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Existence of a positive solution to a class of fractional differential equations

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ABSTRACT

In this paper, we consider a (continuous) fractional boundary value problem of the form $-D_{0+}^{\nu}y(t) = f(t, y(t)), y^{(i)}(0) = 0, \left[D_{0+}^{\alpha}y(t)\right]_{t=1} = 0$, where $0 \le i \le n-2, 1 \le \alpha \le n-2$, $\nu > 3$ satisfying $n-1 < \nu \le n, n \in \mathbb{N}$, is given, and D_{0+}^{ν} is the standard Riemann–Liouville fractional derivative of order ν. We derive the Green's function for this problem and show that it satisfies certain properties. We then use cone theoretic techniques to deduce a general existence theorem for this problem. Certain of our results improve on recent work in the literature, and we remark on the consequences of this improvement.

1. Introduction

While continuous fractional calculus has been studied for as long as ordinary calculus, the progress of research in this area has only recently greatly increased. Some of the current research in fractional differential equations has extended certain of the classical results for ordinary differential equations (ODEs) to the fractional case. For example, Bai and Lü [1] considered a fractional analogue of the well-known two-point conjugate problem. On the other hand, Xu et al. [2] considered a multi-point fractional boundary value problem (BVP). The recent monograph [3] is interesting for its application of fractional derivatives to the spatial spread of an epidemic, and a recent paper by Lakshmikantham and Vatsala [4] is to be recommended for its development of some of the basic theory of fractional initial value problems (IVPs).

In this paper, we are concerned with a partial extension of a problem considered in a very recent paper by Zhang [5]. Zhang considered the problem

$$D_{0+}^{\alpha}u(t) + q(t)f\left(u, u', \ldots, u^{(n-2)}\right) = 0, \quad 0 < t < 1,\ n-1 < \alpha \le n, \tag{1.1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \tag{1.2}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α, q may be singular at $t = 0$, and f may be singular at $u = 0, u' = 0, \ldots, u^{(n-2)} = 0$. As a consequence of the viewpoint assumed by Zhang, it is never addressed whether or not the Green's function associated to (1.1) and (1.2) satisfies a Harnack-like inequality. As is well known from the existing literature, this is a crucial property when seeking the existence of positive solutions by means of cone theory. One may consult from among a great many papers the classic paper of Erbe and Wang [6] to see explicitly this fact. On the other hand, and perhaps surprisingly, it was first shown by Bai and Lü [1] that the fractional analogue of the two-point conjugate BVP does *not* satisfy this property.

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Here we consider, for $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ continuous, a class of (continuous) fractional boundary value problems (FBVPs) of the form

$$-D_{0+}^{\nu} y(t) = f(t, y(t)), \quad 0 < t < 1,\ n-1 < \nu \le n, \tag{1.3}$$

$$y^{(i)}(0) = 0, \quad 0 \le i \le n-2, \tag{1.4}$$

$$\left[D_{0+}^{\alpha} y(t)\right]_{t=1} = 0, \quad 1 \le \alpha \le n-2, \tag{1.5}$$

where $y^{(i)}$ in boundary condition (1.4) represents the ith (ordinary) derivative of y. Clearly, (1.4) and (1.5) generalize the boundary conditions considered in [5]. We shall assume throughout that $n \in \mathbb{N}$ is given subject to the restriction $n > 3$. Note that this problem is not unrelated to the so-called $(k, n-k)$ conjugate BVPs, which have received much attention in recent years; see, for example, the paper by Davis and Henderson [7] and the references therein. Moreover, in the special case when $\nu = 4$, problem (1.3) has been studied with a variety of boundary conditions and nonlinearities; see, for example, [8,9] and the references therein.

Our primary contribution in this paper is that we improve certain of Zhang's results by showing that the Green's function associated to (1.3)–(1.5) satisfies, among other properties, a Harnack-like inequality. Since by putting $\alpha = n-2$ in (1.5) above we get the boundary conditions given by (1.2), our results affirm that the Green's function associated to (1.1) and (1.2) does satisfy a Harnack-like inequality. Finally, in Section 4, we show that problem (1.3)–(1.5) has a positive solution under standard assumptions on the nonlinearity f.

2. Preliminaries

We first wish to collect some basic lemmas that will be important to us in what follows. These and other related results and their proofs can be found, for example, in [1,10].

Definition 2.1. Let $\nu > 0$ with $\nu \in \mathbb{R}$. Suppose that $y : [a, +\infty) \to \mathbb{R}$. Then the νth Riemann–Liouville fractional integral is defined to be

$$D_a^{-\nu} y(t) := \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1} \mathrm{d}s,$$

whenever the right-hand side is defined. Similarly, with $\nu > 0$ and $\nu \in \mathbb{R}$, we define the νth Riemann–Liouville fractional derivative to be

$$D_a^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}} \mathrm{d}s,$$

where $n \in \mathbb{N}$ is the unique positive integer satisfying $n-1 \le \nu < n$ and $t > a$.

Remark 2.2. In what follows, we shall suppress the explicit dependence of D_a^{ν} on a. It will be clear from the context. In fact, in this paper $a = 0$ throughout.

Lemma 2.3. *Let $\alpha \in \mathbb{R}$. Then $D^n D^{\alpha} y(t) = D^{n+\alpha} y(t)$, for each $n \in \mathbb{N}_0$, where $y(t)$ is assumed to be sufficiently regular so that both sides of the equality are well defined. Moreover, if $\beta \in (-\infty, 0]$ and $\gamma \in [0, +\infty)$, then $D^{\gamma} D^{\beta} y(t) = D^{\gamma+\beta} y(t)$.*

Lemma 2.4. *The general solution to $D^{\nu} y(t) = 0$, where $n-1 < \nu \le n$ and $\nu > 0$, is the function $y(t) = c_1 t^{\nu-1} + c_2 t^{\nu-2} + \cdots + c_n t^{\nu-n}$, where $c_i \in \mathbb{R}$ for each i.*

3. Green's function properties

We begin by deriving the Green's function for the operator $-D^{\nu}$ together with the boundary conditions given in (1.4) and (1.5).

Theorem 3.1. *Let $g \in \mathcal{C}^n([0, 1])$ be given. Then the unique solution to problem $-D^{\nu} y(t) = g(t)$ together with the boundary conditions (1.4) and (1.5) is*

$$y(t) = \int_0^1 G(t, s) g(s) \mathrm{d}s, \tag{3.1}$$

where

$$G(t, s) = \begin{cases} \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \le s \le t \le 1 \\[2ex] \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & 0 \le t \le s \le 1 \end{cases} \tag{3.2}$$

is the Green's function for this problem.

Proof. We know from Lemma 2.4 that the general solution to our problem is

$$y(t) = c_1 t^{\nu-1} + c_2 t^{\nu-2} + \cdots + c_n t^{\nu-n} - D^{-\nu} g(t), \tag{3.3}$$

where we note that $-\nu < 0$. We immediately observe that boundary condition (1.4) implies that $c_2 = \cdots = c_n = 0$. On the other hand, recalling (see [10]) that $D^{\alpha}\left[t^{\nu-1}\right] = \frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)} t^{\nu-\alpha-1}$, we find that boundary condition (1.5) implies that

$$0 = c_1 \cdot \frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)} (1)^{\nu-\alpha-1} - \frac{1}{\Gamma(\nu-\alpha)} \int_0^1 (1-s)^{\nu-\alpha-1} g(s) \mathrm{d}s, \tag{3.4}$$

where we have used Lemma 2.3. But (3.4) may be simplified to get that

$$c_1 = \frac{1}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-\alpha-1} g(s) \mathrm{d}s. \tag{3.5}$$

Finally, putting (3.5) into (3.3) and using the fact that $c_i = 0$ for each $i \geq 2$, we find that the general solution to the problem is

$$y(t) = \frac{t^{\nu-1}}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-\alpha-1} g(s) \mathrm{d}s - \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s) \mathrm{d}s, \tag{3.6}$$

from which it is easy to see that (3.1) holds with $G(t, s)$ defined as in (3.2). □

We now state and prove several properties of the Green's function derived in Theorem 3.1. These properties will be crucial when we prove our existence theorem in Section 4. For convenience in what follows, let us put

$$G_1(t, s) := \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, \quad 0 \leq s \leq t \leq 1 \tag{3.7}$$

and

$$G_2(t, s) := \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, \quad 0 \leq t \leq s \leq 1. \tag{3.8}$$

Theorem 3.2. *Let $G(t, s)$ be as given in the statement of Theorem 3.1. Then we find that:*

1. *$G(t, s)$ is a continuous function on the unit square $[0, 1] \times [0, 1]$;*
2. *$G(t, s) \geq 0$ for each $(t, s) \in [0, 1] \times [0, 1]$; and*
3. *$\max_{t \in [0,1]} G(t, s) = G(1, s)$, for each $s \in [0, 1]$.*

Proof. That property (1) holds is trivial. Indeed, it is clear that each of G_1 and G_2 are continuous on their domains and that $G_1(s, s) = G_2(s, s)$, whence (1) follows.

To prove that (3) is true, we begin by noting that, for each fixed admissible s, we have $\frac{\partial G_2}{\partial t} > 0$, clearly. So, in particular, G_2 is increasing with respect to t. On the other hand, note that $\frac{\partial G_1}{\partial t} = \frac{(\nu-1)t^{\nu-2}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-2}(\nu-1)}{\Gamma(\nu)}$. Put $G^*(t, s) := \frac{\partial G_1}{\partial t}$ for each admissible (t, s). Evidently, $G^*(t, s) > 0$ on its domain if and only if $t^{\nu-2}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-2} \geq 0$. But that this latter inequality holds follows from the observation that

$$t^{\nu-2}(1-s)^{\nu-\alpha-1} - t^{\nu-2}\left(1 - \frac{s}{t}\right)^{\nu-2} \geq t^{\nu-2}\left[(1-s)^{\nu-\alpha-1} - (1-s)^{\nu-2}\right] \geq 0, \tag{3.9}$$

where to get the first inequality we use the fact that $0 \leq t \leq 1$, whereas to get the final inequality we use the fact that $\nu - \alpha - 1 \leq \nu - 2$, for each admissible α. Thus, as (3.9) holds, we deduce that $G^*(t, s) \geq 0$ on its domain. In particular, then, G_1 is increasing on its domain, too. Consequently, (3) holds.

Finally, to prove that (2) holds, let us note that, for each fixed and admissible s, we have that $G(0, s) = 0$. So, as (3) implies that $G(t, s)$ is increasing in t for each s, we find at once that $G(t, s) \geq 0$ at each admissible pair (t, s). Thus, (2) holds, and the proof is complete. □

Theorem 3.3. *Let $G(t, s)$ be as given in the statement of Theorem 3.1. Then there exists a constant $\gamma \in (0, 1)$ such that*

$$\min_{t \in \left[\frac{1}{2}, 1\right]} G(t, s) \geq \gamma \max_{t \in [0,1]} G(t, s) = \gamma G(1, s). \tag{3.10}$$

Proof. Notice that Theorem 3.2 implies that

$$\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) = \begin{cases} G_1\left(\frac{1}{2},s\right), & s\in\left(0,\frac{1}{2}\right] \\ G_2\left(\frac{1}{2},s\right), & s\in\left[\frac{1}{2},1\right) \end{cases}$$
$$= \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{\Gamma(\nu)}, & s\in\left(0,\frac{1}{2}\right] \\ \dfrac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & s\in\left[\frac{1}{2},1\right). \end{cases} \tag{3.11}$$

Moreover, observe that

$$\lim_{s\to 0^+}\frac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}\left[1-(1-s)^{\alpha}\right]} \overset{L'H}{=} \lim_{s\to 0^+}\frac{-(\nu-\alpha-1)\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-2}+\left(\frac{1}{2}-s\right)^{\nu-2}(\nu-1)}{-(1-s)^{\nu-\alpha-2}(\nu-\alpha-1)+(1-s)^{\nu-2}(\nu-1)}, \tag{3.12}$$

whence $\lim_{s\to 0^+}\frac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}[1-(1-s)^{\alpha}]} = \frac{1}{\alpha}\left(\frac{1}{2}\right)^{\nu-1}(\nu+\alpha-1) > 0$. It is also the case that for $0 < s \leq \frac{1}{2}$

$$\frac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}\left[1-(1-s)^{\alpha}\right]} \geq \frac{\left(\frac{1}{2}\right)^{\nu-1}\left(1-\frac{1}{2}\right)^{\nu-\alpha-1}-\left(\frac{1}{2}-\frac{1}{2}\right)^{\nu-1}}{\left(1-\frac{1}{2}\right)^{\nu-\alpha-1}\left[1-\left(1-\frac{1}{2}\right)^{\alpha}\right]}$$
$$= \frac{\left(\frac{1}{2}\right)^{\nu-\alpha-1}}{2^{\alpha}-1}. \tag{3.13}$$

Finally, observe that, for $\frac{1}{2} \leq s \leq 1$, we find that

$$\frac{\left(\frac{1}{2}\right)^{\nu-1}}{1-(1-s)^{\alpha}} \geq \left(\frac{1}{2}\right)^{\nu-1}. \tag{3.14}$$

Now, define $\widetilde{\gamma}(s) : [0,1] \to (0,1)$ by

$$\widetilde{\gamma}(s) := \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{\nu-1}(1-s)^{\nu-\alpha-1}-\left(\frac{1}{2}-s\right)^{\nu-1}}{(1-s)^{\nu-\alpha-1}\left[1-(1-s)^{\alpha}\right]}, & 0 < s \leq \frac{1}{2} \\ \dfrac{\left(\frac{1}{2}\right)^{\nu-1}}{1-(1-s)^{\alpha}}, & \frac{1}{2} \leq s \leq 1, \end{cases} \tag{3.15}$$

where $\widetilde{\gamma}(0) := \lim_{s\to 0^+}\widetilde{\gamma}(s)$; note that $\widetilde{\gamma}(0) > 0$ by (3.12). Put

$$\gamma := \min\left\{\frac{\left(\frac{1}{2}\right)^{\nu-\alpha-1}}{2^{\alpha}-1}, \left(\frac{1}{2}\right)^{\nu-1}\right\}, \tag{3.16}$$

where, evidently, $0 < \gamma < 1$. Then, from (3.11)–(3.16), we find that

$$\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) = \widetilde{\gamma}(s)\max_{t\in[0,1]} G(t,s) \geq \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(1,s),$$

as claimed. □

Remark 3.4. Note that, in great contrast to, say, [1], where their γ is a function of s and satisfies $\lim_{s\to 0^+}\gamma(s) = 0$, in our Theorem 3.3 above we are able to take our γ to be a *strictly positive constant*. We believe this to be a very important difference between our results and other work (e.g., [1,2]) on (continuous) FBVPs. Moreover, as pointed out in Section 1, this improves and builds on certain of the results given in [5].

Remark 3.5. It can be shown that, for $0 \leq \alpha < 1$ in (1.5), we find that γ can no longer be taken as a constant and that, moreover, $\lim_{s\to 0^+}\gamma(s) = 0$ in this case.

4. Existence theorem

In this section we deduce the existence of a positive solution to problem (1.3)–(1.5) by assuming some growth conditions on $f(t, y)$. To accomplish this we appeal to cone theoretic techniques. In particular, we shall require the following well-known result due to Krasnosel'skiĭ; see, for example, [11].

Lemma 4.1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T : \mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathcal{K}$ is a completely continuous operator. If either*

1. *$\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$, or*
2. *$\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$,*

then T has at least one fixed point in $\mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

Now, notice that y solves (1.3)–(1.5) if and only if y is a fixed point of the operator $(Ty)(t) := \int_0^1 G(t, s)f(s, y(s))\mathrm{d}s$, where G is the Green's function derived in this paper and $T : \mathcal{B} \to \mathcal{B}$, where $\mathcal{B}$ is the Banach space $\mathcal{C}^n([0, 1])$ equipped with the usual supremum norm, $\|\cdot\|$. Let us also make the following declarations, which will be used in what follows. In particular, we put $\eta := \left[\int_0^1 G(1, s)\mathrm{d}s\right]^{-1}$ and $\lambda := \left[\int_{\frac{1}{2}}^1 G\left(\frac{3}{4}, s\right)\mathrm{d}s\right]^{-1}$, where these are clearly well defined. In addition, let us also introduce two conditions on the behavior of f that will be useful in what follows.

C1: There exists a number $r_1 > 0$ such that $f(t, y) \leq \eta r_1$ whenever $0 \leq y \leq r_1$.
C2: There exists a number $r_2 > 0$ such that $f(t, y) \geq \lambda r_2$ whenever $\gamma r_2 \leq y \leq r_2$, where γ is the constant deduced in Theorem 3.3.

We now can prove the following existence result.

Theorem 4.2. *Suppose that there are numbers $r_2 > r_1 > 0$ such that conditions* (C1) *and* (C2) *hold at r_1 and r_2, respectively. Suppose also that $f(t, y) \geq 0$ and continuous. Then the FBVP* (1.3)–(1.5) *has at least one positive solution.*

Proof. Consider the set $\mathcal{K} := \left\{y \in \mathcal{B} : y(t) \geq 0 \text{ and } \min_{t\in\left[\frac{1}{2},1\right]} y(t) \geq \gamma\|y\|\right\}$, which is a cone with $\mathcal{K} \subseteq \mathcal{B}$. Observe that $T : \mathcal{K} \to \mathcal{K}$, for we observe that

$$\min_{t\in\left[\frac{1}{2},1\right]} (Ty)(t) \geq \gamma \int_0^1 G(1, s)f(s, y(s))\mathrm{d}s = \gamma \max_{t\in[0,1]} \int_0^1 G(t, s)f(s, y(s))\mathrm{d}s = \gamma\|Ty\|,$$

whence $Ty \in \mathcal{K}$, as claimed. Moreover, a standard argument, which we omit, shows that T is equicontinuous and bounded, so the Arzela–Ascoli theorem may be applied to deduce the complete continuity of T.

Now, put $\Omega_1 := \{y \in \mathcal{K} : \|y\| < r_1\}$. Note that, for $y \in \partial\Omega_1$, we have that $\|y\| = r_1$, so condition (C1) holds for all $y \in \partial\Omega_1$. So, for $y \in \mathcal{K} \cap \partial\Omega_1$, we find that

$$\|Ty\| = \max_{t\in[0,1]} \int_0^1 G(t, s)f(s, y(s))\mathrm{d}s \leq \eta r_1 \int_0^1 G(1, s)\mathrm{d}s, \tag{4.1}$$

whence (4.1) implies that $\|Ty\| \leq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_1$. Thus we get that the operator T is a cone compression on $\mathcal{K} \cap \partial\Omega_1$. On the other hand, put $\Omega_2 := \{y \in \mathcal{K} : \|y\| < r_2\}$. Note that, for $y \in \partial\Omega_2$, we have that $\|y\| = r_2$, so condition (C2) holds for all $y \in \partial\Omega_2$. So, for $y \in \mathcal{K} \cap \partial\Omega_2$, we find that

$$\begin{aligned}(Ty)\left(\frac{3}{4}\right) = \int_0^1 G\left(\frac{3}{4}, s\right)f(s, y(s))\mathrm{d}s &\geq \int_{\frac{1}{2}}^1 G\left(\frac{3}{4}, s\right)f(s, y(s))\mathrm{d}s \\ &\geq r_2\lambda \int_{\frac{1}{2}}^1 G\left(\frac{3}{4}, s\right)\mathrm{d}s,\end{aligned} \tag{4.2}$$

whence (4.2) implies that $\|Ty\| \geq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_2$. Thus we get that the operator T is a cone expansion on $\mathcal{K} \cap \partial\Omega_2$. So, it follows by Lemma 4.1 that the operator T has a fixed point. But this means that (1.3)–(1.5) has a positive solution, say y_0, with $r_1 \leq \|y_0\| \leq r_2$, as claimed. And this completes the proof. □

Remark 4.3. The existence result given in Theorem 4.2 is representative rather than definitive. Since γ may be taken to be a constant here, we could argue the existence of a positive solution to (1.3)–(1.5) by assuming that $f(t, y)$ is either sublinear or superlinear in an appropriate way, thus paralleling the assumptions introduced by Erbe and Wang [6]. Again, this relies crucially upon knowing that γ is constant. Indeed, because Theorem 3.3 holds, problem (1.3)–(1.5) admits positive solutions under much weaker conditions on $f(t, y)$ than, say, the two-point conjugate problem studied in [1], at least so far as the author is aware at present.

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