Journal of Combinatorial Theory, Series A 119 (2012) 1132–1157

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Distributive lattices, affine semigroups, and branching rules of the classical groups

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ARTICLE INFO

Article history:
Received 29 July 2010
Available online 24 February 2012

Keywords:
Classical groups
Branching rules
Distributive lattices
Toric deformations

ABSTRACT

We study algebras encoding stable range branching rules for the pairs of complex classical groups of the same type in the context of toric degenerations of spherical varieties. By lifting affine semigroup algebras constructed from combinatorial data of branching multiplicities, we obtain algebras having highest weight vectors in multiplicity spaces as their standard monomial type bases. In particular, we identify a family of distributive lattices and their associated Hibi algebras which can uniformly describe the stable range branching algebras for all the pairs we consider.

1. Introduction

1.1. Let us consider a pair of complex algebraic groups G and H with embedding $H \subset G$ and their completely reducible representations V_G and V_H. If V_H is irreducible, then a description of the multiplicity of V_H in V_G, regarded as a representation of H by restriction, is called a *branching rule for* (G, H). By Schur's lemma, the branching multiplicity is equal to the dimension of the space $\mathrm{Hom}_H(V_H, V_G)$, which we will call the *multiplicity space*.

1.2. In this paper, we shall consider branching rules of the polynomial representations for the following pairs (G, H) of complex classical groups: (GL_m, GL_n), (Sp_{2m}, Sp_{2n}), (SO_p, SO_q). Our goal is to study branching rules for (G, H) collectively in the context of toric degenerations of spherical varieties and to obtain an explicit description of the multiplicity space $\mathrm{Hom}_H(V_H^\mu, V_G^\lambda)$ when the length $\ell(\lambda)$ of highest weight λ for G satisfies the following *stable range condition*:

(1) $\ell(\lambda) \leqslant m$ for (GL_m, GL_n);
(2) $\ell(\lambda) \leqslant n$ for (Sp_{2m}, Sp_{2n}), (SO_{2m}, SO_{2n+1}), (SO_{2m+1}, SO_{2n+1});

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0097-3165/$ – see front matter
doi:10.1016/j.jcta.2012.02.007

(3) $\ell(\lambda) < n$ for (SO_{2m}, SO_{2n}), (SO_{2m+1}, SO_{2n}).

We shall construct an algebra whose graded components are spanned by the highest weight vectors of irreducible representations of H appearing in each irreducible representation of G.

1.3. To give a slightly more detailed overview, let us consider the ring $\mathcal{F}_G$ of regular functions over G/U_G where U_G is a maximal unipotent subgroup of G. This ring is called the *flag algebra for* G, because it can be realized as the multi-homogeneous coordinate ring of the flag variety. As a G-module, the flag algebra $\mathcal{F}_G$ contains exactly one copy of every irreducible representation of G [25,26], and in this context the author studied polynomial models for $\mathcal{F}_G$ and their flat degenerations [18,19].

By highest weight theory, the U_H-invariant subspace of V_G^λ consists of the highest weight vectors of irreducible representations of H appearing in V_G^λ. Therefore, the U_H-invariant subalgebra of $\mathcal{F}_G$ leads us to study the branching rules for (G, H) collectively:

$$\begin{aligned}\mathcal{F}_G^{U_H} &= \sum_{\lambda\in\widehat{G}} \left(V_G^\lambda\right)^{U_H}\\ &= \sum_{\lambda\in\widehat{G}} \sum_{\mu\in\widehat{H}} m\left(V_H^\mu, V_G^\lambda\right)\left(V_H^\mu\right)^{U_H}\end{aligned} \tag{1.3.1}$$

where $m(V_H^\mu, V_G^\lambda)$ is the multiplicity of V_H^μ in V_G^λ.

Moreover, we can impose a graded structure on $\mathcal{F}_G^{U_H}$ so that its graded components correspond to the multiplicity spaces:

$$m\left(V_H^\mu, V_G^\lambda\right)\left(V_H^\mu\right)^{U_H} \cong \mathrm{Hom}_H\left(V_H^\mu, V_G^\lambda\right)$$

for $(\lambda, \mu) \in \widehat{G} \times \widehat{H}$. In this sense, we may call $\mathcal{F}_G^{U_H}$ the *branching algebra for* (G, H). This algebra was introduced by Zelobenko. See [33] and [34].

1.4. In this paper, we describe isomorphisms between stable range branching algebras for the pairs of the symplectic or orthogonal groups and suitable stable range branching algebras for the pairs of the general linear groups. Starting from combinatorial data of stable range branching multiplicities, we shall construct an affine semigroup and its semigroup algebra graded by the pairs of highest weights for the classical groups G and H listed in Section 1.2. This algebra can be realized as a Hibi algebra over a distributive lattice. Then, by using toric deformation techniques, we lift the Hibi algebra to construct a polynomial model of the branching algebra for (G, H). We study its finite presentation and standard monomial type basis. It turns out that there is a particular type of distributive lattices whose Hibi algebras can uniformly describe stable range branching algebras for all the pairs (G, H) we consider. These algebraic statements will enrich well-known combinatorial coincidences between the branching pairs listed in Section 1.2.

Recently, Howe and his collaborators studied branching algebras for classical symmetric pairs, especially their toric degenerations and expressions of branching multiplicities in terms of Littlewood–Richardson coefficients [16,17]. In the cases this paper concerns, using known combinatorics of branching rules, we can explicitly describe the multiplicity spaces and their degenerations. More specifically, we show that the stable range branching algebras are deformations of semigroup algebras of generalized semistandard tableaux or equivalently Gelfand–Tsetlin patterns, and therefore provide a precise connection between the multiplicity space and the combinatorial objects which count its dimension.

We remark that this Hibi algebra structure in branching problems has interesting counterparts in tensor product decomposition problems, which can be explained by reciprocity properties between branchings and tensor products in representation theory. For this direction, we refer readers to [15, 14,20].

1.5. This paper is arranged as follows: In Section 2, we develop the combinatorial tools we will use. In Section 3, we study the branching algebra for (GL_m, GL_n) and its toric degeneration. In Section 4 and Section 5, we study the distributive lattices and affine semigroups associated with the branching rules for (Sp_{2m}, Sp_{2n}) and (SO_p, SO_q), and construct the corresponding stable range branching algebras.

2. Combinatorics of branchings

This section is to prepare us the combinatorial ingredients we will use to construct stable range branching algebras.

2.1. The *Gelfand–Tsetlin* (GT) *poset* for GL_m is the poset

$$\Gamma_m = \left\{x_j^{(i)}\colon\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant i\right\}$$

satisfying $x_j^{(i+1)} \geqslant x_j^{(i)} \geqslant x_{j+1}^{(i+1)}$ for all i and j. The elements of Γ_m can be listed in a reversed triangular array so that $x_j^{(i)}$ are weakly decreasing from left to right along diagonals, as GT patterns are originally drawn [6]. Counting from bottom to top, we will call $x^{(r)} = (x_1^{(r)}, x_2^{(r)}, \ldots, x_r^{(r)})$ the *r-th row* of Γ_m.

Definition 2.1.1.

(1) For $m > n$, the GT poset for (GL_m, GL_n) is the following subposet of Γ_m:

$$\Gamma_m^n = \left\{x_j^{(i)} \in \Gamma_m\colon\ n \leqslant i \leqslant m\right\}.$$

(2) In Γ_m^n, for $m \geqslant k$ we define the GT poset of length k as

$$\Gamma_{m,k}^n = \left\{x_j^{(i)} \in \Gamma_m^n\colon\ j \leqslant k\right\}.$$

For example, $\Gamma_{6,4}^3$ can be drawn as

$$\begin{array}{ccccccccccc}
x_1^{(6)} & & x_2^{(6)} & & x_3^{(6)} & & x_4^{(6)} & & & & \\
& x_1^{(5)} & & x_2^{(5)} & & x_3^{(5)} & & x_4^{(5)} & & & \\
& & x_1^{(4)} & & x_2^{(4)} & & x_3^{(4)} & & x_4^{(4)} & & \\
& & & x_1^{(3)} & & x_2^{(3)} & & x_3^{(3)} & & &
\end{array} \tag{2.1.1}$$

2.2. Next, let us consider the set $\mathcal{L}_m$ of all non-empty subsets of $\{1, 2, \ldots, m\}$. We shall write

$$I = [i_1, \ldots, i_a]$$

for the subset consisting of elements $i_1, \ldots, i_a$ ordered so that $1 \leqslant i_1 < \cdots < i_a \leqslant m$. The *length* $|I| = a$ of I is the number of elements in I.

The following partial order $\preccurlyeq$, called the *tableau order*, can be imposed on $\mathcal{L}_m$: for two elements I and J of $\mathcal{L}_m$, we say $I \preccurlyeq J$, if $|I| \geqslant |J|$ and the c-th smallest element in I is less than or equal to the c-th smallest element in J for $1 \leqslant c \leqslant |J|$. Then, $\mathcal{L}_m$ with $\preccurlyeq$ forms a lattice whose meet $\wedge$ and join $\vee$ are, for $I = [i_1, \ldots, i_a]$ and $J = [j_1, \ldots, j_b]$ with $a \leqslant b$,

$$I \wedge J = \left[\min(i_1, j_1), \ldots, \min(i_a, j_a), i_{a+1}, \ldots, i_b\right]$$
$$I \vee J = \left[\max(i_1, j_1), \ldots, \max(i_a, j_a)\right].$$

Moreover, $\mathcal{L}_m$ is a distributive lattice, i.e., for all $x, y, z \in \mathcal{L}_m$, the following identity holds: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. It is straightforward to check that the following subposets are also distributive lattices.

Definition 2.2.1.

(1) For $m > n$, the distributive lattice $\mathcal{L}_m^n$ for (GL_m, GL_n) is the subposet of $\mathcal{L}_m$ consisting of the following elements:

$$[1, 2, \ldots, r-1, r, a_1, a_2, \ldots, a_s],$$
$$[1, 2, \ldots, r-1, r],$$
$$[a_1, a_2, \ldots, a_s]$$

where $r \leqslant n$ and $n+1 \leqslant a_1 < \cdots < a_s \leqslant m$.

(2) For $k \leqslant m$, we let $\mathcal{L}_{m,k}^n$ denote the subposet of $\mathcal{L}_m^n$ consisting of elements of length not greater than k:

$$\mathcal{L}_{m,k}^n = \{ I \in \mathcal{L}_m^n \colon |I| \leqslant k \}.$$

2.3. Recall that a subset S of a poset $(P, <)$ is called *order increasing*, if the following condition holds: for $x, y \in P$, if $x \in S$ and $x < y$, then $y \in S$. The poset structure of $\mathcal{L}_{m,k}^n$ can be read from the GT poset $\Gamma_{m,k}^n$ of length k. For this, let us impose a partial order on the set of order increasing subsets of $\Gamma_{m,k}^n$ as follows. For two order increasing subsets A and B of $\Gamma_{m,k}^n$, we say A is bigger than B, if $A \subseteq B$ as sets. Note that here we use the reverse inclusion order on sets, because we use order increasing sets instead of order decreasing sets.

Proposition 2.3.1. *There is an order isomorphism between $\mathcal{L}_{m,k}^n$ and the set of order increasing subsets of $\Gamma_{m,k}^n$.*

This is an easy computation similar to [18, Theorem 3.8]. For each $I \in \mathcal{L}_{m,k}^n$, we define the corresponding order increasing subset A_I of $\Gamma_{m,k}^n$ as

$$A_I = \bigcup_{n \leqslant i \leqslant m} \{ x_1^{(i)}, x_2^{(i)} \cdots, x_{s_i}^{(i)} \} \tag{2.3.1}$$

where s_i is the number of entries in I less than or equal to i. For example, the subset of $\Gamma_{6,4}^3$ given in (2.1.1) corresponding to $I = [1, 4, 6] \in \mathcal{L}_{6,4}^3$ is

$$\begin{array}{ccccccc}
x_1^{(6)} & & x_2^{(6)} & & x_3^{(6)} & & \\
& x_1^{(5)} & & x_2^{(5)} & & & \\
& & x_1^{(4)} & & x_2^{(4)} & & \\
& & & x_1^{(3)} & & &
\end{array}$$

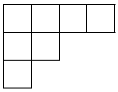
Then, it is straightforward to check that this correspondence gives an order isomorphism. In fact, this proposition gives an example of Birkhoff's representation theorem or the fundamental theorem for finite distributive lattices [29, Theorem 3.4.1]. See [18, §3.3] for further details.

For $k \leqslant n$ and $d \geqslant 0$, we can identify $\Gamma_{m,k}^n$ with $\Gamma_{m+d,k}^{n+d}$ by shifting the i-th row $x^{(i)}$ up to the $(i+d)$-th row $x^{(i+d)}$ for $n \leqslant i \leqslant m$, and then the above proposition gives

Corollary 2.3.2. *For $k \leqslant n$ and $d \geqslant 0$, there is an order isomorphism between distributive lattices*

$$\mathcal{L}_{m,k}^n \cong \mathcal{L}_{m+d,k}^{n+d}.$$

2.4. A *shape* or *Young diagram* is a left-justified array of boxes with weakly decreasing row lengths. We identify a shape with its sequence of row lengths $D = (r_1, r_2, \ldots)$. The following example shows the shape $D = (4, 2, 1)$:



If l is maximal with $r_l \neq 0$, then we call l the *length* of D and write $\ell(D) = l$. If we flip a shape D over its main diagonal that slants down from upper left to lower right, then we obtain its *conjugate* D^t. With the previous example, we have $\ell(D) = 3$ and $D^t = (4, 2, 1)^t = (3, 2, 1, 1)$. For $F = (f_1, f_2, \ldots)$ and $D = (d_1, d_2, \ldots)$, if $f_r \geqslant d_r$ for all r, then we write $F \supseteq D$ and let F/D denote the *skew shape* having F as its outer shape and D as its inner shape.

2.5. Consider a multiset $\{I_1, \ldots, I_s\} \subset \mathcal{L}_m$ with $|I_c| = l_c$ for each c. A concatenation t of its elements is called a *tableau*, if they are arranged so that $l_c \geqslant l_{c+1}$ for all c. The *shape* $sh(\mathrm{t})$ of t is the Young diagram $(l_1, \ldots, l_s)^t$ and the *length* $\ell(\mathrm{t})$ of t is the length of its shape. If $\{I_1, \ldots, I_s\}$ is taken from the subposet $\mathcal{L}_m^n$, then we shall specify the outer and inner shapes of t.

Definition 2.5.1. A standard tableau t for (GL_m, GL_n) is a multiple chain

$$\mathrm{t} = (I_1 \preccurlyeq \cdots \preccurlyeq I_s)$$

in $\mathcal{L}_m^n$. The shape $sh_n(\mathrm{t})$ of t is F/D where

$$F = \big(|I_1|, \ldots, |I_s|\big)^t \quad \text{and} \quad D = (d_1, \ldots, d_n)$$

and d_r is the number of r's in t for $1 \leqslant r \leqslant n$.

For example, the multiple chain $[1, 2, 3, 6] \preccurlyeq [1, 2, 5, 6] \preccurlyeq [1, 2, 6] \preccurlyeq [1, 4] \preccurlyeq [5] \preccurlyeq [5]$ in $\mathcal{L}_{6,4}^3$ forms a standard tableau for (GL_6, GL_3) of shape $(6, 4, 3, 2)/(4, 3, 1)$:

1	1	1	1	5	5
2	2	2	4		
3	5	6			
6	6				

(2.5.1)

Recall that a tableau is called semistandard, if its entries weakly increase along each row and strictly increase along each column (e.g., [30, p. 309]). Then, after erasing $r \leqslant 3$, we can identified the standard tableau (2.5.1) with the following skew semistandard tableau

				5	5
			4		
	5	6			
6	6				

2.6. The following set of pairs of Young diagrams will be used frequently: for $a \geqslant b$,

$$\Lambda_{a,b} = \big\{(F, D)\colon\ \ell(F) \leqslant a,\ \ell(D) \leqslant b,\ F \supseteq D\big\}.$$

We note that if $(F, D) \in \Lambda_{a,b}$, then $\ell(D) \leqslant \min(\ell(F), b)$. This is because $F \supseteq D$ implies $\ell(F) \geqslant \ell(D)$.

2.7. Let $\mathcal{T}_m^n(F, D)$ denote the set of all standard tableaux for (GL_m, GL_n) whose shapes are F/D. For each k with $n \leqslant k \leqslant m$, we consider the following disjoint union over $\Lambda_{k,n}$

$$\mathcal{T}_{m,k}^n = \bigcup_{(F,D)\in\Lambda_{k,n}} \mathcal{T}_m^n(F, D).$$

As illustrated by the example in Section 2.5, if we identify the elements of $\mathcal{L}_m^n$ with single-column tableaux, then our definition of standard tableaux for (GL_m, GL_n) of shape F/D agrees with the usual definition of skew semistandard Young tableaux of shape F/D with entries from $\{n+1, \dots, m\}$.

By setting tableaux in the context of a finite distributive lattice (Definition 2.5.1), we can exploit an additional structure: Proposition 2.3.1 leads us to study $\mathcal{L}_{m,k}^n$ in terms of the order increasing subsets of $\Gamma_{m,k}^n$, and the order increasing subsets of $\Gamma_{m,k}^n$ give rise to the order preserving maps from $\Gamma_{m,k}^n$ to $\{0, 1\}$. More generally,

Definition 2.7.1. A GT pattern for (GL_m, GL_n) is an order preserving map from the GT poset Γ_m^n for (GL_m, GL_n) to the set of non-negative integers:

$$\mathsf{p} : \Gamma_m^n \to \mathbb{Z}_{\geqslant 0}.$$

The r-th row of p is $(\mathsf{p}(x_1^{(r)}), \dots, \mathsf{p}(x_r^{(r)}))$ for $n \leqslant r \leqslant m$. The *type* of p is F/D where F and D are its m-th row and the n-th row respectively.

Note that if $\ell(F) \leqslant k$, then the support of every GT pattern p of type F/D lies in the GT poset $\Gamma_{m,k}^n$ of length k. Therefore, we have GT patterns defined on $\Gamma_{m,k}^n$

$$\mathsf{p} : \Gamma_{m,k}^n \to \mathbb{Z}_{\geqslant 0}.$$

Let $\mathcal{P}_m^n(F, D)$ denote the set of all GT patterns for (GL_m, GL_n) whose type is F/D. Then for each k with $n \leqslant k \leqslant m$, we consider the following disjoint union over $\Lambda_{k,n}$:

$$\mathcal{P}_{m,k}^n = \bigcup_{(F,D)\in\Lambda_{k,n}} \mathcal{P}_m^n(F, D). \tag{2.7.1}$$

2.8. Since the sum of two order preserving maps is an order preserving map, $\mathcal{P}_{m,k}^n$ is a semigroup with function addition as its multiplication, or more precisely a monoid with the zero function as its identity. We further note that $\mathcal{P}_{m,k}^n$ is generated by the order preserving maps from $\Gamma_{m,k}^n$ to $\{0, 1\}$. Then, by identifying each GT pattern p with $(\mathsf{p}(x_j^{(i)})) \in \mathbb{Z}^N$ where N is the number of elements in $\Gamma_{m,k}^n$, we see that $\mathcal{P}_{m,k}^n$ can be understood as an *affine semigroup*, i.e., a finitely generated semigroup which is isomorphic to a subsemigroup of $\mathbb{Z}^N$ containing 0 for some N [3].

This semigroup structure on GT patterns provides a simple bijection between $\mathcal{T}_{m,k}^n$ and $\mathcal{P}_{m,k}^n$.

Proposition 2.8.1. *For each $(F, D) \in \Lambda_{m,n}$, there is a bijection between $\mathcal{T}_m^n(F, D)$ and $\mathcal{P}_m^n(F, D)$.*

Proof. The bijection in Proposition 2.3.1 provides the bijection between $\mathcal{L}_m^n$ and the set of characteristic functions of order increasing subsets of Γ_m^n. This bijection can be extended to multiple chains in $\mathcal{L}_m^n$ as follows. Let $\mathsf{t} = (I_1 \preccurlyeq \cdots \preccurlyeq I_c)$ be a multiple chain in $\mathcal{L}_m^n$ and p_{I_r} be the characteristic function of the order increasing set A_{I_r} corresponding to I_r given in (2.3.1) for each r. Then we can consider the following correspondence:

$$\mathsf{t} = (I_1 \preccurlyeq \cdots \preccurlyeq I_c) \mapsto \mathsf{p}_{\mathsf{t}} = \sum_{r=1}^{c} \mathsf{p}_{I_r}. \tag{2.8.1}$$

Since the order preserving characteristic functions on Γ_m^n generate $\mathcal{P}_m^n$, this correspondence gives a bijection between $\mathcal{T}_m^n(F, D)$ and $\mathcal{P}_m^n(F, D)$. For further details, see [18, §3.2]. □

2.9. We remark that by identifying GT patterns p with their images $(\mathsf{p}(x_j^{(i)}))$, our definition is equivalent to the usual definition of GT patterns. The correspondence given in the above proposition is the same as the well-known conversion procedure between the set of semistandard tableaux and the set of GT patterns (e.g., [9, §8.1.2]), which is usually explained by successive applications of the Pieri's rules.

For example, a pattern $\mathsf{p} \in \mathcal{P}_{6,4}^3$ can be visualized by listing its value at $x_j^{(i)} \in \Gamma_{6,4}^3$

$$\begin{array}{ccccccc} 3 & & 3 & & 3 & & 1 \\ & 3 & & 3 & & 2 & & 0 \\ & & 3 & & 2 & & 1 & & 0 \\ & & & 2 & & 2 & & 1 \end{array} \tag{2.9.1}$$

Then it is the sum of the GT patterns

$$\begin{array}{ccccccc} 1 & & 1 & & 1 & & 1 \\ & 1 & & 1 & & 1 & & 0 \\ & & 1 & & 1 & & 1 & & 0 \\ & & & 1 & & 1 & & 1 \end{array} + \begin{array}{ccccccc} 1 & & 1 & & 1 & & 0 \\ & 1 & & 1 & & 1 & & 0 \\ & & 1 & & 1 & & 0 & & 0 \\ & & & 1 & & 1 & & 0 \end{array} + \begin{array}{ccccccc} 1 & & 1 & & 1 & & 0 \\ & 1 & & 1 & & 0 & & 0 \\ & & 1 & & 0 & & 0 & & 0 \\ & & & 0 & & 0 & & 0 \end{array}$$

corresponding to the elements $[1,2,3,6] \preccurlyeq [1,2,5] \preccurlyeq [4,5,6]$ of $\mathcal{L}_{6,4}^3$. This multiple chain can be identified with the following standard tableau in $\mathcal{T}_{6,4}^3$

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 4 \\ \hline 2 & 2 & 5 \\ \hline 3 & 5 & 6 \\ \hline 6 \\ \cline{1-1} \end{array} \tag{2.9.2}$$

of shape $(3,3,3,1)/(2,2,1)$. Note that to (2.9.2), we can apply the usual conversion procedure (e.g., [9, §8.1.2]) to obtain its corresponding pattern—by successively striking out the boxes with 6, 5, and 4 in the tableau (2.9.2), we obtain each row of the pattern (2.9.1).

2.10. Now we study an algebra constructed from the distributive lattice $\mathcal{L}_{m,k}^n$. In fact, from any distributive lattice one can associate an algebra.

Definition 2.10.1. (See [10].) Let L be a finite distributive lattice. The *Hibi algebra* $\mathcal{H}(L)$ over L is the quotient ring of the polynomial ring $\mathbb{C}[z_\gamma\colon \gamma \in L]$ by the ideal generated by $z_\alpha z_\beta - z_{\alpha\wedge\beta} z_{\alpha\vee\beta}$ for all incomparable pairs (α,β) of L:

$$\mathcal{H}(L) = \mathbb{C}[z_\gamma\colon \gamma \in L] / \langle z_\alpha z_\beta - z_{\alpha\wedge\beta} z_{\alpha\vee\beta} \rangle.$$

Let us consider the Hibi algebra over $\mathcal{L}_{m,k}^n$

$$\mathcal{H}_{m,k}^n = \mathcal{H}\big(\mathcal{L}_{m,k}^n\big).$$

We shall identify the monomials $\prod_r z_{I_r}$ in $\mathcal{H}_{m,k}^n$ with the tableaux consisting of elements $I_r \in \mathcal{L}_{m,k}^n$. For example, the above tableau (2.9.2) will be used to denote the monomial

$$z_{[1236]} z_{[125]} z_{[456]} \in \mathcal{H}_{6,4}^3.$$

Recall that standard tableaux are multiple chains in $\mathcal{L}_{m,k}^n$ (Definition 2.5.1). Then the following property is a consequence of the general theory of Hibi algebras [10,13].

Lemma 2.10.2.

(1) *The set $\mathcal{T}^n_{m,k}$ of all standard tableaux for (GL_m, GL_n) whose shapes are F/D with $\ell(F) \leqslant k$ form a $\mathbb{C}$-basis for the Hibi algebra $\mathcal{H}^n_{m,k}$.*
(2) *In particular, $\mathcal{H}^n_{m,k}$ is graded by $\Lambda_{k,n}$, and the set $\mathcal{T}^n_m(F, D)$ of standard tableaux for (GL_m, GL_n) of shape F/D form a $\mathbb{C}$-basis for the (F, D)-graded component of $\mathcal{H}^n_{m,k}$.*

It is shown in [18, Corollary 3.14] that the Hibi algebra over $\mathcal{L}_m$ is isomorphic to the semigroup algebra of GT patterns defined on Γ_m. This fact combined with the above lemma leads us to study the Hibi algebra $\mathcal{H}^n_{m,k}$ over $\mathcal{L}^n_{m,k}$ in terms of the semigroup algebra $\mathbb{C}[\mathcal{P}^n_{m,k}]$ of the affine semigroup $\mathcal{P}^n_{m,k}$ given in (2.7.1).

Note that for p_1 and $\mathsf{p}_2 \in \mathcal{P}^n_{m,k}$ of types F_1/D_1 and F_2/D_2 respectively, the type of $(\mathsf{p}_1 + \mathsf{p}_2)$ is $(F_1 + F_2)/(D_1 + D_2)$, and therefore $\mathbb{C}[\mathcal{P}^n_{m,k}]$ is graded by pairs of shapes

$$\mathbb{C}\big[\mathcal{P}^n_{m,k}\big] = \bigoplus_{(F,D)\in\Lambda_{k,n}} \mathbb{C}\big[\mathcal{P}^n_m\big]_{(F,D)}$$

where $\mathbb{C}[\mathcal{P}^n_m]_{(F,D)}$ is the space spanned by $\mathcal{P}^n_m(F, D)$.

Proposition 2.10.3.

(1) *The semigroup algebra $\mathbb{C}[\mathcal{P}^n_{m,k}]$ of the GT patterns for (GL_m, GL_n) is isomorphic to the Hibi algebra $\mathcal{H}^n_{m,k}$ over $\mathcal{L}^n_{m,k}$.*
(2) *The set $\mathcal{P}^n_m(F, D)$ of GT patterns for (GL_m, GL_n) of type F/D is a $\mathbb{C}$-basis for the (F, D)-graded component $\mathbb{C}[\mathcal{P}^n_m]_{(F,D)}$.*

Proof. Note that the algebra $\mathbb{C}[\mathcal{P}^n_{m,k}]$ is generated by the set of characteristic functions of order increasing subsets of $\Gamma^n_{m,k}$; and that, for order increasing subsets A and B of $\Gamma^n_{m,k}$, their characteristic functions satisfy $\chi_A + \chi_B = \chi_{A\cap B} + \chi_{A\cup B}$. With this observation, one can show that the isomorphism in Proposition 2.3.1 can be extended to an algebra isomorphism between $\mathbb{C}[\mathcal{P}^n_{m,k}]$ and $\mathcal{H}^n_{m,k}$. For further details, see [13, Theorem 4.3] and [18, §3.2]. The second statement follows from Proposition 2.8.1 and the above lemma. $\square$

3. Branching algebras for (GL_m, GL_n)

In this section, our goal is to construct an algebra encoding branching rules for (GL_m, GL_n) and study its toric degeneration. For later use, we will construct a family of algebras parametrized by the length of highest weights for GL_m.

3.1. Recall that the set of Young diagrams F with $\ell(F) \leqslant m$ can be used as a labeling system of irreducible polynomial representations of GL_m by identifying dominant weights $(f_1 \geqslant \cdots \geqslant f_m) \in \mathbb{Z}^m_{\geqslant 0}$ of GL_m with Young diagrams (cf. [9, §3.1.4]). We let ρ^F_m denote the irreducible polynomial representation of GL_m labeled by Young diagram F.

Then the branching algebra for (GL_m, GL_n) will be graded by the set $\Lambda_{m,n}$ defined in Section 2.6 and its graded components will correspond to the multiplicity spaces $\mathrm{Hom}_{GL_n}(\rho^D_n, \rho^F_m)$ for (F, D).

3.2. For Young diagrams $F = (f_1, f_2, \ldots)$ and $D = (d_1, d_2, \ldots)$, we write

$$F \sqsupseteq D$$

if $f_r \geqslant d_r \geqslant f_{r+1}$ for all r, and say D *interlaces* F.

Proposition 3.2.1. *(See [9, §8.1.1].)*

(1) *For Young diagrams F and D with $\ell(F) \leqslant m$ and $\ell(D) \leqslant m-1$, the multiplicity of ρ_{m-1}^D in ρ_m^F is 1 if $F \sqsupseteq D$, and 0 otherwise.*
(2) *The number of GT patterns in $\mathcal{P}_m^n(F, D)$ is equal to the multiplicity $m(\rho_n^D, \rho_m^F)$ of ρ_n^D in ρ_m^F.*

From Proposition 3.2.1 and Proposition 2.8.1, we have

Corollary 3.2.2. *For $(F, D) \in \Lambda_{m,n}$, the branching multiplicity $m(\rho_n^D, \rho_m^F)$ is equal to the number of standard tableaux for (GL_m, GL_n) whose shapes are F/D.*

3.3. To construct a family of branching algebras for (GL_m, GL_n) parameterized by the length k, let us review a polynomial model for the flag algebra. We assume $m \geqslant k$ and let $GL_m \times GL_k$ act on the space $\mathsf{M}_{m,k} \cong \mathbb{C}^m \otimes \mathbb{C}^k$ of $m \times k$ complex matrices by

$$(g_1, g_2) \cdot Q = \left(g_1^t\right)^{-1} Q\, g_2^{-1} \tag{3.3.1}$$

for $g_1 \in GL_m$, $g_2 \in GL_k$, and $Q \in \mathsf{M}_{m,k}$. Then under the $GL_m \times GL_k$ action, the coordinate ring $\mathbb{C}[\mathsf{M}_{m,k}]$ of $\mathsf{M}_{m,k}$ has the following decomposition:

$$\mathbb{C}[\mathsf{M}_{m,k}] = \sum_{\ell(F) \leqslant k} \rho_m^F \otimes \rho_k^F$$

where the summation is over F with length not more than k. This result is known as $GL_m - GL_k$ *duality* (e.g., [9,12]). If U_k is the subgroup of GL_k consisting of upper triangular matrices with 1's on the diagonal, then by taking $U_k \cong 1 \times U_k$ invariants, we have

$$\mathbb{C}[\mathsf{M}_{m,k}]^{U_k} = \sum_{\ell(F) \leqslant k} \rho_m^F \otimes \left(\rho_k^F\right)^{U_k}.$$

3.4. This representation decomposition turns out to be compatible with the multiplicative structure of the algebra. Since the diagonal subgroup A_k of GL_k normalizes U_k, $\mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$ is stable under the action of A_k. Note that by highest weight theory (e.g., [9, §3.2.1 and §12.1.3]), $(\rho_k^F)^{U_k}$ is the one-dimensional space spanned by a highest weight vector of ρ_k^F, and A_k acts on $(\rho_k^F)^{U_k}$ by the character

$$\phi_F\left(\operatorname{diag}(a_1, \ldots, a_k)\right) = a_1^{f_1} \cdots a_k^{f_k}$$

given by Young diagram $F = (f_1, f_2, \ldots, f_k)$. Thus, $\rho_m^F \simeq \rho_m^F \otimes (\rho_k^F)^{U_k}$ is the space of A_k-eigenvectors of weight ϕ_F in $\mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$ and the $\mathbb{C}$-algebra $\mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$ is graded by the semigroup $\hat{A}_k^+$ of dominant polynomial weights for GL_k, or equivalently the subsemigroup $\hat{A}_k^+ \subset \hat{A}_m^+$ of dominant weights for GL_m:

$$\mathbb{C}[\mathsf{M}_{m,k}]^{U_k} = \sum_{\ell(F) \leqslant k} \rho_k^F,$$

$$\rho_m^{F_1} \cdot \rho_m^{F_2} \subseteq \rho_m^{F_1+F_2} \tag{3.4.1}$$

where we identify $(r_1, \ldots, r_k) \in \mathbb{Z}_{\geqslant 0}^k$ with $(r_1, \ldots, r_k, 0, \ldots, 0) \in \mathbb{Z}_{\geqslant 0}^m$.

3.5. A finite presentation of $\mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$ in terms of generators and relations is well known—all the U_k-invariant minors on $\mathsf{M}_{m,k}$ form a generating set and they satisfy the Plücker relations. To explain more details, let us consider a subposet $\mathcal{L}_{m,k} = \mathcal{L}_{m,k}^1$ of $\mathcal{L}_m$ consisting of elements $I = [i_1, i_2, \ldots, i_r]$ such that $|I| \leqslant k$ (cf. Definition 2.2.1).

For each $Q \in \mathsf{M}_{m,k}$, we let $\delta_I(Q)$ denote the determinant of the submatrix of $Q = (t_{a,b})$ obtained by taking the $i_1, i_2, \ldots, i_r$-th rows and the $1, 2, \ldots, r$-th columns:

$$\delta_I(Q) = \det \begin{bmatrix} t_{i_1 1} & t_{i_1 2} & \cdots & t_{i_1 r} \\ t_{i_2 1} & t_{i_2 2} & \cdots & t_{i_2 r} \\ \vdots & \vdots & \ddots & \vdots \\ t_{i_r 1} & t_{i_r 2} & \cdots & t_{i_r r} \end{bmatrix}. \tag{3.5.1}$$

Definition 3.5.1. A product $\delta_{I_1}\delta_{I_2}\cdots\delta_{I_r}$ is called a *standard monomial* (or GL_m *standard monomial*), if its indices form a multiple chain $\mathsf{t} = (I_1 \preccurlyeq I_2 \preccurlyeq \cdots \preccurlyeq I_r)$ in $\mathcal{L}_{m,k}$. We write

$$\Delta_\mathsf{t} = \delta_{I_1}\delta_{I_2}\cdots\delta_{I_r}.$$

Then we define the *shape* of a standard monomial Δ_t to be the shape of t, i.e., $(|I_1|, |I_2|, \ldots, |I_r|)^t$.

Proposition 3.5.2. *(See [8, pp. 233, 236].)*

(1) *For $I, J \in \mathcal{L}_{m,k}$, the product $\delta_I\delta_J \in \mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$ can be uniquely expressed as a linear combination of standard monomials*

$$\delta_I\delta_J = \sum_r c_r \delta_{S_r}\delta_{T_r} \tag{3.5.2}$$

where, for each r with $c_r \neq 0$, $S_r \preccurlyeq T_r$ in $\mathcal{L}_{m,k}$ and $S_r \dot\cup T_r = I \dot\cup J$ as sets.

(2) *On the right-hand side, $\delta_{I\wedge J}\delta_{I\vee J}$ appears with coefficient* 1, *and $S_r \preccurlyeq I \wedge J$ and $I \vee J \preccurlyeq T_r$ for all r with $c_r \neq 0$. Moreover, for each $(S_r, T_r) \neq (I \wedge J, I \vee J)$, let h be the smallest integer such that the sum s of the h-th entries of S_r and T_r is different from the sum s_0 of the h-th entries of I and J. Then $s > s_0$.*

By applying the straightening relations (3.5.2), we can find a $\mathbb{C}$-basis for $\mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$. The following is well known. See, for example, [3,4,8,11]. For this particular form, see [18, Theorem 4.5, Remark 4.6].

Proposition 3.5.3. *Standard monomials Δ_t associated with multiple chains t in $\mathcal{L}_{m,k}$ form a $\mathbb{C}$-basis for $\mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$. More precisely, standard monomials Δ_t with $sh(\mathsf{t}) = F$ form a weight basis for the GL_m irreducible representation $\rho_m^F \subset \mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$ with highest weight F.*

We specify the following properties of the standard monomial expression of $\delta_I\delta_J$ for $I, J \in \mathcal{L}_{m,k}^n$ of length not more than k, which can be easily derived from the above proposition.

Corollary 3.5.4. *Let I and J be incomparable elements in $\mathcal{L}_{m,k}^n$ with $|I| \geqslant |J|$. Consider the standard monomial expression of the product $\delta_I\delta_J$ given in* (3.5.2)*. Let us denote the standard tableau $S_r \preccurlyeq T_r$ by t_r. Then, for each r with non-zero c_r,*

(1) *the shape $sh_n(\mathsf{t}_r)$ is F/D where $F = (|I|, |J|)^t$ and $D = (d_1, d_2, \ldots)$ where d_h is the number of h's in the disjoint union $I \dot\cup J$ for $1 \leqslant h \leqslant n$;*
(2) *all the entries in the h-th row of t_r are bigger than or equal to h for $1 \leqslant h \leqslant \min(n, |I|)$;*
(3) *if we denote the numbers of entries less than or equal to h in S_r and T_r by α_h and β_h respectively, then $\alpha_h + \beta_h \leqslant 2h$ for $1 \leqslant h \leqslant \min(n, |I|)$.*

Example 3.5.5. For $I = [1, 2, 5, 6]$ and $J = [1, 3, 4]$ from $\mathcal{L}_{6,4}^2$, we have

$$\delta_{[1256]}\delta_{[134]} = \delta_{[1246]}\delta_{[135]} - \delta_{[1236]}\delta_{[145]} + \delta_{[1235]}\delta_{[146]} - \delta_{[1245]}\delta_{[136]} - \delta_{[1234]}\delta_{[156]}.$$

Note that $sh_n(\mathsf{t}_r) = (2, 2, 2, 1)/(2, 1)$ for all the terms t_r on the right-hand side.

3.6. Let $m > n$. To consider the branching rules for (GL_m, GL_n), we use the following embedding of GL_n in GL_m: for $X \in GL_n$,

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \in GL_m$$

where I is the $(m-n) \times (m-n)$ identity matrix and 0's are the zero matrices of proper sizes.

From (3.4.1), by taking U_n-invariants, we have

$$\begin{aligned} \mathbb{C}[\mathsf{M}_{m,k}]^{U_n \times U_k} &= \sum_{\ell(F) \leqslant k} \left(\rho_m^F\right)^{U_n} \\ &= \sum_{\ell(F) \leqslant k} \sum_{D} m\left(\rho_n^D, \rho_m^F\right)\left(\rho_n^D\right)^{U_n} \end{aligned} \tag{3.6.1}$$

where $m(\rho_n^D, \rho_m^F)$ is the multiplicity of ρ_n^D appearing in ρ_m^F, and $(\rho_n^D)^{U_n}$ is the one-dimensional space spanned by a highest weight vector of ρ_n^D.

Definition 3.6.1. For $m \geqslant k$, the length k branching algebra for (GL_m, GL_n) is the $(U_n \times U_k)$-invariant ring of $\mathbb{C}[\mathsf{M}_{m,k}]$

$$\mathcal{B}_{m,k}^n = \mathbb{C}[\mathsf{M}_{m,k}]^{U_n \times U_k}.$$

3.7. Note that for $I \in \mathcal{L}_{m,k}^n$ all the minors δ_I are invariant under the subgroup $U_n \times U_k$ of $GL_n \times GL_k$ with respect to the action (3.3.1). In fact, the length k branching algebra for (GL_m, GL_n) is generated by $\{\delta_I\colon\ I \in \mathcal{L}_{m,k}^n\}$.

Theorem 3.7.1. *For each k with $m \geqslant k$, the branching algebra $\mathcal{B}_{m,k}^n$ for (GL_m, GL_n) is graded by $\Lambda_{k,n}$*

$$\mathcal{B}_{m,k}^n = \bigoplus_{(F,D) \in \Lambda_{k,n}} \mathcal{B}_{m,k}^n(F, D)$$

and the standard monomials Δ_{t} for $\mathsf{t} \in \mathcal{T}_m^n(F, D)$ form a $\mathbb{C}$-basis of the (F, D)-graded component $\mathcal{B}_{m,k}^n(F, D)$.

Proof. For $I \in \mathcal{L}_{m,k}^n$, the determinant functions δ_I, considered as elements of $\mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$, satisfy the relations (3.5.2). Also, by keeping track of the entries of I and J in this relation, we can easily see that all S_r and T_r appearing on the right-hand side of (3.5.2) are elements of $\mathcal{L}_{m,k}^n$, and that all the standard tableaux $\mathsf{t}_r = (S_r \preccurlyeq T_r)$ have the same shape as in the first statement of Corollary 3.5.4. By applying these relations repeatedly, we can express every monomial in $\{\delta_I\colon\ I \in \mathcal{L}_{m,k}^n\}$ as a linear combination of standard monomials of the same shape. In particular, the algebra $\mathcal{B}_{m,k}^n$ is graded by the shapes $sh_n(\mathsf{t}) \in \Lambda_{k,n}$ of standard monomials for (GL_m, GL_n). Now, it is enough to show that for each shape F/D with $(F, D) \in \Lambda_{k,n}$, the number of standard monomials Δ_{t} for $\mathsf{t} \in \mathcal{T}_m^n(F, D)$ is equal to the multiplicity of ρ_n^D in ρ_m^F, which is Corollary 3.2.2. □

Note that the standard monomials Δ_{t} for $\mathsf{t} \in \mathcal{T}_m^n(F, D)$ are invariant under the action of U_n and scaled by the character ϕ_D under the action of the diagonal subgroup of GL_n:

$$\begin{aligned} \mathrm{diag}(a_1, \ldots, a_n) \cdot \Delta_{\mathsf{t}} &= \phi_D\left(\mathrm{diag}(a_1, \ldots, a_n)\right)\Delta_{\mathsf{t}} \\ &= \left(a_1^{d_1} \cdots a_n^{d_n}\right)\Delta_{\mathsf{t}} \end{aligned} \tag{3.7.1}$$

for $D = (d_1, \ldots, d_n)$. This shows that standard monomials Δ_{t} for $\mathsf{t} \in \mathcal{T}_m^n(F, D)$ are the highest weight vectors of the copies of ρ_n^D in ρ_m^F. Accordingly, we have

Proposition 3.7.2. *The standard monomials Δ_{t} with $\mathsf{t} \in \mathcal{T}_m^n(F, D)$, as $\mathbb{C}$-basis elements of $\mathcal{B}_{m,k}^n(F, D)$, are the highest weight vectors of the copies of ρ_n^D in ρ_m^F. Therefore, we have*

$$\mathcal{B}_{m,k}^n(F, D) \cong \mathrm{Hom}_{GL_n}\big(\rho_n^D, \rho_m^F\big).$$

3.8. Toric degenerations of the branching algebras $\mathcal{B}_{m,k}^n$ can be induced by the same methods used for the case of the flag algebra $\mathbb{C}[\mathsf{M}_{m,k}]^{U_k}$ in the literature, for example, [7,18,23,27,31]. See also [32, Theorem 1], for the properties of the algebra of polynomials on a semisimple algebraic group and its associated graded algebra.

Theorem 3.8.1. *The length k branching algebra $\mathcal{B}_{m,k}^n$ for (GL_m, GL_n) is a flat deformation of the Hibi algebra $\mathcal{H}_{m,k}^n$ over $\mathcal{L}_{m,k}^n$.*

Proof. Let us impose a filtration on $\mathcal{B}_{m,k}^n$ by giving the following weight on each monomials. Fix an integer N greater than $2m$, and then define the weight of $I = [i_1, \ldots, i_a] \in \mathcal{L}_{m,k}^n$ as

$$wt(I) = \sum_{r \geqslant 1} i_r N^{m-r}. \tag{3.8.1}$$

The weight of a standard tableau t consisting of I_c is defined to be the sum of individual weights, i.e., $wt(\mathsf{t}) = \sum_c wt(I_c)$. Then we can define a $\mathbb{Z}$-filtration $\mathsf{F}^{wt} = \{\mathsf{F}_d^{wt}\}$ on $\mathcal{B}_{m,k}^n = \mathbb{C}[\mathsf{M}_{m,k}]^{U_n \times U_k}$ with respect to the weight wt. Set $\mathsf{F}_d^{wt}(\mathcal{B}_{m,k}^n)$ to be the space spanned by

$$\big\{\Delta_{\mathsf{t}}\colon\ wt(\mathsf{t}) \geqslant d\big\}.$$

The filtration F^{wt} is well defined, since every product $\prod \delta_{I_c}$ can be expressed as a linear combination of standard monomials with bigger weights by Proposition 3.5.2. For all pairs $A, B \in \mathcal{L}_{m,k}^n$, since $wt(A) + wt(B) = wt(A \wedge B) + wt(A \vee B)$, $\delta_A \delta_B$ and $\delta_{A\wedge B}\delta_{A\vee B}$ belong to the same associated graded component. Therefore, we have $s_A \cdot_{\mathrm{gr}} s_B = s_{A\wedge B} \cdot_{\mathrm{gr}} s_{A\vee B}$ where s_C are elements corresponding to δ_C in the associated graded ring $\mathrm{gr}^{wt}(\mathcal{B}_{m,k}^n)$ of $\mathcal{B}_{m,k}^n$ with respect to the filtration F^{wt}. Then it is straightforward to show that the associated graded ring $\mathrm{gr}^{wt}(\mathcal{B}_{m,k}^n)$ forms the Hibi algebra over $\mathcal{L}_{m,k}^n$. From a general property of the Rees algebras (e.g., [1]), the Rees algebra $\mathcal{R}^t$ of $\mathcal{B}_{m,k}^n$ with respect to F^{wt}:

$$\mathcal{R}^t = \bigoplus_{d \geqslant 0} \mathsf{F}_d^{wt}\big(\mathcal{B}_{m,k}^n\big) t^d$$

is flat over $\mathbb{C}[t]$ with its general fiber isomorphic to $\mathcal{B}_{m,k}^n$ and special fiber isomorphic to the associated graded ring which is $\mathcal{H}_{m,k}^n$. $\square$

We remark that $Spec(\mathcal{H}_{m,k}^n)$ is an affine toric variety in the sense of [31]. Then, the rational polyhedral cone corresponding to the affine toric variety and the integral points therein can be realized from our description of the affine semigroup $\mathcal{P}_{m,k}^n$ given at the beginning of Section 2.8.

4. Stable range branching algebra for (Sp_{2m}, Sp_{2n})

In this section, starting from combinatorial descriptions of stable range branching rules, we study the affine semigroup algebra and its associated Hibi algebra for (Sp_{2m}, Sp_{2n}). Then we construct an explicit model for the stable range branching algebra. Along with these, we also show that these algebraic objects are isomorphic to their (GL_{2m}, GL_{2n}) counterparts with a proper length condition.

Recall that we can label irreducible rational representation of Sp_{2m}, after identifying dominant weights with Young diagrams, by Young diagrams with less than or equal to m rows (cf. [9, §3.1.4]). We let τ_{2m}^F denote the irreducible representation of Sp_{2m} labeled by Young diagram F.

4.1. Let $J_m = (j_{a,b})$ be the $m \times m$ matrix with $j_{a,m+1-a} = 1$ for $1 \leqslant a \leqslant m$ and 0 otherwise. Then we define the symplectic group Sp_{2m} of rank m as the subgroup of GL_{2m} preserving the skew symmetric bilinear form on $\mathbb{C}^{2m}$ induced by

$$\begin{bmatrix} 0 & J_m \\ -J_m & 0 \end{bmatrix}.$$

Note that, for the elementary basis $\{e_i\}$ of the space $\mathbb{C}^{2m}$, e_j and e_{2m+1-j} make an isotropic pair for $1 \leqslant j \leqslant m$ with respect to this bilinear form. Also, the subgroup of upper triangular matrices with 1's on the diagonal can be taken as a maximal unipotent subgroup of Sp_{2m}. We will denoted it by $U_{Sp_{2m}}$.

For $n < m$, we identify Sp_{2n} with the subgroup of Sp_{2m} preserving the skew symmetric bilinear form restricted to the subspace of $\mathbb{C}^{2m}$ spanned by

$$\{e_a, e_{2m+1-a}\colon 1 \leqslant a \leqslant n\}.$$

Then Sp_{2n} can be embedded in Sp_{2m} as follows.

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \mapsto \begin{bmatrix} X & 0 & Y \\ 0 & I & 0 \\ Z & 0 & W \end{bmatrix} \tag{4.1.1}$$

where X, Y, Z, W are $n \times n$ matrices, I is the $2(m-n) \times 2(m-n)$ identity matrix, and 0's are the zero matrices of proper sizes.

4.2. In order to construct an affine semigroup encoding stable range branching rules for (Sp_{2m}, Sp_{2n}), we review the following combinatorial description of branching multiplicities.

Lemma 4.2.1. *(See [9, Theorem 8.1.5].) For Young diagrams F and D with $\ell(F) \leqslant m$ and $\ell(D) \leqslant m-1$, the multiplicity of $\tau^D_{2(m-1)}$ in τ^F_{2m} as an $Sp_{2(m-1)}$ representation is equal to the number of Young diagrams E satisfying the interlacing condition $F \sqsupseteq E \sqsupseteq D$.*

For example, if $F = (5,3,3,2,1)$ and $D = (4,3,2,2)$, then the multiplicity of τ^D_8 in τ^F_{10} is equal to the number of $E = (e_1, e_2, \dots, e_5)$ in

$$\begin{array}{ccccccccc} 5 & & 3 & & 3 & & 2 & & 1 \\ & e_1 & & e_2 & & e_3 & & e_4 & & e_5 \\ & & 4 & & 3 & & 2 & & 2 \end{array}$$

so that the entries are weakly decreasing from left to right along diagonals.

Note that this branching is not multiplicity free and rather similar to the two-step branchings for the general linear groups. To obtain a description of the multiplicity spaces for (Sp_{2m}, Sp_{2n}), we can simply iterate the above lemma. Because of the length condition $\ell(E_k) \leqslant k$ of Sp_{2k} representations $\tau^{E_k}_{2k}$ for $n \leqslant k \leqslant m$, it will be quite different from the (GL_{2m}, GL_{2n}) case (Proposition 3.2.1). Within the stable range $\ell(F) \leqslant n$, however, we have exactly the same description.

In the previous example, if we set $F = (5,3,3,2,0)$ so that $\ell(F) = 4$, then the multiplicity of τ^D_8 in τ^F_{10} is equal to the number of $E = (e_1, e_2, \dots, e_5)$ in

$$\begin{array}{ccccccccc} 5 & & 3 & & 3 & & 2 & & 0 \\ & e_1 & & e_2 & & e_3 & & e_4 & & e_5 \\ & & 4 & & 3 & & 2 & & 2 \end{array}$$

and the interlacing condition makes $e_5 = 0$. Therefore the multiplicity of τ^D_8 in τ^F_{10} is equal to the multiplicity of the GL_8 representation ρ^D_8 in the GL_{10} representation ρ^F_{10}.

Remark 4.2.2.

(1) For complete GT patterns for Sp_{2m}, we refer to [22] and [28]. See also [18, §5] for their ring theoretic interpretation.
(2) The branching algebra for (Sp_{2m}, Sp_{2m-2}) has interesting algebraic and combinatorial properties with an extra structure from the action of $SL_2 \times \cdots \times SL_2$. For this, we refer to [21].

4.3. Recall that $\mathcal{P}_{2m}^{2n}(F, D)$ is the set of all *GT* patterns for (GL_{2m}, GL_{2n}) whose types are F/D. Within the stable range $\ell(F) \leqslant n$, $F \supseteq D$ implies $\ell(D) \leqslant n$, and therefore the support of every GT pattern in $\mathcal{P}_{2m}^{2n}(F, D)$ lies in the GT poset $\Gamma_{2m,n}^{2n}$ of length n:

$$\begin{array}{ccccccccccc}
x_1^{(2m)} & & x_2^{(2m)} & & \cdots & & x_n^{(2m)} & & & & \\
& x_1^{(2m-1)} & & x_2^{(2m-1)} & & \cdots & & x_n^{(2m-1)} & & & \\
& & \ddots & & \ddots & & \cdots & & \ddots & & \\
& & & x_1^{(2n)} & & x_2^{(2n)} & & \cdots & & x_n^{(2n)} &
\end{array}$$

Proposition 4.3.1. *Let F and D be Young diagrams with $F \supseteq D$ and $\ell(F) \leqslant n$. Then the branching multiplicity $m(\tau_{2n}^D, \tau_{2m}^F)$ is equal to the number of elements in $\mathcal{P}_{2m}^{2n}(F, D)$, and therefore it is equal to the number of elements in $\mathcal{T}_{2m}^{2n}(F, D)$.*

Proof. From Lemma 4.2.1, by using the same argument used to prove (2) of Proposition 3.2.1, the set $\mathcal{P}_{2m}^{2n}(F, D)$ of GT patterns of shape F/D counts the multiplicity of τ_{2n}^D in τ_{2m}^F. The last statement follows from Proposition 2.8.1. $\square$

We call the affine semigroup $\mathcal{P}_{2m,n}^{2n}$, defined in (2.7.1), the *semigroup for* (Sp_{2m}, Sp_{2n}) and call its associated semigroup algebra $\mathbb{C}[\mathcal{P}_{2m,n}^{2n}]$ the *semigroup algebra for* (Sp_{2m}, Sp_{2n}). Then it is graded by $\Lambda_{n,n}$ defined in Section 2.6

$$\mathbb{C}\big[\mathcal{P}_{2m,n}^{2n}\big] = \bigoplus_{(F,D)\in\Lambda_{n,n}} \mathbb{C}\big[\mathcal{P}_{2m}^{2n}\big]_{(F,D)}.$$

4.4. To define tableaux and standard monomials for the symplectic groups, we shall use the following ordered letters:

$$\langle 2m\rangle = \{u_1 < v_1 < u_2 < v_2 < \cdots < u_m < v_m\}. \tag{4.4.1}$$

If we let $\mathcal{L}\langle 2m\rangle$ denote the set of all non-empty subsets J of $\langle 2m\rangle$, then on $\mathcal{L}\langle 2m\rangle$ we can impose the tableau order $\preccurlyeq$, as it is done in Section 2.2 for $\mathcal{L}_{2m}$, through the bijection

$$\iota(u_c) = 2c - 1 \quad \text{and} \quad \iota(v_c) = 2c \tag{4.4.2}$$

for $1 \leqslant c \leqslant m$. Then $\mathcal{L}\langle 2m\rangle$ is a distributive lattice isomorphic to $\mathcal{L}_{2m}$.

For $m > n$, we consider the subposet $\mathcal{L}\langle n, 2m\rangle$ of $\mathcal{L}\langle 2m\rangle$ with all the elements $J \subset \langle 2m\rangle$ of the forms

$$\begin{aligned}
&[u_1, u_2, \ldots, u_c, y_1, y_2, \ldots, y_s],\\
&[u_1, u_2, \ldots, u_c],\\
&[y_1, y_2, \ldots, y_s]
\end{aligned} \tag{4.4.3}$$

where $c \leqslant n$ and $u_{n+1} \leqslant y_1 < y_2 < \cdots < y_s \leqslant v_m$. In particular, if $u_c \in J$ for $c \leqslant n$, then $\{u_h\colon 1 \leqslant h \leqslant c\} \subset J$.

Now, for $k \leqslant n$, let $\mathcal{L}\langle n, 2m\rangle_k$ be the subposet of $\mathcal{L}\langle n, 2m\rangle$ consisting of $J \in \mathcal{L}\langle n, 2m\rangle$ with $|J| \leqslant k$. Then, through the map (4.4.2), it is straightforward to see that $\mathcal{L}\langle n, 2m\rangle_k$ is isomorphic to $\mathcal{L}^n_{2m-n,k}$ given in Definition 2.2.1, and therefore isomorphic to $\mathcal{L}^{2n}_{2m,k}$ by Corollary 2.3.2.

Definition 4.4.1.

(1) The distributive lattice for (Sp_{2m}, Sp_{2n}) is

$$\begin{aligned}\mathcal{L}_{Sp} &= \mathcal{L}\langle n, 2m\rangle_n \\ &\cong \mathcal{L}^{2n}_{2m,n}.\end{aligned}$$

(2) The Hibi algebra for (Sp_{2m}, Sp_{2n}), denoted by $\mathcal{H}_{Sp}$, is the Hibi algebra over the distributive lattice $\mathcal{L}_{Sp}$.

Note that from $\mathcal{L}_{Sp} \cong \mathcal{L}^{2n}_{2m,n}$, the Hibi algebra $\mathcal{H}_{Sp}$ for (Sp_{2m}, Sp_{2n}) is isomorphic to $\mathcal{H}^{2n}_{2m,n}$. Then from Proposition 2.10.3 for (GL_{2m}, GL_{2n}) we have

Corollary 4.4.2. *The Hibi algebra for* (Sp_{2m}, Sp_{2n}) *is isomorphic to the semigroup algebra for* (Sp_{2m}, Sp_{2n})*:*

$$\mathcal{H}_{Sp} \cong \mathbb{C}\big[\mathcal{P}^{2n}_{2m,n}\big].$$

4.5. Next, we define standard tableaux for (Sp_{2m}, Sp_{2n}).

Definition 4.5.1.

(1) A standard tableau t for (Sp_{2m}, Sp_{2n}) is a multiple chain in $\mathcal{L}_{Sp}$:

$$\mathrm{t} = (I_1 \preccurlyeq \cdots \preccurlyeq I_s).$$

(2) The shape $sh_n(\mathrm{t})$ of a standard tableau t for (Sp_{2m}, Sp_{2n}) is F/D where

$$F = \big(|I_1|, \ldots, |I_s|\big)^t \quad \text{and} \quad D = (d_1, \ldots, d_n)$$

with d_r being the number of u_h's in t for $1 \leqslant h \leqslant n$.

We write $\mathcal{T}_{Sp}(F, D)$ for the set of all standard tableaux for (Sp_{2m}, Sp_{2n}) whose shapes are F/D, and consider the disjoint union

$$\mathcal{T}_{Sp} = \bigcup_{(F,D)\in\Lambda_{n,n}} \mathcal{T}_{Sp}(F, D)$$

over $\Lambda_{n,n}$. Then as in the case of the general linear groups, $\mathcal{T}_{Sp}$ gives rise to a $\mathbb{C}$-basis for the Hibi algebra for (Sp_{2m}, Sp_{2n}). As in Section 2.10, we shall identify monomials in the Hibi algebra $\mathcal{H}_{Sp}$ with tableaux whose columns are elements of $\mathcal{L}_{Sp}$.

Proposition 4.5.2.

(1) *The Hibi algebra* $\mathcal{H}_{Sp}$ *for* (Sp_{2m}, Sp_{2n}) *is graded by* $\Lambda_{n,n}$*, and for each* $(F, D) \in \Lambda_{n,n}$*,* $\mathcal{T}_{Sp}(F, D)$ *forms a* $\mathbb{C}$*-basis for the graded component* $\mathcal{H}_{Sp}(F, D)$ *of* $\mathcal{H}_{Sp}$*.*
(2) *The number of standard tableaux for* (Sp_{2m}, Sp_{2n}) *of shape* F/D *is equal to the branching multiplicity* $m(\tau^D_{2n}, \tau^F_{2m})$ *of* τ^D_{2n} *in* τ^F_{2m}*.*

Proof. From the isomorphism $\mathcal{L}_{Sp} \cong \mathcal{L}^{2n}_{2m,n}$, we can easily see that there is a bijection between $\mathcal{T}_{Sp}(F, D)$ and $\mathcal{T}^{2n}_{2m}(F, D)$. Then (1) follows from Lemma 2.10.2 and (2) follows from Proposition 4.3.1. □

4.6. We remark that every standard tableau for (Sp_{2m}, Sp_{2n}) of shape F/D can be realized as a skew semistandard tableau of shape F/D having entries from $\{u_{n+1}, v_{n+1}, \ldots, u_m, v_m\}$. For example, for $m = 10$ and $n = 6$, the standard tableau of shape $F = (6, 5, 3, 0, 0)$ and $D = (4, 3, 1)$

$$[u_1, u_2, u_3] \preccurlyeq [u_1, u_2, v_4] \preccurlyeq [u_1, u_2, v_4] \preccurlyeq [u_1, u_4] \preccurlyeq [v_4, u_5] \preccurlyeq [u_5]$$

in $\mathcal{L}_{Sp} = \mathcal{L}\langle 3, 10\rangle_3$ can be identified with the skew semistandard tableau

$$\begin{array}{|c|c|c|c|c|c|}
\hline
 & & & & v_4 & u_5 \\
\hline
 & & & u_4 & u_5 \\
\cline{1-5}
 & v_4 & v_4 \\
\cline{1-3}
\end{array}$$

where the empty boxes in h-th row are considered as the ones with u_h for $1 \leqslant h \leqslant n$.

We also remark that, as it is shown in Proposition 2.3.1, we can attach an order increasing subset A_I of $\Gamma^{2n}_{2m,n}$ to each $I \in \mathcal{L}_{Sp}$:

$$A_I = \bigcup_{2n \leqslant j \leqslant 2m} A_I^{(j)} \tag{4.6.1}$$

where $A_I^{(j)} \subset \Gamma^{2n}_{2m,n}$ are defined as

$$A_I^{(2i-1)} = \left\{x_1^{(2i-1)}, x_2^{(2i-1)}, \ldots, x_{s_i}^{(2i-1)}\right\},$$
$$A_I^{(2i)} = \left\{x_1^{(2i)}, x_2^{(2i)}, \ldots, x_{t_i}^{(2i)}\right\}.$$

Here s_i and t_i are the numbers of elements in I less than or equal to u_i and v_i respectively. Then we can relate every element of $\mathcal{T}_{Sp}$ to a sum of characteristic functions of these order increasing subsets as given in Proposition 2.8.1 and (2.8.1). This gives a direct proof for Corollary 4.4.2.

4.7. Now we want to lift the elements of the Hibi algebra $\mathcal{H}_{Sp}$ to construct the stable range branching algebra for (Sp_{2m}, Sp_{2n}). For this purpose, we briefly review the polynomial model of Sp_{2m}-representation spaces studied in [18].

From (3.4.1), as a GL_{2m} module, $\mathbb{C}[\mathsf{M}_{2m,m}]^{U_m}$ decomposes into irreducible representations ρ^F_{2m} for $\ell(F) \leqslant m$. By taking Sp_{2m} as a subgroup of GL_{2m}, we let $Sp_{2m} \times GL_m$ act on the space $\mathsf{M}_{2m,m} \cong \mathbb{C}^{2m} \otimes \mathbb{C}^m$ as in (3.3.1).

Then we take the quotient of $\mathbb{C}[\mathsf{M}_{2m,m}]^{U_m}$ by the ideal $\mathcal{I}_{Sp} = \sum_F \mathcal{I}^F$ where $\mathcal{I}^F$ is the Sp_{2m}-invariant complement space to τ^F_{2m} in ρ^F_{2m}, i.e., $\rho^F_{2m} = \tau^F_{2m} \oplus \mathcal{I}^F$ for each F (cf. [5, §17.3]). Then this quotient algebra can be taken as a polynomial model of the flag algebra for Sp_{2m} in that it contains exactly one copy of every irreducible representation τ^F_{2m}:

$$\begin{aligned}\mathcal{F}_{Sp} &= \mathbb{C}[\mathsf{M}_{2m,m}]^{U_m}/\mathcal{I}_{Sp} \\ &= \sum_{\ell(F) \leqslant m} \tau^F_{2m}.\end{aligned}$$

Moreover, this decomposition is compatible with the graded structure of the algebra, i.e., $\tau^{F_1}_{2m} \cdot \tau^{F_2}_{2m} \subset \tau^{F_1+F_2}_{2m}$. Therefore, for the stable range $\ell(F) \leqslant n$, we can consider its subalgebra consisting of τ^F_{2m} with $\ell(F) \leqslant n$:

$$\mathcal{F}^{(n)}_{Sp} = \sum_{\ell(F) \leqslant n} \tau^F_{2m}. \tag{4.7.1}$$

4.8. To describe generators of $\mathcal{F}_{Sp}$, to each $I = [w_1, \ldots, w_r] \in \mathcal{L}\langle 2m\rangle$ with $r \leqslant m$, we attach a determinant function $\delta_{I'}$ as follows. For $Q \in \mathsf{M}_{2m,m}$, we let $\delta_{I'}(Q)$ denote the determinant of the submatrix of $Q = (t_{a,b})$ obtained by taking the $i'_1, i'_2, \ldots, i'_r$-th rows and the $1, 2, \ldots, r$-th columns:

$$\delta_{I'}(Q) = \det \begin{bmatrix} t_{i'_1 1} & t_{i'_1 2} & \cdots & t_{i'_1 r} \\ t_{i'_2 1} & t_{i'_2 2} & \cdots & t_{i'_2 r} \\ \vdots & \vdots & \ddots & \vdots \\ t_{i'_r 1} & t_{i'_r 2} & \cdots & t_{i'_r r} \end{bmatrix} \tag{4.8.1}$$

where $\{i'_1, i'_2, \ldots, i'_r\}$ is the image of the set $\{w_1, w_2, \ldots, w_r\} \subset \langle 2m\rangle$ under

$$\psi : \{u_1, v_1, \ldots, u_m, v_m\} \to \{1, 2, \ldots, 2m\}$$
$$\psi(u_c) = c \quad \text{and} \quad \psi(v_c) = 2m + 1 - c \tag{4.8.2}$$

for $1 \leqslant c \leqslant m$.

This conversion procedure is to make the labeling (u_c, v_c) of isotropic pairs compatible with ours $(c, 2m + 1 - c)$ for the skew symmetric form defined in Section 4.1. Note that $(c, \bar{c})$ and $(2c - 1, 2c)$ are used for the isotropic pairs in [2] and [18].

Notation 4.8.1. To avoid a possible ambiguity, we impose a new total order $\lessdot$ on $\{1, 2, \ldots, 2m\}$ induced by ψ in (4.8.2) and the order of $\langle 2m\rangle$ given in (4.4.1):

$$1 \lessdot 2m \lessdot 2 \lessdot 2m - 1 \lessdot \cdots \lessdot m \lessdot m + 1.$$

(1) To emphasize the order $\lessdot$, we shall use the prime symbol as in i'_j for the elements i_j of $\{1, 2, \ldots, 2m\}$.
(2) In the determinant (4.8.1), we may further assume that

$$i'_1 \lessdot i'_2 \lessdot \cdots \lessdot i'_r$$

to fix the sign of the determinant.
(3) We also let I' denote the image of $I \in \mathcal{L}\langle 2m\rangle$ under ψ. Similarly, we let t' denote the multiple chain $(I'_1 \preccurlyeq I'_2 \preccurlyeq \cdots \preccurlyeq I'_c)$ corresponding to the multiple chain $\mathsf{t} = (I_1 \preccurlyeq I_2 \preccurlyeq \cdots \preccurlyeq I_c)$ in $\mathcal{L}\langle 2m\rangle$.

For the flag algebra $\mathcal{F}_{Sp}$, we are interested in $\delta_{I'}$ with $I \in \mathcal{L}\langle 2m\rangle$ whose h-th smallest entry is not less than u_h for all $h \geqslant 0$.

Definition 4.8.2. (See [2,18].) Fix the element $J_0 = [u_1, u_2, \ldots, u_m] \in \mathcal{L}\langle 2m\rangle$ of length m. For a multiple chain $\mathsf{t} = (I_1 \preccurlyeq I_2 \preccurlyeq \cdots \preccurlyeq I_c)$ of $\mathcal{L}\langle 2m\rangle$, its associated monomial

$$\Delta_{\mathsf{t}'} = \delta_{I'_1}\delta_{I'_2} \cdots \delta_{I'_c} \in \mathbb{C}[\mathsf{M}_{2m,m}]^{U_m}$$

is called an *Sp-standard monomial*, if $I_s \succcurlyeq J_0$ for all s.

4.9. To a product of $\delta_{I'}$'s, as an element of $\mathbb{C}[\mathsf{M}_{2m,m}]^{U_m}$, apply the straightening relations in Proposition 3.5.2 to obtain a linear combination of standard monomials for GL_{2m}:

$$\prod_i \delta_{I'_i} = \sum_r c_r \prod_{j \geqslant 1} \delta_{K'_{r,j}}.$$

If there is a non-zero term $\prod_j \delta_{K'_{r,j}}$ which is not an *Sp*-standard monomial, then apply relations from the ideal $\mathcal{I}_{Sp}$. This replaces the entries in $K_{r,j}$'s corresponding to isotropic pairs (u_a, v_a) with the sum of entries corresponding to (u_b, v_b) for $a \leqslant b$, thereby expressing $\prod_j \delta_{K'_{r,j}}$ as a linear combination of *Sp*-standard monomials. For further details, we refer to [18]. A combinatorial description of this procedure in the language of tableaux is given in [2].

Proposition 4.9.1. *(See [18, Theorem 5.20].) Sp-standard monomials project to a $\mathbb{C}$-basis of the flag algebra $\mathcal{F}_{Sp}$ for Sp_{2m}. In particular, for a Young diagram F with $\ell(F) \leqslant m$, Sp-*standard monomials *of shape F project to a weight basis for the Sp_{2m} irreducible representation $\tau_{2m}^F \subset \mathcal{F}_{Sp}$.*

We also note that, from the graded structure $\tau_{2m}^{F_1} \cdot \tau_{2m}^{F_2} \subset \tau_{2m}^{F_1+F_2}$ of $\mathcal{F}_{Sp}$, in order to obtain the subalgebra $\mathcal{F}_{Sp}^{(n)}$ in (4.7.1), it is enough to consider $\delta_{I'}$'s with $I \in \mathcal{L}\langle 2m\rangle$ and $|I| \leqslant n$.

4.10. We want to find an explicit model for the $U_{Sp_{2n}}$-invariant subalgebra of $\mathcal{F}_{Sp}^{(n)}$, which will denote by $\mathcal{B}_{Sp}$. Note that, from (1.3.1) and (4.7.1), we have

$$\begin{aligned}\mathcal{B}_{Sp} &= \sum_{\ell(F)\leqslant n} \left(\tau_{2m}^F\right)^{U_{Sp_{2n}}} \\ &= \sum_{\ell(F)\leqslant n} \sum_{D} m\left(\tau_{2n}^D, \tau_{2m}^F\right)\left(\tau_{2n}^D\right)^{U_{Sp_{2n}}}.\end{aligned}$$

Theorem 4.10.1. *The algebra $\mathcal{B}_{Sp}$ is generated by*

$$\mathcal{G} = \{\delta_{I'} + \mathcal{I}_{Sp}\colon I \in \mathcal{L}_{Sp}\},$$

and it is graded by $\Lambda_{n,n}$. For each $(F, D) \in \Lambda_{n,n}$ the Sp-standard monomials $\Delta_{\mathsf{t}'}$ corresponding to standard tableaux t *for (Sp_{2m}, Sp_{2n}) whose shapes are F/D form a $\mathbb{C}$-basis of the (F, D)-graded component. The dimension of the (F, D)-graded component is equal to the branching multiplicity of τ_{2n}^D in τ_{2m}^F.*

Proof. Let $\mathcal{R}$ be the subalgebra of $\mathcal{F}_{Sp}^{(n)}$ generated by $\mathcal{G}$. We will show that $\mathcal{R} = \mathcal{B}_{Sp}$. Recall that, for $I \in \mathcal{L}_{Sp} \subset \mathcal{L}\langle 2m\rangle$, we defined the polynomial $\delta_{I'}$ on the space $\mathsf{M}_{2m,m}$ in (4.8.1). By (4.4.3) and (4.8.2), it is the determinant of a submatrix of $Q \in \mathsf{M}_{2m,m}$ obtained by taking consecutive columns $\{1, 2, \dots, |I|\}$ and either consecutive rows $\{1, 2, \dots, r\}$ or partially consecutive rows $\{1, 2, \dots, r\} \cup \{b_1, \dots, b_s\}$ or only $\{b_1, \dots, b_s\}$ of Q for $r \leqslant n$ and $b_i \in \{n+1, n+2, \dots, 2m-n\}$ for all i.

Since the left action of $U_{2n} \subset GL_{2m}$ under the embedding (4.1.1) operates the rows of $\mathsf{M}_{2m,m}$, all the elements $\delta_{I'}$ for $I \in \mathcal{L}_{Sp}$ are invariant under the action of U_{2n} and therefor invariant under the action of $U_{Sp_{2n}}$. Since the ideal $\mathcal{I}_{Sp}$ is stable under the action of Sp_{2m}, the generators of the algebra $\mathcal{R}$ are invariant under the unipotent subgroup $U_{Sp_{2n}}$ of Sp_{2n}, and so are their products. Also, since every $I \in \mathcal{L}_{Sp}$ satisfies $|I| \leqslant n$, we have $\mathcal{R} \subseteq \mathcal{B}_{Sp}$.

On the other hand, every element in $\mathcal{L}_{Sp}$ is greater than $J_0 = [u_1, u_2, \dots, u_m]$ with respect to the tableau order, and therefore standard tableaux t for (Sp_{2m}, Sp_{2n}) (Definition 4.5.1 and Proposition 4.9.1) give rise to Sp-standard monomials $\Delta_{\mathsf{t}'}$ (Definition 4.8.2) for $\mathcal{F}_{Sp}$. That is, Sp-standard monomials corresponding to standard tableaux for (Sp_{2m}, Sp_{2n}) project to linearly independent elements in $\mathcal{B}_{Sp} \subseteq \mathcal{F}_{Sp}$. They span the whole algebra $\mathcal{B}_{Sp}$, because for each $(F, D) \in \Lambda_{n,n}$ the number of standard tableaux in $\mathcal{T}_{Sp}(F, D)$ is equal to the multiplicity of τ_{2n}^D in τ_{2m}^F by Proposition 4.3.1. Furthermore, they are scaled by weight D under the action of the diagonal subgroup $\{\mathrm{diag}(a_1, \dots, a_n, a_n^{-1}, \dots, a_1^{-1})\}$ of Sp_{2n} as given in (3.7.1). This shows that standard monomials $\Delta_{\mathsf{t}'}$ with $\mathsf{t} \in \mathcal{T}_{Sp}(F, D)$ are the highest weight vectors of the copies of τ_{2n}^D in τ_{2m}^F. $\square$

In this sense, we call $\mathcal{B}_{Sp}$ the *stable range branching algebra for* (Sp_{2m}, Sp_{2n}). Recall that we obtained $\mathcal{B}_{Sp}$ by lifting the elements of the Hibi algebra $\mathcal{H}_{Sp}$ over the distributive lattice $\mathcal{L}_{Sp}$ which is isomorphic to the distributive lattice $\mathcal{L}_{2m,n}^{2n}$. Now we compare it with the algebra $\mathcal{B}_{2m,n}^{2n}$ (Definition 3.6.1) obtained from the Hibi algebra $\mathcal{H}_{2m,n}^{2n}$ for the general linear groups.

Proposition 4.10.2. *The stable range branching algebra $\mathcal{B}_{Sp}$ for (Sp_{2m}, Sp_{2n}) is isomorphic to the length n branching algebra $\mathcal{B}_{2m,n}^{2n}$ for (GL_{2m}, GL_{2n}).*

Proof. From the isomorphism $\mathcal{L}_{Sp} \cong \mathcal{L}^{2n}_{2m,n}$ of distributive lattices, with $I \mapsto \hat{I}$, we can consider a bijection between the generating set of $\mathcal{B}_{Sp}$ and the generating set of $\mathcal{B}^{2n}_{2m,n}$:

$$\{\delta_{I'} + \mathcal{I}_{Sp}\colon I \in \mathcal{L}_{Sp}\} \quad \longleftrightarrow \quad \{\delta_{\hat{I}}\colon \hat{I} \in \mathcal{L}^{2n}_{2m,n}\}.$$

Then, to see that this bijection gives rise to an algebra isomorphism, let us show that the straightening relations among $\delta_{\hat{I}}$'s in $\mathcal{B}^{2n}_{2m,n}$ agree with those of $(\delta_{I'} + \mathcal{I}_{Sp})$'s in $\mathcal{B}_{Sp} \subset \mathcal{F}_{Sp}$.

As explained in Section 4.9, to express a product of $\delta_{I'}$'s as a linear combination of Sp-standard monomials projecting to the quotient $\mathcal{F}_{Sp} = \mathbb{C}[\mathsf{M}_{2m,m}]^{U_m}/\mathcal{I}_{Sp}$, we first apply the straightening relations in $\mathbb{C}[\mathsf{M}_{2m,m}]^{U_m}$ (Proposition 3.5.2) and then relations from the ideal $\mathcal{I}_{Sp}$.

For elements $I_i \in \mathcal{L}_{Sp} \subset \mathcal{L}\langle 2m\rangle$, the corresponding product $\prod_i \delta_{I'_i}$, as an element in $\mathbb{C}[\mathsf{M}_{2m,m}]^{U_m}$, can be expressed as a linear combination of GL_{2m}-standard monomials:

$$\prod_i \delta_{I'_i} = \sum_r c_r \prod_{j\geqslant 1} \delta_{K'_{r,j}} \tag{4.10.1}$$

in $\mathbb{C}[\mathsf{M}_{2m,m}]^{U_m}$. Now we claim that for each non-zero term $\prod_j \delta_{K'_{r,j}}$, its indices $K_{r,j}$'s form a multiple chain in $\mathcal{L}_{Sp}$, i.e., the monomial $\prod_j \delta_{K'_{r,j}}$ is already Sp-standard. Therefore, the expression (4.10.1) provides the Sp-standard monomial expression of $\prod_i \delta_{I'_i}$ projecting to $\mathcal{B}_{Sp} \subset \mathcal{F}_{Sp}$. This follows directly from the quadratic relation (3.5.2), that is, for $I, J \in \mathcal{L}_{Sp}$,

$$\delta_{I'}\delta_{J'} = \sum_r c_r \delta_{S'_r}\delta_{T'_r}.$$

On the right-hand side, for each non-zero term $\delta_{S'_r}\delta_{T'_r}$, the chain $S_r \preccurlyeq T_r$ satisfies the condition $S_r \not\succcurlyeq J_0$ and $T_r \not\succcurlyeq J_0$ in Definition 4.8.2. This can be easily seen from the statement (2) of Corollary 3.5.4 and the fact that I and J from $\mathcal{L}_{Sp}$ do not contain v_h for $1 \leqslant h \leqslant n$.

Moreover, by Theorem 4.10.1 and Proposition 4.3.1, the (F, D)-graded components of both algebras are of the same dimension, and they have $\mathbb{C}$-bases labeled by the same patterns for all $(F, D) \in \Lambda_{n,n}$. Therefore, two graded algebras are isomorphic to each other. □

With this characterization $\mathcal{B}_{Sp} \cong \mathcal{B}^{2n}_{2m,n}$, from Theorem 3.8.1, we have

Corollary 4.10.3. *The stable range branching algebra $\mathcal{B}_{Sp}$ for (Sp_{2m}, Sp_{2n}) is a flat deformation of the Hibi algebra $\mathcal{H}_{Sp}$ for (Sp_{2m}, Sp_{2n}), which is isomorphic to $\mathcal{H}^{2n}_{2m,n}$.*

5. Stable range branching algebra for (SO_p, SO_q)

Through out this section, for $m > n \geqslant 2$, we set

$$p = 2m+1 \quad \text{or} \quad 2m;$$
$$q = 2n+1 \quad \text{or} \quad 2n;$$
$$k = \begin{cases} n & \text{if } q = 2n+1, \\ n-1 & \text{if } q = 2n. \end{cases}$$

Following the same techniques we developed for the symplectic groups, we construct the stable range branching algebra $\mathcal{B}_{SO}$ for (SO_p, SO_q). The results and their proofs in this section are analogous to the case of (Sp_{2m}, Sp_{2n}).

5.1. Let us review a labeling system for the irreducible rational representations of SO_p (cf. [9, §3.1.4]). For the even orthogonal group O_{2m} of rank m, every Young diagram F with $\ell(F) < m$ can label exactly one irreducible representation σ^F_{2m}, which can be also realized as an SO_{2m} irreducible representation. A diagram of length m labels an irreducible representation of O_{2m} which decomposes into two irreducible representations of SO_{2m}. For the odd special orthogonal group SO_{2m+1} of rank m, every irreducible rational representation σ^F_{2m+1} can be uniquely labeled by a Young diagram F with $\ell(F) \leqslant m$. Then these representations are also O_{2m+1}-irreducible.

5.2. Let $J_m = (j_{a,b})$ be the $m \times m$ matrix such that $j_{a,m+1-a} = 1$ for $1 \leqslant a \leqslant m$ and 0 otherwise. Then we define the special orthogonal groups SO_{2m} and SO_{2m+1} as the subgroups of SL_{2m} and SL_{2m+1} preserving the symmetric bilinear forms on $\mathbb{C}^{2m}$ and $\mathbb{C}^{2m+1}$ induced by

$$\begin{bmatrix} 0 & J_m \\ J_m & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & J_m \\ 0 & 1 & 0 \\ J_m & 0 & 0 \end{bmatrix}$$

respectively where 0's are the zero matrices of proper sizes. Then, the pairs (e_j, e_{p+1-j}) of the elementary basis elements for $\mathbb{C}^p$ make isotropic pairs with respect to the above symmetric bilinear form. Also, the subgroup of upper triangular matrices with 1's on the diagonal can be taken as a maximal unipotent subgroup of SO_p. We will denote it by U_{SO_p}.

For $m > n$, let us identify SO_{2n} as the subgroup of SO_p preserving the symmetric bilinear form on the subspace of $\mathbb{C}^p$ spanned by $\{e_j, e_{p+1-j}\colon 1 \leqslant j \leqslant n\}$. Then we can embed SO_{2n} in SO_p as follows

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \to \begin{bmatrix} X & 0 & Y \\ 0 & I & 0 \\ Z & 0 & W \end{bmatrix}$$

where X, Y, Z, W are blocks of size $n \times n$, I is the $(p-2n) \times (p-2n)$ identity matrix, and 0's are the zero matrices of proper sizes. Similarly, we embed SO_{2n+1} in SO_{2m+1} by considering the $(2n+1)$-dimensional subspace of $\mathbb{C}^{2m+1}$ spanned by $\{e_j, e_{2m+2-j}\colon 1 \leqslant a \leqslant n\}$ and e_{m+1}. For SO_{2n+1} in SO_{2m}, we use the $(2n+1)$-dimensional subspace of $\mathbb{C}^{2m}$ spanned by $\{e_j, e_{2m+1-j}\colon 1 \leqslant j \leqslant n\}$ and $(e_m + e_{m+1})$.

5.3. Our next task is to construct an affine semigroup encoding stable range branching rules for (SO_p, SO_q). Note that $(f_1, \ldots, f_m) \in \mathbb{Z}^m$ is a dominant weight for SO_{2m+1} and SO_{2m}, if $f_1 \geqslant \cdots \geqslant f_m \geqslant 0$ and $f_1 \geqslant \cdots \geqslant f_{m-1} \geqslant |f_m| \geqslant 0$ respectively.

Lemma 5.3.1. *(See [9, Theorems 8.1.3 and 8.1.4].)*

(1) *Let $F = (f_1, \ldots, f_m)$ and $D = (d_1, \ldots, d_m)$ be dominant weights for SO_{2m+1} and SO_{2m} respectively. Then the branching multiplicity of σ^D_{2m} in σ^F_{2m+1} is equal to* 1 *if $(d_1, \ldots, |d_m|)$ interlaces $(f_1, \ldots, f_m)$, i.e.,*

$$\begin{matrix} f_1 & & f_2 & & \cdots & & f_{m-1} & & f_m & \\ & d_1 & & d_2 & & \cdots & & d_{m-1} & & |d_m| \end{matrix}$$

and 0 *otherwise*;

(2) *Let $F = (f_1, \ldots, f_m)$ and $D = (d_1, \ldots, d_{m-1})$ be dominant weights for SO_{2m} and SO_{2m-1} respectively. Then the branching multiplicity of σ^D_{2m-1} in σ^F_{2m} is equal to* 1 *if $(d_1, \ldots, d_m)$ interlaces $(f_1, \ldots, |f_m|)$, i.e.,*

$$\begin{matrix} f_1 & & f_2 & & \cdots & & f_{m-1} & & |f_m| \\ & d_1 & & d_2 & & \cdots & & d_{m-1} & \end{matrix}$$

and 0 *otherwise.*

By iterating these results, we may obtain patterns counting the branching multiplicities for (SO_p, SO_q). Such patterns are different from the GT patterns for (GL_p, GL_q). Within the stable range, however, they are the same as the ones for (GL_p, GL_q) with restrictions on lengths. That is because, as in the case for the symplectic groups, the length restriction $\ell(F) \leqslant k$ forces $\ell(D) \leqslant k$ via the interlacing conditions in Lemma 5.3.1. Therefore, as is shown in Proposition 4.3.1 for the symplectic groups, we have

Proposition 5.3.2. *Let F and D be Young diagrams with $F \supseteq D$ and $\ell(F) \leqslant k$. Then the branching multiplicity $m(\sigma_q^D, \sigma_p^F)$ is equal to the number of elements in $\mathcal{P}_p^q(F, D)$, and therefore it is equal to the number of elements in $\mathcal{T}_p^q(F, D)$.*

As in the case of (GL_p, GL_q) in (2.7.1), we can consider the affine semigroup $\mathcal{P}_{p,k}^q$ of the order preserving maps from the GT poset $\Gamma_{p,k}^q$ of length k:

$$\begin{array}{ccccccccc}
x_1^{(p)} & & x_2^{(p)} & & \cdots & x_k^{(p)} & & & \\
& x_1^{(p-1)} & & x_2^{(p-1)} & & \cdots & x_k^{(p-1)} & & \\
& & \ddots & & \ddots & \cdots & & \ddots & \\
& & & x_1^{(q)} & & x_2^{(q)} & \cdots & & x_k^{(q)}
\end{array}$$

to non-negative integers. We call $\mathcal{P}_{p,k}^q$ the *semigroup for* (SO_p, SO_q), and define its associated semigroup algebra:

$$\mathbb{C}\big[\mathcal{P}_{p,k}^q\big] = \bigoplus_{(F,D)\in\Lambda_{k,k}} \mathbb{C}\big[\mathcal{P}_p^q\big]_{(F,D)}$$

and call it the *semigroup algebra for* (SO_p, SO_q).

5.4. Let us define the distributive lattice for (SO_p, SO_q) and study its Hibi algebra. We shall closely follow the construction developed in Section 4.4 for the symplectic groups. Consider the ordered letters:

$$\begin{aligned}
&\langle 2m\rangle = \{u_1 < v_1 < u_2 < v_2 < \cdots < u_m < v_m\}, \qquad (5.4.1)\\
&\langle 2m+1\rangle = \{u_1 < v_1 < u_2 < v_2 < \cdots < u_m < v_m < \infty\}
\end{aligned}$$

for $p = 2m$ and $2m+1$ respectively.

If we let $\mathcal{L}\langle p\rangle$ denote the set of all non-empty subsets J of $\langle p\rangle$, then on $\mathcal{L}\langle p\rangle$ we can also impose the tableau order $\preccurlyeq$ as in Section 2.2 and Section 4.4. Then $\mathcal{L}\langle p\rangle$ is a distributive lattice isomorphic to $\mathcal{L}_p$, as in the case of the symplectic groups, through the bijection (4.4.2) (and $\iota(\infty) = 2m+1$ for $p = 2m+1$).

Then, we define $\mathcal{L}\langle n, q, p\rangle$ to be the set of non-empty subsets J of $\mathcal{L}\langle p\rangle$ of the following forms:

$$\begin{aligned}
&[u_1, u_2, \ldots, u_c, y_1, y_2, \ldots, y_s],\\
&[u_1, u_2, \ldots, u_c],\\
&[y_1, y_2, \ldots, y_s] \qquad (5.4.2)
\end{aligned}$$

where $c \leqslant n$ and, for $q = 2n$ and $2n+1$,

$$\begin{aligned}
&u_{n+1} \leqslant y_1 < y_2 < \cdots < y_s;\\
&v_{n+1} \leqslant y_1 < y_2 < \cdots < y_s,
\end{aligned}$$

respectively. In particular, if $u_c \in J$ for $c \leqslant n$, then $\{u_h\colon 1 \leqslant h \leqslant c\} \subset J$.

Now, let $\mathcal{L}\langle n, q, p\rangle_k$ be the subset of $\mathcal{L}\langle n, q, p\rangle$ consisting of J with $|J| \leqslant k$. Then, as is the case for the symplectic groups (Section 4.4), we can identify $\mathcal{L}\langle n, q, p\rangle_k$ with the distributive lattice $\mathcal{L}_{p-q+n,k}^n$, and therefore with $\mathcal{L}_{p,k}^q$ by Corollary 2.3.2.

Definition 5.4.1. The distributive lattice for (SO_p, SO_q) is $\mathcal{L}\langle n, q, p\rangle_k$, and it will be denoted by $\mathcal{L}_{SO}$.

$$\begin{aligned}
\mathcal{L}_{SO} &= \mathcal{L}\langle n, q, p\rangle_k\\
&\cong \mathcal{L}_{p,k}^q.
\end{aligned}$$

Then we define the *Hibi algebra for* (SO_p, SO_q), denoted by $\mathcal{H}_{SO}$, to be the Hibi algebra over the distributive lattice $\mathcal{L}_{SO}$. From the isomorphism of distributive lattices, we have $\mathcal{H}_{SO} \cong \mathcal{H}^q_{p,k}$. Then from Proposition 2.10.3 for (GL_p, GL_q), we have

Corollary 5.4.2. *There is an algebra isomorphism*

$$\mathcal{H}_{SO} \cong \mathbb{C}\big[\mathcal{P}^q_{p,k}\big].$$

5.5. As in the previous cases (Section 2.10), we shall identify the monomials in the Hibi algebra $\mathcal{H}_{SO}$ with tableaux whose columns are elements of $\mathcal{L}_{SO}$.

Definition 5.5.1. A standard tableau t for (SO_p, SO_q) is a multiple chain $I_1 \preccurlyeq \cdots \preccurlyeq I_s$ in $\mathcal{L}_{SO}$. The shape $sh_n(\mathrm{t})$ of t is F/D where $F = (|I_1|, \ldots, |I_s|)^t$ and $D = (d_1, \ldots, d_n)$ with d_r being the number of u_r's in t for $1 \leqslant r \leqslant n$.

We write $\mathcal{T}_{SO}(F, D)$ for the set of all standard tableaux for (SO_p, SO_q) whose shapes are F/D, and set

$$\mathcal{T}_{SO} = \bigcup_{(F,D)\in\Lambda_{k,k}} \mathcal{T}_{SO}(F, D).$$

Then, as in the case of the symplectic groups, $\mathcal{T}_{SO}$ gives rise to a $\mathbb{C}$-basis for the Hibi algebra for (SO_p, SO_q).

Proposition 5.5.2.

(1) *The Hibi algebra $\mathcal{H}_{SO}$ for (SO_p, SO_q) is graded by $\Lambda_{k,k}$ and $\mathcal{T}_{SO}(F, D)$ forms a $\mathbb{C}$-basis of the graded component $\mathcal{H}_{SO}(F, D)$.*
(2) *For $(F, D) \in \Gamma_{k,k}$, the number of standard tableaux for (SO_p, SO_q) of shape F/D is equal to the branching multiplicity $m(\sigma^D_q, \sigma^F_p)$ of σ^D_q in σ^F_p.*

Proof. From the isomorphism $\mathcal{L}_{SO} \cong \mathcal{L}^q_{p,k}$, it is straightforward to see that there is a bijection between $\mathcal{T}_{SO}(F, D)$ and $\mathcal{T}^q_p(F, D)$. Then (1) follows from Lemma 2.10.2 and (2) follows from Proposition 5.3.2. □

5.6. We can also find a correspondence between $\mathcal{L}_{SO}$ and the set of order increasing subsets of the GT poset $\Gamma^q_{p,k}$ in the same way explained in Section 4.6. Namely, define the order increasing subset A_I of $\Gamma^q_{p,k}$ corresponding to $I \in \mathcal{L}_{SO}$ as

$$A_I = \bigcup_{q\leqslant j\leqslant p} \big\{x^{(j)}_1, x^{(j)}_2, \ldots, x^{(j)}_{s_j}\big\} \tag{5.6.1}$$

where, for $n+1 \leqslant h \leqslant m$, s_{2h-1} and s_{2h} are the numbers of elements in I less than or equal to u_h and v_h respectively; and s_{2n} is the number of elements in I less than v_n and s_{2m+1} is the number of elements in I. Then every element of $\mathcal{T}_{SO}$ can be related to a sum of characteristic functions of these order increasing subsets as given in Proposition 2.8.1 and (2.8.1). This gives a direct proof for Corollary 5.4.2.

5.7. To construct the stable range branching algebra for (SO_p, SO_q), we review the polynomial model of SO_p-representation spaces studied in [19].

From (3.4.1), $\mathbb{C}[\mathsf{M}_{p,m}]^{U_m}$ consists of GL_p-irreducible representations ρ^F_p with $\ell(F) \leqslant m$. By taking O_p as a subgroup of GL_p, we let $O_p \times GL_m$ act on the space $\mathsf{M}_{p,m} \cong \mathbb{C}^p \otimes \mathbb{C}^m$ via the action of $GL_p \times GL_m$ given in (3.3.1). Then we take the quotient of $\mathbb{C}[\mathsf{M}_{p,m}]^{U_m}$ by the ideal $\mathcal{I}_O = \sum_F \mathcal{I}^F$ where

$\mathcal{I}^F$ is the O_p-invariant complement space to the O_p-irreducible representation σ_p^F in ρ_p^F, i.e., $\rho_p^F = \sigma_p^F \oplus \mathcal{I}^F$ for each F (cf. [5, §19.5]).

Then [19] shows that this quotient algebra can be taken as a polynomial model for the flag algebra for SO_p in that it contains exactly one copy of each irreducible representation σ_p^F with $\ell(F) \leqslant m$:

$$\begin{aligned}\mathcal{F}_{SO} &= \mathbb{C}[\mathsf{M}_{p,m}]^{U_m}/\mathcal{I}_O \\ &= \sum_{\ell(F)\leqslant m} \sigma_p^F\end{aligned}$$

and it is graded by Young diagrams, i.e., $\sigma_p^{F_1} \cdot \sigma_p^{F_2} \subset \sigma_p^{F_1+F_2}$. We note that σ_{2m}^F with $\ell(F) = m$ are irreducible O_{2m} representations, but they are not irreducible as SO_{2m} representations.

To take the stable range $\ell(F) \leqslant k$, we consider its subalgebra consisting of σ_p^F with $\ell(F) \leqslant k$:

$$\mathcal{F}_{SO}^{(k)} = \sum_{\ell(F)\leqslant k} \sigma_p^F. \tag{5.7.1}$$

5.8. To describe generators of $\mathcal{F}_{SO}$, to each $I = [w_1, \ldots, w_r] \in \mathcal{L}\langle p\rangle$, we attach a determinant function $\delta_{I'}$ as follows.

For $Q \in \mathsf{M}_{p,m}$, we let $\delta_{I'}(Q)$ denote the determinant of the submatrix of $Q = (t_{a,b})$ obtained by taking the $i'_1, i'_2, \ldots, i'_r$-th rows and the $1, 2, \ldots, r$-th columns:

$$\delta_{I'}(Q) = \det \begin{bmatrix} t_{i'_1 1} & t_{i'_1 2} & \cdots & t_{i'_1 r} \\ t_{i'_2 1} & t_{i'_2 2} & \cdots & t_{i'_2 r} \\ \vdots & \vdots & \ddots & \vdots \\ t_{i'_r 1} & t_{i'_r 2} & \cdots & t_{i'_r r} \end{bmatrix} \tag{5.8.1}$$

where is $\{i'_1, i'_2, \ldots, i'_r\}$ is the image of the set $\{w_1, w_2, \ldots, w_r\} \subset \langle p\rangle$ under ψ_p:

$$\begin{aligned}&\psi_{2m} : \{u_1, v_1, \ldots, u_m, v_m\} \to \{1, 2, \ldots, 2m\}, \\ &\psi_{2m}(u_c) = c \quad \text{and} \quad \psi_{2m}(v_c) = 2m + 1 - c, \\ &\psi_{2m+1} : \{u_1, v_1, \ldots, u_m, v_m, \infty\} \to \{1, 2, \ldots, 2m, 2m + 1\}, \\ &\psi_{2m+1}(u_c) = c \quad \text{and} \quad \psi_{2m+1}(v_c) = 2m + 2 - c\end{aligned} \tag{5.8.2}$$

for $p = 2m$ and $2m + 1$ respectively, for $1 \leqslant c \leqslant m$ and $\psi_{2m+1}(\infty) = m + 1$.

Then, with the bijection ψ_p, we can impose a new order $\lessdot$ on $\{1, 2, \ldots, p\}$ induced by the order on $\langle p\rangle$ in (5.4.1):

$$\begin{aligned}&1 \lessdot 2m \lessdot 2 \lessdot 2m - 1 \lessdot \cdots \lessdot m \lessdot m + 1; \\ &1 \lessdot 2m + 1 \lessdot 2 \lessdot 2m \lessdot \cdots \lessdot m \lessdot m + 2 \lessdot m + 1\end{aligned}$$

and we keep using the convention of I', $\delta_{I'}$ and $\Delta_{\mathsf{t}'}$ used for the symplectic groups (Notation 4.8.1). This conversion procedure is to make our labeling (u_c, v_c) of isotropic pairs (Section 5.2) compatible with those used in [24,19].

To $I = [w_1, \ldots, w_s] \in \mathcal{L}\langle p\rangle$, we attach a determinant function $\delta_{I'}$ as we define in (5.8.1). For a multiple chain $\mathsf{t} = (I_1 \preccurlyeq \cdots \preccurlyeq I_r)$ of $\mathcal{L}\langle p\rangle$, let $\mathsf{t}(a, b)$ denote the a-th smallest element in the b-th column I_b of the tableau t. Also, let α_{2c} and β_{2c} be the numbers of elements less than or equal to v_c in I_1 and I_2 respectively.

Definition 5.8.1. (Cf. [24,28].) Then the corresponding monomial

$$\Delta_{\mathsf{t}'} = \delta_{I'_1}\delta_{I'_2}\cdots\delta_{I'_r} \in \mathbb{C}[\mathsf{M}_{p,m}]^{U_m}$$

is called an *O-standard monomial*, if, in the chain $\mathsf{t} = (I_1 \preccurlyeq \cdots \preccurlyeq I_r)$,

(1) $\alpha_{2c}+\beta_{2c}\leqslant 2c$ for $1\leqslant c\leqslant m$, and
(2) if $\alpha_{2c}+\beta_{2c}=2c$ for some c with $\mathrm{t}(\alpha_{2c},1)=u_c$ and $\mathrm{t}(\beta_{2c},b)=v_c$ for some b, then $\mathrm{t}(\beta_{2c}-1, b)=u_c$.

In [24] and [28], the above conditions (1) and (2) are used to define Young tableaux describing weight basis elements of irreducible O_p representations.

5.9. To a product of $\delta_{I'}$'s in $\mathbb{C}[\mathsf{M}_{p,m}]^{U_m}$, we apply the straightening relations in Proposition 3.5.2 to obtain a linear combination of standard monomials for GL_p:

$$\prod_i \delta_{I'_i}=\sum_r c_r\prod_{j\geqslant 1}\delta_{K'_{r,j}}.$$

If there is a non-zero term $\prod_j \delta_{K'_{r,j}}$ which is not an O-standard monomial, then apply relations from the ideal $\mathcal{I}_O$. This replaces the entries of $K_{r,j}$'s corresponding to isotropic pairs (u_a, v_a) with the sum of pairs (u_b, v_b)'s (and (∞,∞) for $p=2m+1$) for $a\leqslant b$, thereby expressing $\prod_j \delta_{K'_{r,j}}$ as a linear combination of O-standard monomials. For further details, we refer to [19]. A combinatorial description of this straightening procedure in the language of tableaux is given in [24].

The following is shown in [19]. See also [24] and [28].

Proposition 5.9.1. *(See [19, Theorem 3.6, Proposition 3.9].) O-standard monomials project to a $\mathbb{C}$-basis of the flag algebra $\mathcal{F}_{SO}$ for SO_p. In particular, for a Young diagram F with $\ell(F)\leqslant m$, O-*standard monomials *of shape F form a weight basis for the O_p-irreducible representation $\sigma_p^F\subset\mathcal{F}_{SO}$.*

5.10. Our next task is to find an explicit model for the U_{SO_q}-invariant subalgebra of $\mathcal{F}_{SO}^{(k)}$, which we will denote by $\mathcal{B}_{SO}$. Then, from (1.3.1) and (5.7.1), we have

$$\begin{aligned}\mathcal{B}_{SO}&=\sum_{\ell(F)\leqslant k}\left(\tau_p^F\right)^{U_{SO_q}}\\&=\sum_{\ell(F)\leqslant k}\sum_D m\left(\tau_q^D,\tau_p^F\right)\left(\tau_q^D\right)^{U_{SO_q}}.\end{aligned}$$

Theorem 5.10.1. *The algebra $\mathcal{B}_{SO}$ is generated by*

$$\mathcal{G}=\{\delta_{I'}+\mathcal{I}_{SO}\colon I\in\mathcal{L}_{SO}\},$$

and it is graded by $\Lambda_{n,n}$. For each $(F,D)\in\Lambda_{k,k}$ the O-standard monomials $\Delta_{\mathrm{t}'}$ corresponding to standard tableaux t *for (SO_p,SO_q) whose shapes are F/D form a $\mathbb{C}$-basis of the (F,D)-graded component. The dimension of the (F,D)-graded component is equal to the branching multiplicity of τ_q^D in τ_p^F.*

Proof. Let $\mathcal{R}$ be the subalgebra of $\mathcal{F}_{SO}^{(k)}$ generated by $\mathcal{G}$. We will show that $\mathcal{R}=\mathcal{B}_{SO}$. For $I\in\mathcal{L}_{SO}\subset\mathcal{L}\langle p\rangle$, we defined the polynomial $\delta_{I'}$ on the space $\mathsf{M}_{p,m}$ in (5.8.1). By (5.4.2) and (5.8.2), it is the determinant of a submatrix of $Q\in\mathsf{M}_{2m,m}$ obtained by taking consecutive columns $\{1,2,\dots,|I|\}$, and either consecutive rows $\{1,2,\dots,r\}$ or partially consecutive rows $\{1,2,\dots,r\}\cup\{b_1,\dots,b_s\}$ or only $\{b_1,\dots,b_s\}$ of Q for $r\leqslant n$ and $b_i\in\{n+1,n+2,\dots,p-n\}$.

Since the left action of $U_q\subset GL_p$, under the embedding given in Section 5.2, operates the rows of $\mathsf{M}_{p,m}$, all the determinants $\delta_{I'}$ for $I\in\mathcal{L}_{SO}$ are invariant under the action of U_q, and therefor invariant under the action of U_{SO_q}. Since the ideal $\mathcal{I}_O$ is stable under the action of O_p, the generators of the algebra $\mathcal{R}$ are invariant under the unipotent subgroup U_{SO_q} of SO_q, and so are their products. Also, since every $I\in\mathcal{L}_{SO}$ satisfies $|I|\leqslant k$, we have $\mathcal{R}\subseteq\mathcal{B}_{SO}$.

On the other hand, for every chain $I\preccurlyeq J$ in $\mathcal{L}_{SO}$, $\delta_{I'}\delta_{J'}$ satisfies the conditions (1) and (2) in Definition 5.8.1. This can be easily seen from the statement (3) of Corollary 3.5.4 and the fact that I

and J from $\mathcal{L}_{SO}$ do not contain v_h for $1 \leqslant h \leqslant n$. This implies that standard monomials $\Delta_{t'}$ corresponding to standard tableaux t for (SO_p, SO_q) project to linearly independent elements in the algebra $\mathcal{B}_{SO} \subseteq \mathcal{F}_{SO}$. They span the whole algebra $\mathcal{B}_{SO}$, because for each $(F, D) \in \Lambda_{k,k}$ the number of standard tableaux in $\mathcal{T}_{SO}(F, D)$ is equal to the multiplicity of τ_q^D in τ_p^F by Proposition 5.3.2. Furthermore, they are scaled by weight D under the action of the diagonal subgroup $\{\mathrm{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1})\}$ or $\{\mathrm{diag}(a_1, \ldots, a_n, 1, a_n^{-1}, \ldots, a_1^{-1})\}$ of SO_q. This shows that standard monomials $\Delta_{t'}$ with $t \in \mathcal{T}_{SO}(F, D)$ are the highest weight vectors of the copies of τ_q^D in τ_p^F. $\square$

In this sense, we call $\mathcal{B}_{SO}$ the *stable range branching algebra for* (SO_p, SO_q). Recall that we obtained $\mathcal{B}_{SO}$ by lifting the elements of the Hibi algebra $\mathcal{H}_{SO}$ over the distributive lattice $\mathcal{L}_{SO}$ which is isomorphic to the distributive lattice $\mathcal{L}_{p,k}^q$. Now we compare it with the algebra $\mathcal{B}_{p,k}^q$ (Definition 3.6.1) obtained from the Hibi algebra $\mathcal{H}_{p,k}^q$ for the general linear groups.

Proposition 5.10.2. *The stable range branching algebra $\mathcal{B}_{SO}$ for (SO_p, SO_q) is isomorphic to the length k branching algebra $\mathcal{B}_{p,k}^q$ for (GL_p, GL_q).*

Proof. From the isomorphism $\mathcal{L}_{SO} \cong \mathcal{L}_{p,k}^q$ of distributive lattices, with $I \mapsto \hat{I}$, we can consider a bijection between the generating set of $\mathcal{B}_{SO}$ and the generating set of $\mathcal{B}_{p,k}^q$:

$$\{\delta_{I'} + \mathcal{I}_O \colon I \in \mathcal{L}_{SO}\} \quad \longleftrightarrow \quad \big\{\delta_{\hat{I}} \colon \hat{I} \in \mathcal{L}_{p,k}^q\big\}.$$

Then, to see that this bijection gives rise to an algebra isomorphism, let us show that the straightening relations among $\delta_{\hat{I}}$'s in $\mathcal{B}_{p,k}^q$ agree with those of $(\delta_{I'} + \mathcal{I}_O)$'s in $\mathcal{B}_{SO} \subset \mathcal{F}_{SO}$.

As explained in Section 5.9, to express a product of $\delta_{I'}$'s as a linear combination of O-standard monomials projecting to the quotient $\mathcal{F}_{SO} = \mathbb{C}[\mathsf{M}_{p,m}]^{U_m}/\mathcal{I}_O$, we first apply the straightening relations in $\mathbb{C}[\mathsf{M}_{p,m}]^{U_m}$ (Proposition 3.5.2) and then relations from the ideal $\mathcal{I}_O$.

A product of representatives $\prod_i \delta_{I'_i}$, as an element in $\mathbb{C}[\mathsf{M}_{p,m}]^{U_m}$, can be expressed as a linear combination of GL_p-standard monomials:

$$\prod_i \delta_{I'_i} = \sum_r c_r \prod_{j \geqslant 1} \delta_{K'_{r,j}} \tag{5.10.1}$$

in $\mathbb{C}[\mathsf{M}_{p,m}]^{U_m}$.

Now we claim that for each non-zero term $\prod_j \delta_{K'_{r,j}}$, the indices $K_{r,j}$'s form a multiple chain in $\mathcal{L}_{SO}$, therefore (5.10.1) gives O-standard monomial expression of $\prod_i \delta_{I'_i}$ projecting to $\mathcal{B}_{SO} \subset \mathcal{F}_{SO}$. This follows directly from the quadratic relation (3.5.2). For every chain $I \preccurlyeq J$ in $\mathcal{L}_{SO}$, $\delta_{I'}\delta_{J'}$ satisfies the conditions (1) and (2) in Definition 5.8.1, which can be easily seen from the statement (3) of Corollary 3.5.4 and the fact that I and J from $\mathcal{L}_{SO}$ do not contain v_h for $1 \leqslant h \leqslant n$.

Moreover, from Theorem 5.10.1 and Proposition 5.3.2, the (F, D)-graded components of both algebras are of the same dimension with bases labeled by the same patterns for all (F, D). This shows that two graded algebras are isomorphic to each other. $\square$

With this characterization $\mathcal{B}_{SO} \cong \mathcal{B}_{p,k}^q$, from Theorem 3.8.1, we have

Corollary 5.10.3. *The stable range branching algebra $\mathcal{B}_{SO}$ for (SO_p, SO_q) is a flat deformation of the Hibi algebra $\mathcal{H}_{SO}$ for (SO_p, SO_q), which is isomorphic to $\mathcal{H}_{p,k}^q$.*

Acknowledgments

The author thanks Roger Howe and Steven Glenn Jackson for insightful conversations regarding several aspects of this work. He also thanks the two anonymous reviewers for their constructive comments and suggestions.

References

[1] V. Alexeev, M. Brion, Toric degenerations of spherical varieties, Selecta Math. (N.S.) 10 (4) (2004) 453–478.
[2] A. Berele, Construction of Sp-modules by tableaux, Linear Multilinear Algebra 19 (4) (1986) 299–307.
[3] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge Stud. Adv. Math., vol. 39, Cambridge University Press, Cambridge, 1993.
[4] C. De Concini, D. Eisenbud, C. Procesi, Hodge Algebras, Astérisque, vol. 91, Société Mathématique de France, Paris, 1982.
[5] W. Fulton, J. Harris, Representation theory. A first course, in: Readings in Mathematics, in: Grad. Texts in Math., vol. 129, Springer-Verlag, New York, 1991.
[6] I.M. Gelfand, M.L. Tsetlin, Finite-dimensional representations of the group of unimodular matrices, Doklady Akad. Nauk SSSR (N.S.) 71 (1950) 825–828 (in Russian); English translation in: I.M. Gelfand, Collected Papers, vol. II, Springer-Verlag, Berlin, 1988, pp. 653–656, edited by S.G. Gindikin, V.W. Guillemin, A.A. Kirillov, B. Kostant and S. Sternberg.
[7] N. Gonciulea, V. Lakshmibai, Degenerations of flag and Schubert varieties to toric varieties, Transform. Groups 1 (3) (1996) 215–248.
[8] N. Gonciulea, V. Lakshmibai, Flag Varieties, Hermann-Acutalities Mathematiques, 2001.
[9] R. Goodman, N.R. Wallach, Symmetry, Representations, and Invariants, Grad. Texts in Math., vol. 255, Springer, Dordrecht, 2009.
[10] T. Hibi, Distributive lattices, affine semigroup rings and algebras with straightening laws, in: Commutative Algebra and Combinatorics, Kyoto, 1985, in: Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 93–109.
[11] W.V.D. Hodge, Some enumerative results in the theory of forms, Proc. Cambridge Philos. Soc. 39 (1943) 22–30.
[12] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, in: The Schur Lectures, Tel Aviv, 1992, in: Israel Math. Conf. Proc., vol. 8, Bar-Ilan Univ., Ramat Gan, 1995, pp. 1–182.
[13] R. Howe, Weyl Chambers and standard monomial theory for poset lattice cones, Q. J. Pure Appl. Math. 1 (1) (2005) 227–239.
[14] R. Howe, S. Kim, S.T. Lee, Iterated Pieri algebras and Hibi algebras, in preparation.
[15] R. Howe, S.T. Lee, Bases for some reciprocity algebras. I, Trans. Amer. Math. Soc. 359 (9) (2007) 4359–4387.
[16] R. Howe, S. Jackson, S.T. Lee, E.-C. Tan, J. Willenbring, Toric degeneration of branching algebras, Adv. Math. 220 (6) (2009) 1809–1841.
[17] R. Howe, E.-C. Tan, J. Willenbring, Stable branching rules for classical symmetric pairs, Trans. Amer. Math. Soc. 357 (4) (2005) 1601–1626.
[18] S. Kim, Standard monomial theory for flag algebras of $GL(n)$ and $Sp(2n)$, J. Algebra 320 (2) (2008) 534–568.
[19] S. Kim, Standard monomial bases and degenerations of $SO_m(\mathbb{C})$ representations, J. Algebra 322 (11) (2009) 3896–3911.
[20] S. Kim, S.T. Lee, Pieri algebras for the orthogonal and symplectic groups, Israel J. Math., in press.
[21] S. Kim, O. Yacobi, A basis for the symplectic group branching algebra, J. Algebraic Combin. 35 (2) (2012) 269–290.
[22] A.A. Kirillov, A remark on the Gelfand–Tsetlin patterns for symplectic groups, J. Geom. Phys. 5 (3) (1988) 473–482.
[23] M. Kogan, E. Miller, Toric degeneration of Schubert varieties and Gelfand–Tsetlin polytopes, Adv. Math. 193 (1) (2005) 1–17.
[24] R.C. King, T.A. Welsh, Construction of orthogonal group modules using tableaux, Linear Multilinear Algebra 33 (3–4) (1993) 251–283.
[25] G. Lancaster, J. Towber, Representation-functors and flag-algebras for the classical groups. I, J. Algebra 59 (1) (1979) 16–38.
[26] G. Lancaster, J. Towber, Representation-functors and flag-algebras for the classical groups. II, J. Algebra 94 (2) (1985) 265–316.
[27] E. Miller, B. Sturmfels, Combinatorial Commutative Algebra, Grad. Texts in Math., vol. 227, Springer-Verlag, New York, 2005.
[28] R.A. Proctor, Young tableaux, Gelfand patterns, and branching rules for classical groups, J. Algebra 164 (2) (1994) 299–360.
[29] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Stud. Adv. Math., vol. 49, Cambridge University Press, Cambridge, 1997.
[30] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Stud. Adv. Math., vol. 62, Cambridge University Press, Cambridge, 1999.
[31] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lecture Ser., vol. 8, American Mathematical Society, Providence, RI, 1996.
[32] E.B. Vinberg, The asymptotic semigroup of a semisimple Lie group, in: Semigroups in Algebra, Geometry and Analysis, Oberwolfach, 1993, in: de Gruyter Exp. Math., vol. 20, de Gruyter, Berlin, 1995, pp. 293–310.
[33] D.P. Zelobenko, Classical groups. Spectral analysis of finite-dimensional representations, Usp. Mat. Nauk 17 (1(103)) (1962) 27–120.
[34] D.P. Zelobenko, Compact Lie Groups and Their Representations, Transl. Math. Monogr., vol. 40, American Mathematical Society, Providence, RI, 1973.