhaps those coming from convolution operators on free products of discrete groups. Let us consider here non-selfadjoint convolution operators supported on the generators of free products of cyclic groups. For a survey on spectra of such operators on more general finitely generated discrete groups see [9, 10]. Working with finite groups has the advantage that the involved equations are algebraic and the powerful machinery of algebraic geometry is available to obtain quite explicit results. Let us therefore recall some facts from algebraic geometry. For more background on algebraic equations we refer to the excellent survey [8] and text book [7].

We used `Singular` [16] for the algebraic computations, `GNU octave` [13] for the eigenvalue computations and `Mathematica` [28] for visualizations.

5.1. *Eigenvalue Approach to Polynomial Equations*

The most efficient numerical approach seems to be the *matrix eigenvalue method* following [8, Section 2.4]. Recall that systems of polynomial equations are in one-to-one correspondence with polynomials ideals. Let $f_1, ..., f_m \in \mathbf{C}[x_1, ..., x_n]$ be polynomials and denote by $V = V(f_1, ..., f_m)$ the set of solutions (also called *algebraic set* or *algebraic variety*) of the system of polynomial equations

$$f_1 = 0, ..., f_m = 0. \tag{5.1}$$

On the other hand, on can associate to this system the ideal

$$I = \langle f_1, \ldots, f_m \rangle = \left\{ \sum g_i f_i \colon g_i \in \mathbf{C}[x_1, \ldots, x_n] \right\}$$

it generates. Then it is easy to see that the variety V does not depend on the generators, but only on the ideal I. That is, $V = V(I) = \{ x \in \mathbf{C} : f(x) = 0 \ \forall f \in I \}$ (note however, that different ideals can lead to the same variety V).

There are close relations between the properties of the ideal and its variety. In particular, one can show that the system (5.1) has finitely many solutions (V is *zero-dimensional*) if and only if the quotient algebra $A = \mathbf{C}[x_1, x_2, \ldots, x_n]/I$ is finite dimensional. Then for any polynomial $h \in \mathbf{C}[x_1, \ldots, x_n]$, the eigenvalues of multiplication map

$$M_h \colon A \to A$$

$$[g] \mapsto [hg]$$

coincide with the values of the polynomial $h(\xi)$, evaluated at $\xi \in V$. In particular, choosing $h = x_i$ one can compute the coordinates of the solutions; in fact the eigenvalues and eigenvectors of a single matrix M_h are sufficient to determine all the solutions, see [8, Section 2.4]. Such an approach avoids the propagation of rounding errors which are common to elimination methods.

In order to do concrete computations in the quotient algebra A one needs a normal form for its elements. This can be accomplished using Gröbner bases. Given any (linear) order on the monomials, one can use a generalized Euclid's algorithm to write any $f \in \mathbf{C}[x_1, \ldots, x_n]$ as a sum

$$f = q_1 f_1 + \cdots + q_m f_m + r.$$

The remainder r of this division is a natural candidate for a normal form of f modulo the ideal I, however it may not be unique and there may exist $f \in I$ with $r \neq 0$. Fortunately there always exist generating sets which are nice in this respect. A *Gröbner basis* of the ideal I is a generating set $G = \{g_1, \ldots, g_p\}$ s.t. the remainder r of Euclidean division is unique, i.e., $f \in I$ if and only if $r = 0$. Gröbner bases always exist and can be constructed with Buchberger's algorithm. With such bases the remainder r can serve as a normal form for the elements of A. Another characterizing property of Gröbner bases is the following. For a polynomial $f \in \mathbf{C}[x_1, \ldots, x_n]$ denote $L(f)$ the leading monomial of f (with respect to the chosen monomial order) and let $L(I)$ be the ideal generated by the leading monomials of all elements of I. Then G is a Gröbner basis if and only if $\langle L(G) \rangle = L(I)$. In the case of a zero dimensional ideal, a natural basis of $A=\mathbf{C}[x_1, ..., x_n]/I$ are the equivalence classes

$$B=\{[x^\alpha]: x^\alpha \notin \langle L(G)\rangle\}$$

i.e. the (finitely many) equivalence classes of monomials which are reduced with respect to the Gröbner basis G.

5.2. *Elimination*

Gröbner bases are also needed for elimination. Roughly speaking, elimination of variables corresponds to projection of algebraic varieties. Elimination is done by computing the intersection $I_k=I\cap\mathbf{C}[x_{k+1}, ..., x_n]$. For suitable monomial orders (e.g. lexicographical order), one can use the fact that a Gröbner basis G of I has the property that $G_k=G\cap\mathbf{C}[x_{k+1}, ..., x_n]$ is a Gröbner basis for I_k. However, lexicographical Gröbner bases are expensive to compute and it is usually more efficient to work with other monomial orders, like *degree reverse lexicographical order* and use other methods for elimination.

5.3. *Free Product Groups*

For simplicity, let us consider the sum of two convolution operators u_m+v_n, where u_m, v_n are the generators of $C^*_\lambda(\mathbf{Z}_m * \mathbf{Z}_n)$. The moments of u_m are

$$\tau(u_m^k)=\begin{cases}1 & k=0 \mod m\\ 0 & \text{otherwise}\end{cases}$$

and the inverse of $1-su_m$ is

$$(1-su_m)^{-1}=\frac{1+su_m+\cdots+s^{m-1}u_m^{m-1}}{1-s^m}.$$

Thus for the moment generating function we obtain

$$f(s)=\tau((1-su_m)^{-1})=\frac{1}{1-s^m}.$$

From Proposition 4.2 we infer that if the Eqs. (4.1) and (4.2) satisfied, which in our particular example read

$$\frac{s}{1-s^m}=\frac{t}{1-t^n}$$

$$\lambda=\frac{1}{t}+s^{m-1}$$

and if s and t are such that the L^2-norm condition

$$\frac{|s|^2+|s|^4+\cdots+|s|^{2m-2}}{|1-s^m|^2}\,\frac{|t|^2+|t|^4+\cdots+|t|^{2n-2}}{|1-t^n|^2}<\frac{1}{|1-s^m|^2}\,\frac{1}{|1-t^n|^2}$$

holds, then λ is not in the spectrum of u_m+u_n. In order to find the outer border of the spectrum (that is, the curve where the L^2-norm becomes infinite), we have to solve the equations

$$s(1-t^n)-t(1-s^m)=0 \tag{5.2a}$$

$$\lambda t-1-s^{m-1}t=0 \tag{5.2b}$$

$$(|s|^2+|s|^4+\cdots+|s|^{2m-2})(|t|^2+|t|^4+\cdots+|t|^{2n-2})<1 \tag{5.2c}$$

and look where the inequality (5.2c) becomes an equality. There are two approaches to cope with the non-algebraic part (5.2c). One is to solve the algebraic part (5.2a), (5.2b) first and verify (5.2c) numerically. The alternative approach is to introduce new variables for the real and imaginary parts of $\lambda=x+iy$, $s=s_1+is_2$ and $t=t_1+it_2$, and to separate real and imaginary parts of the equations, which now become purely algebraic, but with more unknowns. The real solutions of the new system correspond to the complex solutions of the original one.

The second approach makes sense if the enlarged system is sufficiently small to allow complete algebraic elimination of the parameters s_i and t_i; for a numerical solution the first approach is preferable.

Example 5.1. ["u_2+v_3"] Let us consider the simplest non-trivial example ($m=2$ and $n=3$) and compute the border of the spectrum of u_2+v_3. If for a given λ the system

$$s(1-t^3)-t(1-s^2)=0 \tag{5.3a}$$

$$\lambda t-1-st=0 \tag{5.3b}$$

$$|s|^2\,(|t|^2+|t|^4)<1 \tag{5.3c}$$

has a solution, then λ is not in the spectrum of u_2+v_3.

In order to get the border of the spectrum, we replace the inequality (5.3c) by an equality and separate real and imaginary part, which leads to the system

$$
\begin{aligned}
-s_1t_1^3+3s_2t_1^2t_2+3s_1t_1t_2^2-s_2t_2^3+s_1^2t_1-s_2^2t_1-2s_1s_2t_2+s_1-t_1&=0\\
-s_2t_1^3-3s_1t_1^2t_2+3s_2t_1t_2^2+s_1t_2^3+2s_1s_2t_1+s_1^2t_2-s_2^2t_2+s_2-t_2&=0\\
-xt_1+s_1t_1+yt_2-s_2t_2+1&=0\\
-yt_1+s_2t_1-xt_2+s_1t_2&=0\\
s_1^2t_1^4+s_2^2t_1^4+2s_1^2t_1^2t_2^2+2s_2^2t_1^2t_2^2+s_1^2t_2^4+s_2^2t_2^4&\\
+s_1^2t_1^2+s_2^2t_1^2+s_1^2t_2^2+s_2^2t_2^2-1&=0.
\end{aligned}
\tag{5.4}
$$

Using degrevlex order there is a Gröbner basis of size 34 for the ideal I generated by the left hand sides of these equations and elimination of s_i and t_i succeeds: the ideal $I\cap \mathbf{C}[x, y]$ is generated by a single polynomial and we get the implicit equation

$$
\begin{aligned}
&x^{16}+8x^{14}y^2+28x^{12}y^4+56x^{10}y^6+70x^8y^8+56x^6y^{10}+28x^4y^{12}\\
&\quad+8x^2y^{14}+y^{16}-6x^{15}+54x^{13}y^2+226x^{11}y^4+238x^9y^6-18x^7y^8\\
&\quad-158x^5y^{10}-74x^3y^{12}-6xy^{14}+5x^{14}-65x^{12}y^2+305x^{10}y^4\\
&\quad+435x^8y^6-226x^6y^8-179x^4y^{10}+107x^2y^{12}+y^{14}+32x^{13}\\
&\quad-400x^{11}y^2-400x^9y^4+480x^7y^6+384x^5y^8-80x^3y^{10}-16xy^{12}\\
&\quad-59x^{12}+74x^{10}y^2-1033x^8y^4-548x^6y^6+547x^4y^8-70x^2y^{10}+y^{12}\\
&\quad-40x^{11}+776x^9y^2-1008x^7y^4+112x^5y^6-104x^3y^8+8xy^{10}\\
&\quad+136x^{10}+48x^8y^2+736x^6y^4-176x^4y^6+24x^2y^8-32x^9\\
&\quad+224x^7y^2-224x^5y^4+32x^3y^6-48x^8-32x^6y^2+16x^4y^4=0.
\end{aligned}
$$

This implicit equation defines the curve shown in Fig. 1. The figure does not show the isolated solutions 2, $\frac{1}{2}\pm(\sqrt{3/2})\,i$, $-\frac{3}{2}\pm(\sqrt{3/2})\,i$, which however are not in the spectrum, as they also come from solutions of (5.3a, 5.3b) which satisfy (5.3c).

The intersection with the x-axis are the solutions at $y=0$:

$$
\begin{aligned}
&x^{16}-6x^{15}+5x^{14}+32x^{13}-59x^{12}-40x^{11}+136x^{10}-32x^9-48x^8\\
&\quad=(x^4+2x^3-3x^2-8x-3)(x-2)^4\,x^8=0.
\end{aligned}
$$

In particular, the spectral radius is the positive solution of

$$x^4+2x^3-3x^2-8x-3=0$$

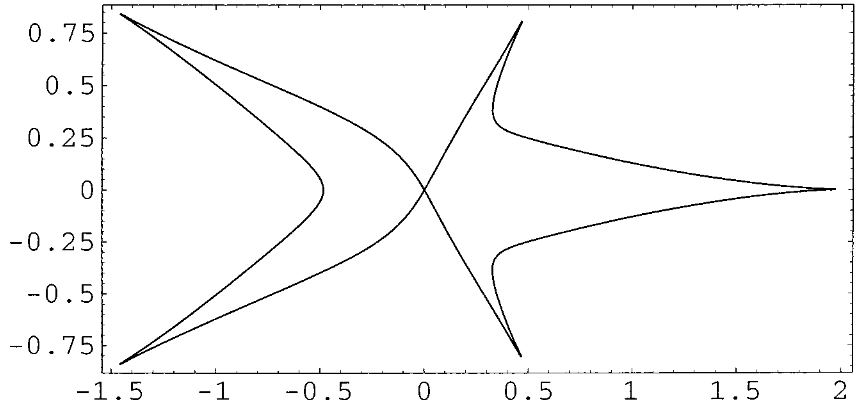

FIG. 1. Spectrum of u_2+u_3.

which is

$$\begin{aligned}\rho(u_2+u_3) &= \frac{1}{6}\Bigg\{-3+\sqrt{3(9+(135-6\sqrt{249})^{1/3}+(135+6\sqrt{249})^{1/3})}\\&\quad+3\Bigg(6-\frac{(135-6\sqrt{249})^{1/3}}{3}-\frac{(135+6\sqrt{249})^{1/3}}{3}\\&\quad+8\sqrt{\frac{3}{9+(135-6\sqrt{249})^{1/3}+(135+6\sqrt{249})^{1/3}}}\Bigg)^{1/3}\Bigg\}\\&\simeq 1.97148. \qquad (5.5)\end{aligned}$$

With the exception of the double point at 0 the apparent singularities of the curve only appear as such and the curve is actually smooth; for example, the curvature at the point of maximal modulus calculated in (5.5) is approximately 166053.0, so the radius of curvature is about $1/166053.0 \simeq 6.02219\times 10^{-6}$.

EXAMPLE 5.2. $[u+u^*+i(v+v^*)]$ Let u and v be freely independent Haar unitaries (e.g., $u=\lambda(g_1)$, $v=\lambda(g_2)$), where g_1 and g_2 are the generators of the free group $\mathbf{F}_2$ on two generators. The distribution of the selfadjoint operator $u+u^*$ is the arcsine distribution with Cauchy transform $G(\zeta)=1/\zeta\sqrt{1-4/\zeta^2}$. The operator $u+u^*+i(v+v^*)$ recently came up as a candidate for a counterexample to the invariant subspace conjecture in the von Neumann setting; however, its "cousin", the *circular element* turned out to have plenty of invariant subspaces [11]. Let us compute the border of its spectrum here. The moment generating function of $u+u^*$ is

$$f(x)=\frac{1}{s}G\left(\frac{1}{s}\right)=\frac{1}{\sqrt{1-4s^2}};$$

from this we easily obtain the moment generating function of $i(v+v^*)$

$$g(t)=f(it)=\frac{1}{\sqrt{1+4t^2}}.$$

Now the L^2-norm of the resolvent of a selfadjoint operator a is

$$\begin{aligned}\|(1-sa)^{-1}\|_2^2&=\frac{1}{|s|^2}\left\|\left(\frac{1}{s}-a\right)^{-1}\right\|_2^2\\&=-\frac{1}{|s|^2}\frac{G\left(\frac{1}{s}\right)-G\left(\frac{1}{\bar{s}}\right)}{\frac{1}{s}-\frac{1}{\bar{s}}}\\&=\begin{cases}\dfrac{\bar{s}f(\bar{s})-sf(s)}{\bar{s}-s} & \operatorname{Im} s\neq 0\\ f(s)+sf'(s) & \operatorname{Im} s=0\end{cases}\end{aligned}$$

and we have the equations

$$\begin{aligned}\lambda sf-f-g+1&=0\\ sf-tg&=0\\ f^2(1-4s^2)-1&=0\\ g^2(1+4t^2)-1&=0.\end{aligned}$$

A Gröbner basis with respect to degrevlex order on (s,t,f,g) has 7 elements and the quotient algebra $\mathbf{C}[s,t,f,g]/I$ is four dimensional with basis $([g],[f],[t],[1])$. Elimination did not succeed in this example and we chose the eigenvalue approach of Section 5.1. With the aid of the Gröbner basis the matrix of multiplication by $[s]$ is computed as

$$M_s = \begin{bmatrix} \frac{4z^2-48}{z^5+24x^3+16z} & \frac{1}{2} & \frac{z^2+16}{z^4+24z^2+16} & \frac{z^6+16z^4-48z^2+128}{z^7+20z^5-80z^3-64z} \\ \frac{z^4-24z^2-48}{z^5+24z^3+16z} & \frac{1}{z} & -\frac{2z^2-8}{z^4+24z^2+16} & -\frac{z^6-16z^4-48z^2-128}{z^7+20z^5-80z^3-64z} \\ -\frac{z^4-24z^2+16}{z^4+24z^2+16} & 0 & \frac{3z^3+12z}{z^4+24z^2+16} & \frac{z^6-20z^4-80z^2+64}{z^6+20z^4-80z^2-64} \\ \frac{z^4+16z^2+48}{z^5+24z^3+16z} & -\frac{1}{z} & -\frac{z^2+12}{z^4+24z^2+16} & \frac{8z^2+32}{z^5+24z^3+16z} \end{bmatrix}.$$

The numerical result is shown in Fig. 2.

EXAMPLE 5.3. As an example with more than two summands, let us consider $u_2+u_3+u_4$. Generally, the equations for a sum $\sum u_k$, where u_k

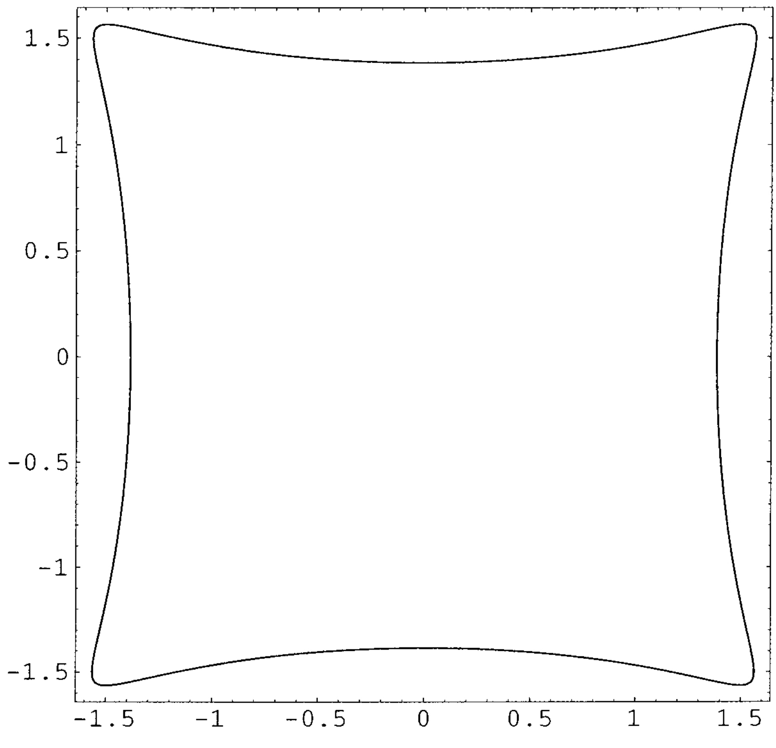

FIG. 2. Spectrum of $u+u^*+i(v+v^*)$.

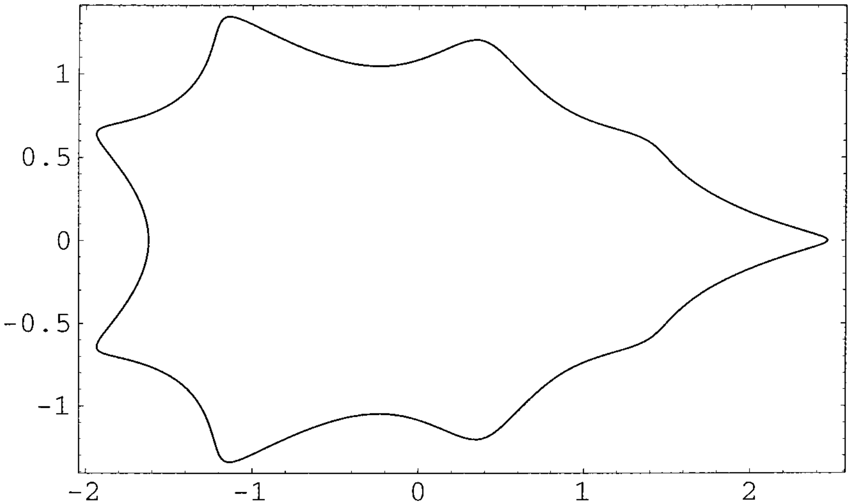

FIG. 3. Spectrum of $u_2+u_3+u_4$.

are free unitaries with $u_k^{n_k}=1$, are as follows. If, for given λ, there is a solution s_j for the system

$$\lambda=\frac{1}{z}+\sum\left(\frac{1}{s_k}-\frac{1}{z}\right)$$

$$\frac{s_k}{1-s_k^{n_k}}=z$$

$$\sum_k \frac{|s_k|^2+|s_k|^4+\cdots+|s_k|^{2n_k-2}}{|1-s_k^{n_k}|^2+|s_k|^2+|s_k|^4+\cdots+|s_k|^{2n_k-2}}<1$$

then λ is not in the spectrum. The example with $n_1=2$, $n_2=3$, $n_3=4$ is shown in Fig. 3.

ACKNOWLEDGMENTS

We are grateful to U. Haagerup and P. Biane for useful discussions. This work was done at Odense Universitet and Université d'Orléans under the EU-network "Non-commutative geometry" ERB FMRX CT960073; the final part was done at the Institut Henri Poincaré in Paris with a grant from the Ostrowski fundation. We express our gratitude to all these institutions.

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