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Large time behavior to the system of incompressible non-Newtonian fluids in $\mathbf{R}^2$ ☆

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Abstract

In this paper, the authors study the large time behavior for the weak solutions to a class system of the incompressible non-Newtonian fluids in $\mathbf{R}^2$. It is proved that the weak solutions decay in L^2 norm at $(1+t)^{-1/2}$ and the estimate for the decay rate is sharp in the sense that it coincides with the decay rate of a solution to the heat equation.

Keywords: Decay rate; Weak solutions; Non-Newtonian fluids

1. Introduction

In this paper we consider the optimal decay rate of global solutions to the Cauchy problem for 2-dimensional non-Newtonian fluids

$$u_t - \Delta u + (u \cdot \nabla)u - \nabla \cdot \left(\left|e(u)\right|^{p-2} e(u)\right) + \nabla \pi = 0, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$u(x,0) = u_0. \tag{1.3}$$

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Here, $u = u(x,t) = (u_1, u_2)$ and π denote the unknown velocity vector and pressure of the fluids as the point $(x,t) \in \mathbf{R}^2 \times (0,\infty)$, while u_0 is the given initial velocity vector field. For simplicity, we assume that the external force has a scalar potential and it is included into the pressure gradient. $e(u) = (e_{ij}(u))$ is the symmetric deformations velocity tensor whose components are given

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and $|e(u)| = (e_{ij}(u)e_{ij}(u))^{1/2}$. The non-Newtonian model above includes shear thinning $(p < 2)$ and shear thickening $(p > 2)$. When $p = 2$, as we know, the system turns out to be the famous Navier–Stokes equations (see [4,10]).

There is an extensive literature on the solutions of the incompressible non-Newtonian fluids. Ladyzhenskaya [5] and Lions [6] first discussed the existence and uniqueness for weak solutions of the sort model, Bellout et al. [3] studied the existence of Young measure solutions of non-Newtonian bipolar fluids. Bae and Choe [1,2] obtained the existence, uniqueness, regularity and decay rates of solutions to the monopolar fluids. Pokorny [8] investigated the Cauchy problem for both monopolar and bipolar fluids. As for the large time behavior of solutions, Nečasová and Penel [7] recently studied uniformly algebraic decay in $\mathbf{R}^3$ and logarithmic decay in $\mathbf{R}^2$ by using Fourier splitting methods which were developed by Schonbek [9] and Wiegner [11], for example, they got the decay of the L^2 norm is $(\ln(t+e))^{-m}$, $m \in N$, in $\mathbf{R}^2$, assuming $L^1 \cap L^2$ integrability of the initial data u_0.

The purpose of this paper is also to investigate the L^2 decay rates for the weak solutions of this incompressible non-Newtonian fluid in $\mathbf{R}^2$. By improving the Fourier splitting method [12], we will show that if the initial data $u_0 \in L^2 \cap L^1$, then the solutions decay in L^2 norm at $(1+t)^{-1/2}$, the decay rates are optimal in the sense that they coincide with the decay rates of the solution to the heat equation.

The remains of this paper are organized as follows. In Section 2, we introduce the mathematical preliminaries and state our main results. In Section 3, we prove some auxiliary lemma and finally we apply this result to prove the main theorems in Sections 4 and 5.

2. Statement of the main results

Throughout this paper, we denote by $L^q(\mathbf{R}^2)$ the usual Lebesgue space with the norm $\|\cdot\|_q$. In particular $\|\cdot\| = \|\cdot\|_2$. $W^{m,p}(\mathbf{R}^2)$ is the usual Sobolev space with the norm $\|\cdot\|_{m,p}$. For a Banach space X, $X^2 = \{u = (u_1,u_2)\colon u_i \in X\}$, $L^q(0,T;X)$ is the space of all measurable functions $u : (0,T) \mapsto X$ with the norm $\|u\|^q_{L^q(0,T;X)} = \int_0^T \|u\|^q_X \, dt$. When $q = \infty$, $\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|u\|_X$. $C(I;X)$ denotes the space of continuous functions from the interval I to X. We denote by $\mathbf{H} = \{\varphi \in L^2(\mathbf{R}^2)^2,\ \operatorname{div}\varphi = 0\}$. The Fourier transformation of a function f is denoted by

$$F[f](\xi) = \hat{f}(\xi) = \int_{\mathbf{R}^2} f(x) \cdot e^{-ix\cdot\xi} \, dx, \quad \xi = (\xi_1, \xi_2) \in \mathbf{R}^2.$$

Denote by $C > 0$ any constant appearing in our paper, which may only depend on the initial data u_0, but never depends on $t > 0$.

By a weak solution of the Cauchy problem (1.1)–(1.3) we mean a function

$$u \in L^2\big((0,T);(W^{1,2})^2\big) \cap L^\infty\big([0,T);\mathbf{H}\big) \cap L^p\big((0,T);(W^{1,p})^2\big) \quad (\forall T > 0),$$

which satisfies

$$\int_{\mathbf{R}^2} u(t)\cdot\varphi(t)\,dx - \int_0^t\int_{\mathbf{R}^2} u\cdot\frac{\partial\varphi}{\partial t}\,dx\,dt + \int_0^t\int_{\mathbf{R}^2} u_j\frac{\partial u_i}{\partial x_j}\varphi_i\,dx\,dt + \int_0^t\int_{\mathbf{R}^2}\big(1+|e(u)|^{p-2}\big)e_{ij}(u)\cdot e_{ij}(\varphi)\,dx\,dt = \int_{\mathbf{R}^2} u_0\cdot\varphi(0)\,dx \tag{2.1}$$

a.e. $t \in (0,T)$ for every $\varphi \in C^1([0,T),\mathbf{H}) \cap C([0,T), W^{1,2}(\mathbf{R}^2)^2 \cap W^{1,p}(\mathbf{R}^2)^2)$.

We remark that the weak solution $u(x,t)$ of (1.1)–(1.3) satisfies the following energy inequality:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2}|u|^2\,dx + \int_{\mathbf{R}^2}|\nabla u|^2\,dx + \int_{\mathbf{R}^2}|\nabla u|^p\,dx \leqslant 0. \tag{2.2}$$

Our main theorems are the following

Theorem 2.1. *Let $u_0 \in \mathbf{H} \cap L^1$. Then we have the decay estimate for the solutions of the problem* (1.1)–(1.3),

$$\big\|u(t)\big\| \leqslant C(1+t)^{-1/2} \quad \forall t > 0. \tag{2.3}$$

The above result is sharp in the following sense.

Theorem 2.2. *Let $u(t)$ be a weak solution of* (1.1)–(1.3), *and $v(t)$ the solution of the linear heat equation with the same initial value $u_0 \in \mathbf{H} \cap L^1$. Then*

$$\big\|u(t) - v(t)\big\| \leqslant C(1+t)^{-3/4} \quad \textit{for } t > 1. \tag{2.4}$$

3. Some auxiliary lemmas

In this section, we prove an auxiliary lemma, which will be employed in the proof of the main theorems.

Lemma 3.1 (Gronwall inequality). *Let $f(t), g(t), h(t)$ be nonnegative continuous functions and satisfy the inequality*

$$g(t) \leqslant f(t) + \int_0^t g(s)h(s)\,ds \quad \forall t > 0,$$

where $f'(t) \geqslant 0$. *Then*

$$g(t) \leqslant f(t) \exp\left(\int_0^t h(s)\,ds\right) \quad \forall t > 0. \tag{3.1}$$

Lemma 3.2. *Let* $u_0 \in \mathbf{H} \cap L^1$, *and* u *be a weak solution of* (1.1)–(1.3). *Then*

$$\sup_{0 \leqslant t \leqslant \infty} \|u(t)\| \leqslant \|u_0\|, \tag{3.2}$$

$$\left|\hat{u}(\xi, t)\right| \leqslant C + C|\xi| + C|\xi| \int_0^t \|u(s)\|^2\,ds. \tag{3.3}$$

Proof. From the energy inequality (2.2), it is easy to get (3.2).

Applying the Fourier transformation to (1.1), we have

$$\hat{u}_t + |\xi|^2 \hat{u} = F\left[-(u \cdot \nabla)u - \nabla\pi + \nabla \cdot \left(\left|e(u)\right|^{p-2} e(u)\right)\right] =: G(\xi, t). \tag{3.4}$$

Now we estimate $G(\xi, t)$.

$$\left|F\left[(u \cdot \nabla)u\right]\right| = \left|F\left[\operatorname{div}(u \otimes u)\right]\right| \leqslant \sum_{i,j} \int_{\mathbf{R}^n} |u_i u_j||\xi_j|\,dx \leqslant |\xi|\|u\|^2, \tag{3.5}$$

$$\left|F\left[\nabla \cdot \left(\left|e(u)\right|^{p-2} e(u)\right)\right]\right| \leqslant |\xi|\left|F\left[\left|e(u)\right|^{p-2} e(u)\right]\right| \leqslant |\xi|\|\nabla u\|_{p-1}^{p-1}. \tag{3.6}$$

Taking divergence of (1.1), we get

$$\Delta\pi = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\left(-u_i u_j + \left|e(u)\right|^{p-2} e_{ij}(u)\right).$$

The Fourier transformation yields

$$|\xi|^2 F[\pi] = \sum_{i,j} \xi_i \xi_j F\left[-u_i u_j + \left|e(u)\right|^{p-2} e_{ij}(u)\right],$$

and thus

$$\left|F[\nabla\pi]\right| \leqslant |\xi|\left|F[\pi]\right| \leqslant |\xi|\|u\|^2 + |\xi|\|\nabla u\|_{p-1}^{p-1}. \tag{3.7}$$

Inserting (3.5)–(3.7) into $G(\xi, t)$ we have

$$\left|G(\xi, t)\right| \leqslant C|\xi|\|u\|^2 + C|\xi|\|\nabla u\|_{p-1}^{p-1}. \tag{3.8}$$

From (3.4),

$$\hat{u}(\xi, t) = \hat{u}_0(\xi) e^{-|\xi|^2 t} + \int_0^t G(\xi, t) e^{-|\xi|^2 (t-s)}\,ds. \tag{3.9}$$

Since the weak solution $u(x,t)$ of system (1.1)–(1.3) satisfies $\nabla u \in L^2(0,\infty;L^2) \cap L^p(0,\infty;L^p)$ for $p \geqslant 3$, by using interpolation technology, we have

$$\int_0^\infty \|\nabla u\|_{p-1}^{p-1}\,ds \leqslant C, \tag{3.10}$$

where C is independent of time. Hence (3.9) imply, noting that $|\hat{u}_0(\xi)| \leqslant \|u_0\|_1 \leqslant C$ if $u_0 \in L^1$,

$$\begin{aligned}\big|\hat{u}(\xi,t)\big| &\leqslant \|u_0\|_1 + C|\xi| \int_0^t \big(\big\|u(s)\big\|^2 + \|\nabla u\|_{p-1}^{p-1}\big)\,ds \\ &\leqslant C + C|\xi| + C|\xi| \int_0^t \big\|u(s)\big\|^2\,ds.\end{aligned}$$

The proof of this lemma is completed. □

4. Proof of Theorem 2.1

From the energy inequality (2.2), it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2} |u|^2\,dx + \int_{\mathbf{R}^2} |\nabla u|^2\,dx \leqslant 0. \tag{4.1}$$

Applying Plancherel's theorem to (4.1) yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2} \big|\hat{u}(\xi,t)\big|^2\,d\xi + \int_{\mathbf{R}^2} |\xi|^2\big|\hat{u}(\xi,t)\big|^2\,d\xi \leqslant 0. \tag{4.2}$$

Let $f(t)$ be a continuous function of t with $f(0)=1$, $f(t)>0$ and $f'(t)>0$, then we have

$$\frac{d}{dt}\left(f(t)\int_{\mathbf{R}^2} \big|\hat{u}(\xi,t)\big|^2\,d\xi\right) + 2f(t)\int_{\mathbf{R}^2} |\xi|^2\big|\hat{u}(\xi,t)\big|^2\,d\xi \leqslant f'(t)\int_{\mathbf{R}^2} \big|\hat{u}(\xi,t)\big|^2\,d\xi.$$

Let $B(t) = \{\xi \in \mathbf{R}^2\colon\ 2f(t)|\xi|^2 \leqslant f'(t)\}$; then

$$\begin{aligned}&2f(t)\int_{\mathbf{R}^2} |\xi|^2\big|\hat{u}(\xi,t)\big|^2\,d\xi \\ &\quad = 2f(t)\int_{B(t)^c} |\xi|^2\big|\hat{u}(\xi,t)\big|^2\,d\xi + 2f(t)\int_{B(t)} |\xi|^2\big|\hat{u}(\xi,t)\big|^2\,d\xi\end{aligned}$$

$$\geqslant 2f(t)\int_{B(t)^c}|\xi|^2\big|\hat{u}(\xi,t)\big|^2\,d\xi$$

$$\geqslant f'(t)\int_{\mathbf{R}^2}\big|\hat{u}(\xi,t)\big|^2\,d\xi - f'(t)\int_{B(t)}\big|\hat{u}(\xi,t)\big|^2\,d\xi.$$

Therefore we get

$$\frac{d}{dt}\left(f(t)\int_{\mathbf{R}^2}\big|\hat{u}(\xi,t)\big|^2\,d\xi\right)\leqslant f'(t)\int_{B(t)}\big|\hat{u}(\xi,t)\big|^2\,d\xi.$$

Integrating in time yields

$$f(t)\int_{\mathbf{R}^2}\big|\hat{u}(\xi,t)\big|^2\,d\xi\leqslant\int_{\mathbf{R}^2}|\hat{u}_0|^2\,d\xi + C\int_0^t f'(s)\int_{B(s)}\big|\hat{u}(\xi,s)\big|^2\,d\xi\,ds. \tag{4.3}$$

Let $A^2 = f'(t)/(2f(t))$ and apply to Lemma 3.2. Then (4.3) implies

$$\begin{aligned} f(t)\int_{\mathbf{R}^2}\big|\hat{u}(\xi,t)\big|^2\,d\xi \\ &\leqslant C + C\int_0^t f'(s)\int_0^{2\pi}d\theta\int_0^A\left(1+\rho+\rho\int_0^s\big\|u(\tau)\big\|^2\,d\tau\right)^2\rho\,d\rho\,ds \\ &\leqslant C + C\int_0^t f'(s)\left\{\frac{f'(s)}{2f(s)}+\left(\frac{f'(s)}{2f(s)}\right)^2+\left(\frac{f'(s)}{2f(s)}\right)^2\left(\int_0^s\big\|u(\tau)\big\|^2\,d\tau\right)^2\right\}ds. \end{aligned} \tag{4.4}$$

Now we prove the algebraic decay rate of Theorem 2.1 by two steps. Firstly, applying (3.2) to (4.4) implies

$$f(t)\int_{\mathbf{R}^2}\big|\hat{u}(\xi,t)\big|^2\,d\xi\leqslant C + C\int_0^t f'(s)\left\{\frac{f'(s)}{2f(s)}+\left(\frac{f'(s)}{2f(s)}\right)^2+s^2\left(\frac{f'(s)}{2f(s)}\right)^2\right\}ds.$$

Choose $f(t) = (\ln(e+t))^3$, then

$$f'(t) = \frac{3(\ln(e+t))^2}{e+t}, \qquad \frac{f'(t)}{2f(t)} = \frac{3}{(e+t)\ln(e+t)},$$

by direct calculation we have

$$\big\|\hat{u}(t)\big\| = \big\|u(t)\big\| \leqslant C\big(\ln(e+t)\big)^{-1}. \tag{4.5}$$

Next we choose $f(t) = (1+t)^2$ again, from (4.4) and using Hölder inequality we get

$$(1+t)^2 \int_{\mathbf{R}^2} \left|\hat{u}(\xi,t)\right|^2 d\xi \leqslant C + C(1+t) + C \int_0^t (1+s)^{-1} \left(\int_0^s \left\|u(\tau)\right\|^2 d\tau \right)^2 ds$$
$$\leqslant C(1+t) + C \int_0^t \int_0^s \left\|u(\tau)\right\|^4 d\tau\, ds$$
$$\leqslant C(1+t) + C(1+t) \int_0^t \left\|u(s)\right\|^4 ds.$$

Applying (4.5) to the above inequality we infer that

$$(1+t) \int_{\mathbf{R}^2} \left|\hat{u}(\xi,t)\right|^2 d\xi \leqslant C + C \int_0^t \left\|u(s)\right\|^2 (1+s)\big((1+s)^{-1}\big(\ln(e+s)\big)^{-2}\big)\, ds.$$

Let

$$g(t) = (1+t) \int_{\mathbf{R}^2} \left|\hat{u}(\xi,t)\right|^2 d\xi = (1+t) \int_{\mathbf{R}^2} \left|u(x,t)\right|^2 dx,$$
$$f(t) = C, \qquad h(t) = (1+t)^{-1}\big(\ln(e+t)\big)^{-2},$$

it is easy to see that

$$\int_0^\infty h(t)\, dt < \infty,$$

so, applying Lemma 3.1 we have

$$g(t) \leqslant C \exp\left(\int_0^\infty h(t)\, dt \right) \leqslant C,$$

and thus

$$\left\|u(t)\right\| \leqslant C(1+t)^{-1/2}. \tag{4.6}$$

The proof of Theorem 2.1 is completed. □

5. Proof of Theorem 2.2

Denote $w(t) = u(t) - v(t)$ the difference of $u(t)$ and $v(t)$, where $u(t)$ is the solution of the system (1.1)–(1.3) and $v(t) = e^{t\Delta}u_0$ is the solution of the heat system with the same initial data u_0 as $u(t)$. Thus $w(t)$ satisfies the equation

$$w_t - \Delta w = -(u \cdot \nabla)u + \nabla \cdot \big(\left|e(u)\right|^{p-2} e(u)\big) - \nabla\pi \tag{5.1}$$

with a homogeneous initial condition $w(x,0) = 0$. Since $u_0 \in \mathbf{H}$, v is divergence free, and so is w. Similar to Lemma 3.2, we have

Lemma 5.1. *Let w be the solution of the above problem. Then*

$$\left|\hat{w}(\xi,t)\right| \leqslant C|\xi|\ln(1+t)+C|\xi|. \tag{5.2}$$

Proof. Applying Fourier transform to (5.1) we have

$$\hat{w}_t+|\xi|^2\hat{w}=F\left[-(u\cdot\nabla)u+\nabla\cdot\left(\left|e(u)\right|^{p-2}e(u)\right)-\nabla\pi\right]=G(\xi,t).$$

By the formula of constant variation and according to (3.8) and (3.10), we have

$$\begin{aligned}\left|\hat{w}(\xi,t)\right| &\leqslant \int_0^t \left|G(\xi,t)e^{-|\xi|^2t}\right|ds \leqslant C|\xi|\int_0^t \left\|u(s)\right\|^2 ds + C|\xi|\int_0^\infty \|\nabla u\|_{p-1}^{p-1}ds\\ &\leqslant C|\xi|\ln(1+t)+C|\xi|. \qquad \square\end{aligned} \tag{5.3}$$

Now we prove Theorem 2.2. Multiplying by w and integrating over $\mathbf{R}^2$ yield

$$\frac{d}{dt}\|w\|^2+2\|\nabla w\|^2=2B(u,v,w), \tag{5.4}$$

where

$$\begin{aligned}B(u,v,w)&=-\left((u\cdot\nabla)u,w\right)-\left(\left|e(u)\right|^{p-1}e(u),\nabla w\right)\\ &=-\left((u\cdot\nabla)(w+v),w\right)-\left(\left|e(u)\right|^{p-1}e(u),e(w)\right)\\ &=-\left((u\cdot\nabla)v,w\right)-\left(\left|e(u)\right|^{p-1}e(u),e(u-v)\right)\\ &\quad \left(\text{by } \left((u\cdot\nabla)w,w\right)=0\right)\\ &=\left((u\cdot\nabla)w,v\right)+\left(\left|e(u)\right|^{p-1}e(u),e(v)\right)-\left(\left|e(u)\right|^{p-1}e(u),e(u)\right).\end{aligned} \tag{5.5}$$

As we know, $v(x,t)$ satisfies the following decay estimate for $1\leqslant q\leqslant\infty$, $k\in\mathbf{N}$, and $t>0$:

$$\left\|D^k v(t)\right\|_q \leqslant t^{-\frac{n}{2}(1-\frac{1}{q})-\frac{k}{2}}\|u_0\|_1. \tag{5.6}$$

Now we estimate the first two terms of $B(u,v,w)$,

$$\begin{aligned}&\left|2\left(\left((u\cdot\nabla)w,v\right)-\left(\left|e(u)\right|^{p-1}e(u),e(v)\right)\right)\right|\\ &\quad\leqslant 2\|\nabla w\|\,\|u\|\,\|v\|_\infty+2\|\nabla v\|_\infty\,\|\nabla u\|_{p-1}^{p-1}\\ &\quad\leqslant \|\nabla w\|^2+\|u\|^2\|v\|_\infty^2+2\|\nabla v\|_\infty\,\|\nabla u\|_{p-1}^{p-1}\\ &\quad\leqslant \|\nabla w\|^2+C(1+t)^{-1}t^{-2}+Ct^{-3/2}\|\nabla u\|_{p-1}^{p-1}\\ &\quad\leqslant \|\nabla w\|^2+C(1+t)^{-3}+C(1+t)^{-3/2}\|\nabla u\|_{p-1}^{p-1} \quad \text{for } t>1.\end{aligned} \tag{5.7}$$

Since $(|e(u)|^{p-1}e(u),e(u))>0$, thus (5.4)–(5.7) imply

$$\frac{d}{dt}\left\|w(t)\right\|^2+\left\|\nabla w(t)\right\|^2 \leqslant C(1+t)^{-3}+C(1+t)^{-3/2}\|\nabla u\|_{p-1}^{p-1}.$$

Let $f(t)$ and $B(t)$ be the same as in the proof of Theorem 2.1. Similar to (4.3), we have

$$f(t)\|\hat{w}(\xi,t)\|^2 \leqslant \int_0^t f'(s)\,ds \int_{B(s)} |\hat{w}(\xi,s)|^2\,d\xi + C\int_0^t f(s)\big((1+s)^{-3} + (1+s)^{-3/2}\|\nabla u(s)\|_{p-1}^{p-1}\big)\,ds. \tag{5.8}$$

Noting that $B(t) = \{\xi \in \mathbf{R}^2:\ 2f(t)|\xi|^2 \leqslant f'(t)\}$, letting $f(t) = (1+t)^4$ and applying (5.2) to the first term of the right-hand side to get

$$\int_0^t f'(s)\,ds \int_{B(s)} |\hat{w}(\xi,s)|^2\,d\xi \leqslant C(1+t)^2\big(\ln(1+t)\big)^2 + C(1+t)^2,$$

and to the second term of the right-hand side to get

$$\int_0^t f(s)\big((1+s)^{-3} + (1+s)^{-3/2}\|\nabla u(s)\|_{p-1}^{p-1}\big)\,ds \leqslant C(1+t)^2 + C(1+t)^{5/2}\int_0^t \|\nabla u(s)\|_{p-1}^{p-1}\,ds \leqslant C(1+t)^{5/2}.$$

Hence, inserting the above estimates to (5.8),

$$\|\hat{w}(t)\|^2 = \|w(t)\|^2 \leqslant C(1+t)^{-3/2} \quad \text{for } t > 1. \tag{5.9}$$

So the proof of Theorem 2.2 is completed. □

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