On the Inverse Sturm–Liouville Problem for Spatially Symmetric Operators, I

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1. INTRODUCTION

In the previous work [11], we studied the inverse Sturm–Liouville problem. We simplified the proofs of theorems due to Borg, Levinson, Hochstadt, and Lieberman [1, 5, 3, 4]. In the present articles we study Sturm–Liouville operators of spatially symmetric type.

For \( p \in C^1[0, 1], h \in \Theta \) and \( H \in \Theta \), \( A_{p,h,H} \) denotes the realization in \( L^2(0, 1) \) of the differential operator \(- (d^2/dx^2) + p(x)\) with the boundary condition \( -(d/dx) + h \cdot 1 \cdot x = 1 \cdot x = 0 \). Let \( \sigma(A_{p,h,H}) = \{ \lambda_n \}_{n=0}^{\infty} \) be the eigenvalues of \( A_{p,h,H} \). As is well known, each \( \lambda_n \) is simple: \(-\infty < \lambda_0 < \lambda_1 < \cdots \to \infty \). Put

\[
C^1_\gamma[0, 1] = \{ p \in C^1[0, 1] \mid p(1-x) = p(x) \ (0 < x < 1) \}. \tag{1.1}
\]

We say that \( A_{p,h,H} \) is a (spatially) symmetric operator iff \( p \in C^1_\gamma[0, 1] \) and \( h = H \).

The following theorem was obtained by Borg [1], whose proof was later simplified by Levinson [5] and Hochstadt [3].

Let a symmetric operator \( A_{p,h,h} \) be given, and let \( \{ \lambda_n \}_{n=0}^{\infty} \ (-\infty < \lambda_0 < \lambda_1 < \cdots \to \infty) \) be \( \sigma(A_{p,h,h}) \), the eigenvalues of \( A_{p,h,h} \). Furthermore let \( \{ \mu_m \}_{m=0}^{\infty} \ (-\infty < \mu_0 < \mu_1 < \cdots \to \infty) \) be the eigenvalues of another symmetric operator \( A_{q,j,j} \). Then,

**THEOREM 0.** The relation

\[
\mu_n = \lambda_n \quad (n = 1, 2, \ldots) \tag{1.2}
\]

combined with

\[
j = h \tag{1.3}
\]

implies

\[
q = p. \tag{1.4}
\]

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In other words, a symmetric operator is, roughly speaking, determined by its eigenvalues.

We note here that $\mu_0 = \lambda_0$ is not assumed in (1.2), while (1.3) is assumed in Theorem 0, and arrive at the following questions:

(a) Without (1.3), does $P_\nu = 1\nu$ imply $q = P_j = h$?

(b) For $n_1 \in \mathcal{N} \equiv \{0, 1, 2, \ldots\}$ with $n_1 \neq 0$, does (1.3) and $P_n \equiv 1$ imply (1.4)?

Our theorems show that (a) is always affirmative, while (b) is, in general, negative. In the next section, we shall give precise statements of them.

2. Summary

Recall $C^1[0, 1] = \{ p \in C[0, 1] \mid p(1 - x) = p(x) \ (0 \leq x \leq 1) \}$. We henceforth adopt the notation:

For $p \in C^1[0, 1]$ and $h \in \mathcal{R}$, $A_{p, h, h}$ denotes the symmetric operator $-(d^2/dx^2) + p(x)$ in $L^2(0, 1)$ with the boundary condition $(-(d/dx) + h) \cdot x = 0 = (d/dx) + h) \cdot x = 0$. $\sigma(A_{p, h, h}) = \{ \lambda_n \}_{n=0}^\infty (-\infty < \lambda_0 < \lambda_1 < \cdots \rightarrow \infty)$, the eigenvalues of $A_{p, h, h}$, are denoted by $\lambda_n = \lambda_n(p, h) (n = 0, 1, 2, \ldots)$.

Let a symmetric operator $A_{p, h, h}$ ($p \in C^1[0, 1], h \in \mathcal{R}$) and a set $\Sigma \subset \mathcal{N} \equiv \{0, 1, 2, \ldots\}$ be given, and put

$$\hat{\mathcal{Q}}_{p, h, \Sigma} = \{ (q, j) \in C^1[0, 1] \times \mathcal{R} \mid \lambda_n(q, j) = \lambda_n(p, h) (n \in \mathcal{N} \setminus \Sigma) \}. \quad (2.1)$$

$\hat{\mathcal{Q}}_{p, h, \Sigma}$ denotes the totality of symmetric operators $A_{q, j, j}$, whose eigenvalues $\lambda_n(q, j)$ coincide with those of $A_{p, h, h}$, except for $n \in \Sigma$. Our answer to question (a) is contained in

**Theorem A.**

$$\hat{\mathcal{Q}}_{p, h, \Sigma} = \{(p, h)\} \quad (2.2)$$

if and only if $\Sigma = \phi$. 
Namely, for symmetric operators, (1.2') implies (1.4'), while (1.2"") without (1.3) does not imply (1.4') for any \( n_1 \in \mathcal{N} \). Therefore, \( \sigma(A_{p,h,h}) \) completely characterizes \( A_{p,h,h} \) in the case of symmetric operators. In Theorem 0, the assumption \( j = h \) recovers the condition \( \lambda_{n_1}(q, j) = \lambda_{n_1}(p, h) \) for \( n_1 = 0 \), and derives \( (q, j) = (p, h) \) with (1.2).

To answer question (\( \beta \)), we set

\[
Q_{p,h,n_1} = \{ q \in C^1_t[0, 1] | \lambda_n(q, h) = \lambda_n(p, h) \ (n \neq n_1) \}
\]

(\( = \text{Proj}_1[\hat{Q}_{p,h,(n_1)}] \cap \{(q, j)|j = h\} \)), for \( p \in C^1_s[0, 1] \), \( h \in \mathcal{R} \) and \( n_1 \in \mathcal{N} \). Then, we have

**Theorem B.** (i) If \( n_1 = 0 \), then

\[
Q_{p,h,n_1} = \{ p \}.
\]

(ii) If \( n_1 \geq 1 \), then

\[
Q_{p,h,n_1} = \{ p, p - 2(d^2/dx^2) \log(W) \},
\]

where \( W = W(x; p, h, n_1) \) is a function defined below.

**Proof.** Part (i) of Theorem B is nothing but Theorem 0, while (ii) of Theorem B is new. Our proof of (i) is different from those of [1, 5, 3], and is related to the proof of (ii).

To define \( W \), we need a few notations, which also will be used frequently in later sections.

**Notation 2.** For \( p \in C^1[0, 1] \), \( h \in \mathcal{R} \), and \( \lambda \in \mathcal{R} \), \( \phi = \phi(x; p, h, \lambda) \) and \( \phi^* - \phi^*(x; p, \lambda) \) denote the solutions of

\[
\left( -\frac{d^2}{dx^2} + p(x) \right) \phi = \lambda \phi \quad (0 \leq x \leq 1), \ \phi(0) = 1, \ \phi'(0) = h,
\]

and

\[
\left( -\frac{d^2}{dx^2} + p(x) \right) \phi^* = \lambda \phi^* \quad (0 \leq x \leq 1), \ \phi^*(0) = 0, \ \phi^*(0) = 1,
\]

respectively. Henceforth, ' means \( d/dx \).

For \( \lambda = \lambda_n(p, h) \ (n \in \mathcal{N}) \), \( \phi(\cdot; p, h, \lambda) \) becomes an eigenfunction of \( A_{p,h,h} \). On the other hand, \( \langle \phi, \phi^* \rangle \) gives a fundamental system of the solutions of \( (-(d^2/dx^2) + p(x)) \psi = \lambda \psi \).

**Notation 3.** For \( p \in C^1[0, 1] \), \( A_p^* \) denotes the realization in \( L^2(0, 1) \) of the differential operator \( -(d^2/dx^2) + p(x) \) with the (Dirichlet) boundary
The eigenvalues of $A_p^*$, $\sigma(A_p^*) = \{\lambda_n^*\}_{n=1}^{\infty}$ ($-\infty < \lambda_1^* < \lambda_2^* < \cdots \to \infty$), are denoted by $\lambda_n^* = \lambda_n^*(p)$ for $n \in \mathcal{N}^* \equiv \{1, 2, \ldots\}$.

Note that in our notation, $\sigma(A_{p,h,h})$ are numbered from 0, while $\sigma(A_p^*)$ are numbered from 1. For $\lambda = \lambda_n^*(p) (n \geq 1)$, $\phi^*(\cdot; p, \lambda)$ becomes an eigenfunction of $A_p^*$.

Now, the function $W$ is defined as

Notation 4. For $n_1 \geq 1$, $p \in C^1[0, 1]$, and $h \in \mathcal{R}$,

$$W = W(x; p, h, n_1) = \phi^*(x; p, \lambda_n^*(p)) \phi(x; p, h, \lambda_n(p, h)) - \phi^*(x; p, \lambda_n^*(p)) \phi'(x; p, h, \lambda_n(p, h)).$$ (2.8)

As will be shown in later sections, $p \in C^1[0, 1]$ implies $(d^2/dx^2) \log(W) \in C_c[0, 1]$. In the case of $\lambda_n^*(p) = \lambda_n(p, h)$, $W$ becomes the Wronskian for $-\psi'' + p\psi = \lambda \psi$ ($\lambda = \lambda_n^*(p) = \lambda_n(p, h)$), hence $W' = 0$ and so $(d^2/dx^2) \log(W) = (W'/W)' = 0$. We have, conversely, that $(W'/W)' = 0$ implies $\lambda_n^* = \lambda_n(p, h)$, which will be also proved later, so that we obtain

**Corollary.**

$$Q_{p,h,n_1} = \{ p \}$$ (2.9)

if and only if either

(i) $n_1 = 0$

or

(ii) $n_1 \geq 1$ and $\lambda_n^*(p) = \lambda_n(p, h)$.

For example, if $p \equiv 0$ and $h = 0$, then $\lambda_n(p, h) = (n\pi)^2 (n = 0, 1, 2, \ldots)$ and $\lambda_n^*(p) = (n\pi)^2 (n = 1, 2, \ldots)$, hence we get

$$Q_{p,h,n_1} = \{ p \}$$

for any $n_1 \in \mathcal{N}$. In this way, our question ($\beta$) is about the operator $A_{p,h,h}$, while its answer is given in connection with the operator $A_p^*$.

This article is composed of six sections. We discuss the preliminaries in Section 3 and we prove Theorems A and B in Sections 4 and 5, respectively. Section 6 is devoted to concluding remarks.
3. Preliminaries

Let \( \Omega \subset \mathbb{R}^2 \) be the interior of a triangle \( \triangle ABC \) with \( \overline{AC} = \overline{BC} \), \( \angle ACB = \pi/2 \), \( AB \) being parallel to either the \( x \) axis or the \( y \) axis, and let \( r \in C^1(\overline{\Omega}) \) be given (see Fig. 1). The following propositions on the hyperbolic equation

\[
K_{xx} - K_{yy} = r(x, y) K \quad \text{(on } \overline{\Omega})
\]  

(3.1)

are obtained by Picard's method [9]. In fact, Propositions 1–3 are proved in Picard [9], while the proof of Proposition 4 is given in Suzuki [11, Appendix B]. Henceforth \( v \) will denote the outer unit normal vector on \( \partial \Omega \).

**Proposition 1.** For each \( f \in C^2(\overline{AC}) \) and \( g \in C^2(\overline{BC}) \) with \( f|_C = g|_C \), there exists a unique \( K = K(x, y) \in C^2(\overline{\Omega}) \) such that (3.1) and

\[
K|_{AC} = f, \quad K|_{BC} = g.
\]  

(3.2)

**Proposition 2.** For each \( f \in C^2(\overline{AB}) \) and \( g \in C^4(\overline{AB}) \), there exists a unique \( K = K(x, y) \in C^2(\overline{\Omega}) \) such that (3.1) and

\[
K|_{AB} = f, \quad \frac{\partial}{\partial v} K|_{AB} = g.
\]  

(3.3)

**Proposition 3.** For each \( f \in C^2(\overline{AC}) \) and \( g \in C^2(\overline{AB}) \) with \( f|_A = g|_A \), there exists a unique \( K = K(x, y) \in C^2(\overline{\Omega}) \) such that

\[
K|_{AC} = f, \quad K|_{AB} = g.
\]  

(3.4)

**Proposition 4.** For each \( f \in C^2(\overline{AC}) \), \( g \in C^4(\overline{AB}) \) and \( h \in \mathcal{A} \), there exists a unique \( K = K(x, y) \in C^2(\overline{\Omega}) \) such that

\[
K|_{AC} = f, \quad \frac{\partial}{\partial v} K + hK|_{AB} = g.
\]  

(3.5)
Let us now review briefly the proof of these propositions. By the d’Alembert formula, we can give the solution 
\[ K_0 = K_0(x, y) \in C^2(\Omega) \]
of 
\[ K_{xx} - K_{yy} = 0 \quad (\text{on } \Omega), \tag{3.1}' \]
satisfying the boundary conditions given in these propositions. On the other hand, for each 
\[ F = F(x, y) \in C^0(\Omega), \]
the solution 
\[ K = K(x, y) \in C^2(\Omega) \]
of 
\[ K_{xx} - K_{yy} = F(x, y) \quad (\text{on } \Omega), \tag{3.1}'' \]
satisfying the boundary conditions for \( f = g = 0 \), is also given in a similar way. From these formulas, our problems are shown to be equivalent to certain integral equations of Volterra type, which are solved by the iteration. For details, we refer to \([9, 10, 11, \text{Appendix B}]\).

We now describe the “first deformation formula” found by \([10]\). It connects \( \phi(\cdot; p, h, \lambda) \) and \( \phi(\cdot; q, j, \lambda) \) through an integral transformation, whose kernel \( K \) is independent of \( \lambda \). The conditions \((1.2') \) and \((1.2'') \) on eigenvalues will be rewritten later in terms of \( K \).

**Lemma 1.** Given \( p, q \in C^1[0, 1] \) and \( h, j \in \mathbb{R} \), there exists a unique 
\( K = K(x, y) = K(x, y; q, j; p, h) \in C^2(\tilde{D}) \) such that 
\[ K_{xx} - K_{yy} + p(y) K = q(x) K \tag{3.6.1} \]
\[ K(x, x) = (j - h) + \frac{1}{2} \int_0^x (q(s) - p(s)) \, ds \quad (0 \leq x \leq 1), \tag{3.6.2} \]
\[ K_y(x, 0) = h K(x, 0) \quad (0 \leq x \leq 1). \tag{3.6.3} \]

**Lemma 2 (First deformation formula).** Recall the function \( \phi = \phi (\cdot; p, h, \lambda) \) defined in Notation 2. Then, for \( K = K(\cdot; q, j; p, h) \) in Lemma 1, the identity 
\[ \phi(x; q, j, \lambda) = \phi(x; p, h, \lambda) + \int_0^x K(x, y; q, j; p, h) \phi(y; p, h, \lambda) \, dy \tag{3.7} \]
holds for \( q, p \in C^1[0, 1], j, h \in \mathbb{R}, \) and \( \lambda \in \mathbb{R} \).

**Proof of Lemma 1.** To show the existence of \( K = K(\cdot; q, j, p, h) \), we extend \( q \in C^1[0, 1] \) to \( \bar{q} \in C^1[0, 2] \). Set \( D = \{(x, y) \mid 0 < y < x < 2 - y\} \). By Proposition 4, there exists \( \bar{K} = \bar{K}(x, y) \in C^2(\tilde{D}) \) such that 
\[ K_{xx} - K_{yy} + p(y) K = \bar{q}(x) \bar{K} \tag{3.6.1'} \]
\[ R(x, X) = (j - h) + \int_{0}^{X} (q(s) - p(t)) \, ds \quad (0 \leq x \leq 1), \quad (3.6.2') \]

\[ \hat{R}(x, 0) = h \hat{R}(x, 0) \quad (0 \leq x \leq 2). \quad (3.6.3') \]

The restriction \( K = \hat{R}|_{D} \in C^{2}(\overline{D}) \) satisfies (3.6). To verify the uniqueness, we divide \( D \) into \( Q_{1} = \{(x, y) | 0 < y < x < 1 - y\} \) and \( Q_{2} = D \setminus Q_{1} \). We prove that (3.6.1), \( K(x, x) = 0 \) (0 \( \leq x \leq 1\), (3.6.3) and \( K \in C^{2}(\overline{D}) \) imply \( K \equiv 0 \). In fact, we first have \( K = 0 \) on \( Q_{1} \) by Proposition 4, so that we next have \( K = 0 \) on \( Q_{2} \) by Proposition 1.

**Proof of Lemma 2.** We have only to show that the right-hand side of (3.7), which we set \( \psi(x) \), satisfies

\[ \left(-\frac{d^{2}}{dx^{2}} + q(x)\right)\psi = \lambda \psi \quad (0 \leq x \leq 1), \psi(0) = 1, \psi'(0) = j. \]

See [10], for this elementary calculation.

We finally note the following facts on the eigenfunctions \( \{\phi(\cdot; p, h, \lambda(p, h))\}_{n=0}^{\infty} \) of a symmetric operator \( A_{p, h, h} \). Put, for the moment,

\[ \phi_{n}(x) = \phi(x; p, h, \lambda(p, h)) \quad (n = 0, 1,...) \]

for \( p \in C_{1}^{1}[0, 1] \) and \( h \in \mathfrak{H} \).

**Lemma 3.** We have

\[ \phi_{n}(1 - x) = (-1)^{n} \phi_{n}(x) \quad (n \in \mathbb{N}, x \in [0, 1]). \]

**Proof.** Since \( A_{p, h, h} \) is symmetric,

\[ \phi_{n}(1 - x) = c_{n} \phi_{n}(x) \quad (0 \leq x \leq 1) \]

holds for some \( c_{n} \in \mathfrak{H} \), because of the uniqueness of the Cauchy problem (2.6). From \( c_{n} \neq 1 \) and \( c_{n} \neq -1 \) follow \( \phi_{n}(\frac{1}{2}) = 0 \) and \( \phi_{n}'(\frac{1}{2}) = 0 \), respectively, while \( \phi_{n}'(\frac{1}{2}) = \phi_{n}'(\frac{1}{2}) = 0 \) implies \( \phi_{n} \equiv 0 \), which is a contradiction. Therefore, \( \phi_{n}(\frac{1}{2}) = 0 \) and \( \phi_{n}'(\frac{1}{2}) = 0 \) cannot occur simultaneously, hence \( c_{n} = \pm 1 \). By Sturm–Liouville's theorem (see, e.g., Levitan–Sargsjan [6]), \( \phi_{n} \) has \( n \)-zeros in \((0, 1)\), hence (3.8) holds.

**Lemma 4.** Both \( \{\phi_{2n}\}_{n=0}^{\infty} \) and \( \{\phi_{2n+1}\}_{n=0}^{\infty} \) are complete and orthogonal systems in \( L^{2}(0, \frac{1}{2}) \).

**Proof.** By (3.8), \( \phi_{2n} \) satisfies

\[ \left(-\frac{d^{2}}{dx^{2}} + p(x)\right)\phi = \lambda \phi \quad (0 \leq x \leq \frac{1}{2}), \phi'(0) - h\phi(0) = \phi'(\frac{1}{2}) = 0 \]

(3.9)
for \( \lambda = \lambda_{2n}(p, h) \), and has \( n \)-zeros in \((0, \frac{1}{2})\). Therefore, by Sturm–Liouville's theorem \( \{ \phi_{2n} \}_{n=0}^{\infty} \) coincides with the totality of eigenfunctions of the eigenvalue problem (3.9), so that it is a complete and orthogonal system in \( L^2(0, \frac{1}{2}) \). Similarly, \( \{ \phi_{2n+1} \}_{n=0}^{\infty} \) is so. \( \square \)

4. PROOF OF THEOREM A

Recall

\[ \hat{Q}_{p, h, \Sigma} = \{(q, j) \in C^1_\Sigma[0, 1] \times \mathcal{R} | \lambda_n(q, j) = \lambda_n(p, h) \ (n \in \mathcal{N} \setminus \Sigma) \}, \]

where \( p \in C^1_\Sigma[0, 1] \), \( h \in \mathcal{R} \), and \( \Sigma \subset \mathcal{N} \). Theorem A is reduced to the following two theorems:

**THEOREM 1.** We have

\[ \hat{Q}_{p, h, \phi} = \{(p, h)\}. \]

**THEOREM 2.** We have

\[ \hat{Q}_{p, h, \{n_1\}} \not\subseteq \{(p, h)\}, \]

for each \( n_1 \in \mathcal{N} \).

**Proof of Theorem 1.** Take \( (q, j) \in \hat{Q}_{p, h, \phi} \). By definition we have

\[ \lambda_n(q, j) = \lambda_n(p, h) \quad (n = 0, 1, \ldots), \]

and by Lemma 3 we get

\[ \phi(1; q, j, \lambda_n(q, j)) - \phi(1; p, h, \lambda_n(p, h)) = (-1)^n \quad (n = 0, 1, \ldots). \]

The fact that (4.4) and (4.5) imply

\[ (q, j) = (p, h) \]

has been proved by Murayama [8] and Suzuki [10]. \( \square \)

**Remark 1.** This simple proof is essentially due to Professor A. Mizutani [7]. (Theorem 1 can be proved in another way similar to the proof of Theorem 2.) The author wishes to thank Professor A. Mizutani for permission to use his unpublished arguments here.

We prepare two lemmas to prove Theorem 2. We henceforth write

\[ \phi_n(x) = \phi(x; p, h, \lambda_n(p, h)) \quad (n \in \mathcal{N}, x \in [0, 1]) \]

for simplicity. First, the deformation formula (3.7) yields
**Lemma 5.** For $q \in C^1([0, 1])$ and $j \in \mathbb{R}$, $(q, j) \in \mathcal{Q}_{p, h, \{n_1\}}$ holds if and only if there exists $K \in C^2(\overline{D})$ and $c \in \mathbb{R}$ such that (3.6),

\[ j = h - K(1, 1), \]  
\[ \int_0^1 \{ K_x(1, y) + jK(1, y) \} \phi_n(y) \, dy = 0 \quad (n \neq n_1), \]

and

\[ K(\frac{1}{2}, y) = 0, \quad K_x(\frac{1}{2}, y) = c\phi_{n_1}(y) \quad (0 \leq y \leq \frac{1}{2}), \]

if $n_1$ is even, and

\[ K(\frac{1}{2}, y) = c\phi_{n_1}(y), \quad K_x(\frac{1}{2}, y) = 0 \quad (0 \leq y \leq \frac{1}{2}), \]

if $n_1$ is odd.

**Proof.** Assume $(q, j) \in \mathcal{Q}_{p, h, \{n_1\}}$ and take $K = K(\cdot; q, j; p, h) \in C^2(\overline{D})$ of Lemma 1. By the first deformation formula,

\[ \phi(x; q, j, \lambda) = \phi(x; p, h, \lambda) + \int_0^x K(x, y) \phi(y; p, h, \lambda) \, dy \]

holds. On the other hand, $(q, j) \in \mathcal{Q}_{p, h, \{n_1\}}$ means $\lambda_n(q, j) = \lambda_n(p, h)$ ($n \in \mathcal{N} \setminus \{n_1\}$), hence in particular

\[ \phi'(1; q, j, \lambda) + j\phi(1; q, j, \lambda) = 0 \]

follows for $\lambda = \lambda_n(p, h)$ ($n \in \mathcal{N} \setminus \{n_1\}$). Therefore, we have

\[ (j - h + K(1, 1)) \phi_n(1) + \int_0^1 \{ K_x(1, y) + jK(1, y) \} \phi_n(y) \, dy = 0 \]

\[ (n \in \mathcal{N} \setminus \{n_1\}). \]  

Put $\rho_n = \int_0^1 \phi_n(x)^2 \, dx$, $g(y) = K_x(1, y) + jK(1, y)$ and $a_n = \int_0^1 g(y) \phi_n(y) \, dy$. Since $K \in C^2(\overline{D})$, $g \in L^2(0, 1)$ holds. Therefore, Parseval's identity yields $\sum_{n=0}^\infty a_n^2/\rho_n = \|g\|_{L^2(0, 1)}^2 < \infty$, and so, $\lim_{n \to \infty} a_n / \sqrt{\rho_n} = 0$. On the other hand, the asymptotic behaviors

\[ \rho_n = \frac{1}{2} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty) \]

and

\[ \phi_n(x) = \cos \pi n \pi x + O\left(\frac{1}{n}\right) \quad (n \to \infty) \]
are known for each $x \in [0, 1]$ (see Levitan–Sargsjan [6]). We have, therefore
\[ 0 = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \int_0^1 \{ K_x(1, y) + jK(1, y) \} \phi_n(y) \, dy = -\lim_{n \to \infty} (j - h + K(1, 1))(-1)^n, \]
so that each term of (4.12) must be zero. Thus, the relations (4.8) follow.

Similarly, since
\[ \phi((\frac{1}{2}; p, h, \lambda_{2n}(p, h)) = \phi((\frac{1}{2}; q, j, \lambda_{2n}(q, j)) = 0 \]
and
\[ \phi((\frac{1}{2}; p, h, \lambda_{2n+1}(p, h)) = \phi((\frac{1}{2}; q, j, \lambda_{2n+1}(q, j)) = 0 \] (m, n = 0, 1, ...)
by Lemma 3, the equalities
\[ K(\frac{1}{2}, \frac{1}{2}) \phi_n(\frac{1}{2}) + \int_0^{1/2} K_x(\frac{1}{2}, y) \phi_n(y) \, dy = 0 \] (n: even, n \neq n_1) \hspace{1cm} (4.15)
and
\[ \int_0^{1/2} K(\frac{1}{2}, y) \phi_n(y) \, dy = 0 \] (n: odd, n \neq n_1) \hspace{1cm} (4.16)
follow from (4.10) and \( \lambda_n(q, j) = \lambda_n(p, h) \) (n \neq n_1). By the same reasoning as above, (4.15) is shown to be equivalent to
\[ K(\frac{1}{2}, \frac{1}{2}) = 0 \] \hspace{1cm} (4.17)
and
\[ \int_0^{1/2} K_x(\frac{1}{2}, y) \phi_n(y) \, dy = 0 \] (n: even, n \neq n_1). \hspace{1cm} (4.18)

First, assume that n_1 is even. Then, by Lemma 4, (4.16), and (4.18) give (4.9.e) for some \( c \in \mathcal{R} \). Similarly, if n_1 is odd, (4.16) and (4.18) give (4.9.o). (In both cases, (4.17) is automatically satisfied by (4.9) because \( \phi_n(\frac{1}{2}) = 0 \) if n is odd.)

Suppose, conversely, that there exists \( K \in C^2(\overline{D}) \) and \( c \in \mathcal{R} \) such that (3.6), (4.8), and (4.9), for \( (q, j) \in C'_2[0, 1] \times \mathcal{R} \). Since \( K \) satisfies (3.6), \( K = K(\cdot, \cdot; q, j; p, h) \) follows, and the first deformation formula (4.10) holds. In virtue of the assumptions (4.8), we get (4.11) for \( \lambda = \lambda_n(p, h) \) (n \in \mathcal{N} \setminus \{n_1\} ), which means \( \lambda_n(p, h) \in \sigma(A_{q, j, j}) \) (n \neq n_1). Namely, there exists some \( m(n) \in \mathcal{N} \) for each n \neq n_1, such that
\[ \lambda_n(p, h) = \lambda_{m(n)}(q, j) \] (n \neq n_1). \hspace{1cm} (4.19)

We now show
\[ m(n) = n \] (n \neq n_1) \hspace{1cm} (4.20)
to prove \( (q, j) \in \mathcal{Q}_{p, h, \{n_1\}} \). In fact, from the assumption (4.9), we get
\[ \phi((\frac{1}{2}; q, j, \lambda_n(p, h)) = 0 \] (n: even, n \neq n_1),
\[ \phi((\frac{1}{2}; q, j, \lambda_n(p, h)) = 0 \] (n: odd, n \neq n_1), \hspace{1cm} (4.21)
by (4.10) and Lemma 4, which gives

\[ m(n) \equiv n \pmod{2; n \neq n_1} \quad (4.22) \]

by Lemma 3. On the other hand, since the asymptotic behaviors

\[ \lambda_n(p, h)^{1/2} = n\pi + O\left(\frac{1}{n}\right) \quad (n \to \infty) \quad (4.23.1) \]

and

\[ \lambda_m(q, j)^{1/2} = m\pi + O\left(\frac{1}{m}\right) \quad (m \to \infty) \quad (4.23.2) \]

are known (see Levitan–Sargsjan [6]), (4.19) implies that there exists some \( n_0 \in \mathcal{N} \) such that

\[ m(n) = n \quad (n \geq n_0). \quad (4.24) \]

Noting that the mapping \( n \to m(n) \) is an injective order-homeomorphism, we see that (4.22) and (4.24) imply (4.20).

Now we prove

**Lemma 6.** For each \( p, q \in C_1^1[0, 1] \) and \( h, j, c \in \mathbb{R} \), the solution \( K = K(x, y) \in C^2(D) \) of (3.6.1) with (3.6.3), (4.8.2), and (4.9.e) is unique, and is given by

\[ K(x, y) = g(x) \phi_n(y), \quad (4.25) \]

where \( g = g(x) \in C^2[0, 1] \) is the solution of

\[ \left( - \frac{d^2}{dx^2} + q(x) \right) g = \lambda_n(p, h) g \quad (0 \leq x \leq 1), \quad (4.26.e) \]

\[ g'(\tfrac{1}{2}) = c, \quad g(\tfrac{1}{2}) = 0. \]

Similarly, for each \( p, q \in C_1^1[0, 1] \) and \( h, j, c \in \mathbb{R} \), the solution \( K = K(x, y) \in C^2(D) \) of (3.6.1) with (3.6.3), (4.8.2), and (4.9.o) is unique, and is given by (4.25), where \( g = g(x) \) is the solution of

\[ \left( - \frac{d^2}{dx^2} + q(x) \right) g = \lambda_n(p, h) g \quad (0 \leq x \leq 1), \quad (4.26.o) \]

\[ g'(\tfrac{1}{2}) = 0, \quad g(\tfrac{1}{2}) = c. \]
Proof. It is obvious that $K \in C^2(\overline{D})$ defined by (4.25) with (4.26) satisfies (3.6.1), (3.6.3), (4.8.2), and (4.9). We show the uniqueness of such $K$. To this end, we have only to derive $K = 0$ from these relations, assuming $c = 0$.

In Section 3, we divided $D = \{(x, y)|0 < y < x < 1\}$ into $\Omega_1(x, y)|0 < y < x < 1-y\}$ and $\Omega_2 = D \cap \overline{\Omega}_1$. Furthermore, we now divide $\Omega_1$ into $\Omega_3 = \{(x, y)|0 < y < x < 1-y\}$, $\Omega_4 = \{(x, y)|0 < y < x-\frac{1}{2} < 1-y\}$, and $\Omega_5 = \Omega_1 \cap \overline{\Omega}_5 \cap \overline{\Omega}_4$. Set $A = (0, 0), B = (1, 0), C = (1, 1), E = (\frac{1}{2}, 0), F = (\frac{1}{2}, \frac{1}{4}), G = (\frac{3}{4}, \frac{1}{4}),$ and $H = (\frac{1}{2}, \frac{1}{2})$. By means of the uniqueness assertion of Proposition 2, (3.6.1) and (4.9) with $c = 0$ imply $K = 0$ on $\overline{\Omega}_5$. Similarly, by Proposition 4, (3.6.1), (3.6.3), and $K = 0$ on $\overline{EF}$ and $\overline{EG}$ imply $K = 0$ on $\overline{\Omega}_3 \cup \overline{\Omega}_4$ (see Fig. 2). Now, (4.8.2) means

$$K_x(1, y) + jK(1, y) = d\phi_n(y) \quad (0 \leq y \leq 1) \quad (4.27)$$

for some $d \in \mathcal{R}$, while $K = 0$ on $\overline{\Omega}_4$ gives $K_x(1, 0) = K(1, 0) = 0$, hence $d = 0$. Again by Proposition 4, $K = 0$ on $\overline{\Omega}_2$ follows from (3.6.1), (4.27) with $d = 0$, and $K = 0$ on $\overline{BH}$. Thus, $K = 0$ on $\overline{D}$ has been verified.  

From Lemmas 5 and 6 follows

**Theorem 3.** Given $p \in C^1_s[0, 1], h \in \mathcal{R}$, and $n_1 \in \mathcal{N}, (q, j) \in \mathcal{Q}_{p, h, \{n_1\}}$ if and only if there exists $g = g(x) \in C^2[0, 1]$ with

$$\frac{d^2}{dx^2} g = \left(2 \frac{d}{dx} (g\phi_n) + p - \lambda_n(p, h)\right) g \quad (0 \leq x \leq 1), \quad (4.28.1)$$

$$g(1 - x) = (-1)^{n_1 + 1} g(x) \quad (0 \leq x \leq 1), \quad (4.28.2)$$

such that

$$q = p + 2 \frac{d}{dx} (g\phi_n), \quad j = h + g(0). \quad (4.29)$$
Furthermore, such \( g \in C^2[0, 1] \) is unique for each \( (q, j) \in \mathcal{Q}_{p,h,\{n_1\}} \). In particular, \( A_{q,j,j} = A_{p,h,h} \) if and only if \( g \equiv 0 \).

**Proof.** Assume, for the moment, that \( n_1 \) is even. By virtue of Lemmas 5 and 6, \( (q, j) \in \mathcal{Q}_{p,h,\{n_1\}} \) if and only if there exists \( g = g(x) \in C^2[0, 1] \) such that

\[
\left( -\frac{d^2}{dx^2} + q(x) \right) g = \lambda_{n_1}(p, h) g \quad (0 \leq x \leq 1), \tag{4.30.e}
\]

\[
g(\frac{1}{2}) = 0,
\]

\[
g(x) \phi_{n_1}(x) = (j - h) + \frac{1}{2} \int_0^x (q(s) - p(s)) \, ds \quad (0 \leq x \leq 1) \tag{4.31}
\]

and

\[
j = h - g(1) \phi_{n_1}(1). \tag{4.32}
\]

Eliminating \( q \) and \( j \) in (4.30) and (4.31), we conclude that if \( (q, j) \in \mathcal{Q}_{p,h,\{n_1\}} \), there exists \( g \in C^2[0, 1] \) satisfying (4.28.1) and (4.29). Noting \( q(1 - x) = q(x) \) and \( g(\frac{1}{2}) = 0 \), we have

\[
g(1 - x) = -g(x) \quad (0 \leq x \leq 1)
\]

from (4.30.e), which means (4.28.2). Conversely, if \( q \in C^1[0, 1] \) and \( j \in \mathcal{D} \) are given as (4.29) through \( g \in C^2[0, 1] \) satisfying (4.28), then \( (q, j, g) \in C^1[0, 1] \times \mathcal{D} \times C^2[0, 1] \) satisfies (4.30) and (4.31). The condition (4.32) is automatically fulfilled because of \( (g \times \phi_{n_1})(1 - x) = -(g \phi_{n_1})(x) \), which also gives \( g(1 - x) = g(x) \) \( (0 \leq x \leq 1) \). Hence \( (q, j) \in \mathcal{Q}_{p,h,\{n_1\}} \) follows. In this way the first part of the theorem has been verified for even \( n_1 \). Similarly, it can be proved for odd \( n_1 \).

To show the second part, let \( g \in C^2[0, 1] \) and \( \tilde{g} \in C^2[0, 1] \) satisfy (4.28) and (4.29) for some \( q \in C^1[0, 1] \) and \( j \in \mathcal{D} \). Then, both

\[
K(x, y) = g(x) \phi_{n_1}(y)
\]

and

\[
\tilde{K}(x, y) = \tilde{g}(x) \phi_{n_1}(y)
\]

satisfy (3.6). By means of the uniqueness of the solution of (3.6),

\[
K(x, y) = \tilde{K}(x, y) \quad ((x, y) \in \mathcal{D})
\]

holds. Hence

\[
g(x) = \tilde{g}(x) \quad (0 \leq x \leq 1). \]
We now give the

Proof of Theorem 2. For given \( p \in C^1_\ast[0, 1], h \in \mathcal{R}, \) and \( n_1 \in \mathcal{N}, \) taking \( c \in \mathcal{R} \setminus \{0\} \) close to zero, we obtain a solution \( g \in C^2[0, 1] \) of the nonlinear equation (4.28.1) satisfying \( g(\frac{1}{2}) = 0, \) \( g'(\frac{1}{2}) = c \) if \( n_1 \) is even, and \( g(\frac{1}{2}) = c, \) \( g'(\frac{1}{2}) = 0 \) if \( n_1 \) is odd. Then, (4.28.2) and \( g \neq 0 \) holds. Therefore, \( q \in C^1_\ast[0, 1] \) and \( j \in \mathcal{R} \) given by (4.29) satisfy \((q, j) \in \mathcal{Q}_{p, h, n_1} \) and \((q, j) \neq (p, h)\) by Theorem 3.

Remark 2. Hochstadt [3] showed that if \((q, j) \in \mathcal{Q}_{p, h, n_1} \) with \( j = h \) holds, then there exists \( g \in C^2[0, 1] \) such that (4.28.1) and \( q = p + 2(d/dx)(g, n_1) \) \((g, n_1)\). He, however, seems to have been unaware of the relation (4.28.2) and the second equality of (4.29). His method is based on the construction of the Green function of the boundary value problem \((- (d'/dx') + p(x) \lambda) \phi = f(x) \) \( (0 \leq x \leq 1), \) \( \phi'(0) = \phi'(1) + h \phi(1) = 0 \) \((\lambda \in \sigma(A_{p, h, h}))\) (see Coddington–Levinson [2, Chap. 7]), and is different from that of ours.

5. PROOF OF THEOREM B AND COROLLARY

Throughout this section, we set

\[
\lambda_{n_1} = \lambda_{n_1}(p, h) \quad \text{and} \quad \phi_{n_1} = \phi(\cdot; p, h, \lambda_{n_1}(p, h)),
\]

for simplicity. Consider

\[
\frac{d^2}{dx^2} g = \left(2 \frac{d}{dx} (g, n_1) + p - \lambda_{n_1}\right) g \quad (0 \leq x \leq 1),
\]

\[
g(1 - x) = (-1)^{n_1 + 1} g(x) \quad (0 \leq x \leq 1).
\]

By virtue of Theorem 3, the equality

\[
Q_{p, h, n_1} = \{ q \in C^1_\ast[0, 1] | \lambda_{n}(q, h) = \lambda_{n}(p, h) \quad (n \in \mathcal{N} \setminus \{n_1\}) \}
\]

\[
= \left\{ p + 2 \frac{d}{dx}(g, n_1) | g \in C^2[0, 1] \text{ satisfies } (5.2) \text{ and } g(0) = 0 \right\}
\]

holds.

We first show (i) of Theorem B; that is,

**Theorem 4.** If \( n_1 = 0, \) then \( Q_{p, h, n_1} = \{ p \}. \)

To this end, we prepare
LEMMA 7. If \( g \in C^2[0, 1] \) satisfies (5.2.1), then
\[
\phi_n g' - \phi'_n g = (\phi_n g)^2 + \gamma \quad (0 \leq x \leq 1)
\] (5.4)
holds, where \( \gamma \in \mathcal{R} \) is a constant.

Proof: Recalling the notation (2.6), we have
\[
\left(-\frac{d^2}{dx^2} + p(x)\right) \phi_n = \lambda_n \phi_n, \quad (0 \leq x \leq 1),
\] (5.5)
\[
\phi_n(0) = 1, \quad \phi'_n(0) = h.
\]
From (5.2.1) and (5.5), we get
\[
(\phi_n g' - \phi'_n g)' = \phi_n g'' - \phi''_n g = 2(g \phi_n)' g \phi_n = ((\phi_n g)^2)',
\]
hence (5.4) follows. \( \square \)

We now give the

Proof of Theorem 4. We have only to derive \( g \equiv 0 \) from (5.2), \( g(0) = 0 \) and \( n_1 = 0 \). If \( \gamma = 0 \) in (5.4), then \( g(0) = g'(0) = 0 \), so that \( g \equiv 0 \) follows because \( g \) satisfies (5.2.1). On the other hand, if \( \gamma \neq 0 \), then for any \( x_0 \in [0, 1] \) with \( g(x_0) = 0 \), \( g'(x_0) \) has the same sign as that of \( \gamma \), because \( \phi_n > 0 \) holds on \([0, 1]\) (see Fig. 3) in the case of \( n_1 = 0 \). This, however, contradicts \( g \in C^2[0, 1] \) and \( g(0) = g(1) = 0 \), the latter being derived from \( g(0) = 0 \) and (5.2.2). \( \square \)

Before starting the proof of (ii) of Theorem B, we note that the following theorems are obtained in the same way as the proofs of Theorems 1–3. Recall that \( A^*_\omega \) denotes the differential operator \(- (d^2/dx^2) + p(x)\) with the Dirichlet boundary condition \( \cdot|_{x=0} = \cdot|_{x=1} = 0 \) and that \( \lambda^*_n(p) \) \( (n \in \mathcal{N}^* \equiv \{1, 2, \ldots\}) \) denotes its eigenvalues.

**Theorem 1**. For \( q, p \in C^1_j[0, 1] \), the relation
\[
\lambda^*_n(q) = \lambda^*_n(p) \quad (n \in \mathcal{N}^*)
\] (5.6)
implies \( q \equiv p \).
THEOREM 2*. For \( p \in C^1_t[0, 1] \) and \( n_1 \in \mathcal{N}^* \), let
\[
Q^*_{p,n_1} = \{ q \in C^1_t[0, 1] \mid \lambda^*_n(q) = \lambda^*_n(p) \ (n \in \mathcal{N}^* \setminus \{ n_1 \}) \}. \tag{5.7}
\]
Then, we have \( Q^*_{p,n_1} \supseteq \{ p \} \).

THEOREM 3*. Recall the notation \( \phi^*(\cdot; p, \lambda) \) of (2.7), and put
\[
\lambda^*_n = \lambda^*_n(p) \quad \text{and} \quad \phi^*_n = \phi^*(\cdot; p, \lambda^*_n(p)). \tag{5.8}
\]
Then, \( q \in Q^*_{p,n_1} \) if and only if there exists \( f = f(x) \in C^2[0, 1] \) with
\[
\begin{align*}
\frac{d^2}{dx^2} f &= \left(2 \frac{d}{dx} (\phi^*_n) + p - \lambda^*_n\right) f \quad (0 \leq x \leq 1), \tag{5.9.1} \\
f(1 - x) &= (-1)^n f(x) \quad (0 \leq x \leq 1), \tag{5.9.2}
\end{align*}
\]
such that
\[
q = p + 2 \frac{d}{dx} (f\phi^*_n). \tag{5.10}
\]

Furthermore, such \( f \in C^2[0, 1] \) is unique for each \( q \in Q^*_{p,n_1} \). In particular, \( q \equiv p \) if and only if \( f \equiv 0 \).

The proof of these theorems are based on the following lemmas as well as on Propositions 1–3. Recall \( D = \{(x, y)|0 < y < x < 1\} \).

LEMMA 1*. Given \( p, q \in C^1_t[0, 1] \), there exists a unique \( F(x, y) \in C^2(D) \) such that
\[
\begin{align*}
F_{xx} - F_{yy} + p(y) F &= q(x) F \quad (D), \tag{5.11.1} \\
F(x, x) &= \frac{1}{2} \int_0^x (q(s) - p(s)) \, ds \quad (0 \leq x \leq 1), \tag{5.11.2} \\
F(x, 0) &= 0 \quad (0 \leq x \leq 1). \tag{5.11.3}
\end{align*}
\]

LEMMA 2*. For \( \phi^* = \phi^*(\cdot; p, \lambda) \) defined through (2.7), the identity
\[
\phi^*(x; q, \lambda) = \phi^*(x; p, \lambda) + \int_0^x F(x, y; q; p) \phi^*(y; p, \lambda) \, dy \tag{5.12}
\]
holds for \( q, p \in C^1_t[0, 1] \) and \( \lambda \in \mathcal{K} \).
LEMMA 3*. For simplicity, we put \( \phi_n^*(x) = \phi^*(x; p, \lambda_n^*(p)) \) \((0 \leq x \leq 1)\) for \( n \in \mathcal{N}^* \). We then have
\[
\phi_n^*(1-x) = (-1)^{n+1} \phi_n^*(x) \quad (x \in [0, 1], n \in \mathcal{N}^*).
\]

LEMMA 4*. Both \( \{\phi_n^*\}_{n=1}^\infty \) and \( \{\phi_{n-1}^*\}_{n=1}^\infty \) are complete and orthogonal systems in \( L^2(0, \frac{1}{2}) \).

Note that \( F(0, 0) = 0 \) follows from (5.11.3) in Lemma 1*, while \( K(0, 0) \) is not determined by (3.6.3) in Lemma 1. This causes the difference between (5.11.2) and (3.6.2). However, the similarity of Theorems 1*-3* to Theorems 1-3, turns out in spite of the fact that the set \( Q_{p,n}^* \) is defined just for the fixed boundary condition \( y|_{x=0} = y|_{x=1} = 0 \).

Now, we show the following theorem, which is not only the key to the proof of (ii) of Theorem B, but is also of interest by itself.

THEOREM 5. For each \( p, q \in C^1[0, 1] \) and \( n_1 \in \mathcal{N}^* = \{1, 2, \ldots\} \),
\[
\lambda_{n_1}^*(p) = \lambda_{n_1}^*(q) \quad (n \in \mathcal{N}^* \setminus \{n_1\})
\]
holds if and only if there exists \( h \in \mathfrak{H} \) such that
\[
\lambda_n(p, h) = \lambda_n(q, h) \quad (n \in \mathcal{N} \setminus \{n_1\}).
\]

In the case of \( q \neq p \), we have furthermore
\[
\lambda_{n_1}^*(p) = \lambda_{n_1}^*(q), \quad \lambda_n(p, h) = \lambda_n(q).
\]

Proof. We first show that (5.14) implies (5.15). To this end, we have only to consider the case
\[
q \neq p,
\]
which we henceforth assume. Since (5.14) means \( q \in Q_{p,n_1}^* \), by Theorem 3* there exists some \( f \in C^2[0, 1] \) with (5.9) and \( f \neq 0 \), such that
\[
q = p + 2 \frac{d}{dx} (f\phi_{n_1}^*) \quad \text{(recall } \phi_{n_1}^* = \phi^*(\cdot; p, \lambda_{n_1}^*(p)))\).
\]

Let us show
\[
f(0) \neq 0.
\]
In fact, if \( f(0) = 0 \), then \( f(1) = 0 \) by (5.9.2), so that \( \lambda_{n_1}^* (= \lambda_{n_1}^*(p)) \in \sigma(A_q^*) \) because of (5.9.1), (5.18), \( f \neq 0 \) and \( f(0) = f(1) = 0 \). Combining this with (5.14), we get
\[
\sigma(A_{p}^*) \subset \sigma(A_q^*).
\]
Namely, for each $n \in \mathcal{N}^*$, there exists $m(n) \in \mathcal{N}^*$ such that
\[ \lambda_n^*(p) = \lambda_{m(n)}^*(q) \quad (n = 1, 2, \ldots). \]

Since the asymptotic behaviors
\[ \lambda_n^*(p)^{1/2} = n\pi + O\left(\frac{1}{n}\right) \quad (n \to \infty) \]
and
\[ \lambda_{m(n)}^*(q)^{1/2} = m\pi + O\left(\frac{1}{m}\right) \quad (m \to \infty) \]
are known (see Levitan-Sargsjan [6]), $m(n) = n$ ($n \geq n_0$) holds for sufficiently large $n_0$. Therefore, since the mapping $n \mapsto m(n)$ is an injective order-homeomorphism, we have $m(n) = n$ ($n \in \mathcal{N}^*$), hence
\[ \sigma(A_p^*) = \sigma(A_q^*). \quad (5.21) \]

Now Theorem 1* implies $q \equiv p$, which contradicts the assumption (5.17). In this way, (5.19) has been established, and we can define
\[ h = f'(0)/f(0) \in \mathcal{R}. \]

From (5.9.2) it follows that
\[ f'(0) - hf(0) = f'(1) + hf(1) = 0. \quad (5.22) \]

We show that (5.15) holds for $h \in \mathcal{R}$ taken in this way. Since $f$ satisfies (5.9.1), (5.10), $f \neq 0$, and (5.22), the equalities
\[ \lambda_{n_1}^* = \lambda_{m_1}(q, h) \quad \text{(recall } \lambda_{n_1}^* = \lambda_{n_1}^*(p)) \quad (5.23) \]
and
\[ f = \text{const.} \times \phi(\cdot; q, h, \lambda_{m_1}(q, h)) \quad (5.24) \]
holds for some $m_1 \in \mathcal{N} \equiv \{0, 1, 2, \ldots\}$. We now prove

**Lemma 8**. $m_1$ is equal to $n_1$.

**Proof of Lemma 8**. We prove that $f$ has $n_1$-zeros in $(0, 1)$. Then, from Sturm–Liouville’s theorem, $m_1 = n_1$ follows. Recall that $\phi_{n_1}^* = \phi(\cdot; p, \lambda_{n_1}^*(p))$ has $(n_1 - 1)$-zeros in $(0, 1)$, hence has $(n_1 + 1)$-zeros on $[0, 1]$. Let them be $x_{1}^* = 0 < x_{2}^* < \cdots < x_{n_1 - 1}^* < x_{n_1}^* = 1$. We wish to show
(i) \( f(x*) \neq 0 \) \((0 \leq i \leq n_1)\),
(ii) \( f \) has at least one zero in \( I_i = (x_i^*, x_{i+1}^*) \) \((0 \leq i \leq n_1 - 1)\)
and
(iii) \( f \) has at most one zero in \( I_i \) \((0 \leq i \leq n_1 - 1)\).

In the same way as in the proof of Lemma 7, we get
\[
(\phi_{n_1}^* f' - \phi_{n_1}^* f')' = \phi_{n_1}^* f'' - \phi_{n_1}^* f
= 2(\phi_{n_1}^* f')' \phi_{n_1}^* f = ((\phi_{n_1}^* f)^2)',
\]
because \( f \) satisfies (5.9.1) and \( \phi_{n_1}^* \) satisfies
\[
\left(- \frac{d^2}{dx^2} + p(x)\right) \phi_{n_1}^* = \lambda_{n_1} \phi_{n_1}^* \quad \text{for} \quad 0 \leq x \leq 1, \quad \phi_{n_1}^*(0) = 0, \quad \phi_{n_1}^*(0) = 1, \quad (5.25)
\]
so that
\[
\phi_{n_1}^* f' - \phi_{n_1}^* f = (\phi_{n_1}^* f)^2 + \gamma \quad (5.26)
\]
follows, \( \gamma \in \mathcal{R} \) being a constant.

(i) If \( f(x^*) = 0 \), then \( \gamma = 0 \) in (5.26). On the other hand, (5.9.2) and (5.13) imply \( \phi_{n_1}^*(\frac{1}{2}) f(\frac{1}{2}) = 0 \), hence \( f(\frac{1}{2}) = f'(\frac{1}{2}) = 0 \) from (5.26) with \( \gamma = 0 \), (5.9.2) and (5.13). Therefore, \( f \equiv 0 \) because \( f \) satisfies (5.9.1), which contradicts (5.17).

(ii) By (5.26), we have
\[
[\phi_{n_1}^* f' - \phi_{n_1}^* f]_{x = x_i^*}^{x = x_{i+1}^*} = [(\phi_{n_1}^* f)^2]_{x = x_i^*}^{x = x_{i+1}^*} = 0,
\]
so that
\[
\phi_{n_1}^*(x^*) f(x^*) = \phi_{n_1}^*(x_{i+1}^*) f(x_{i+1}^*) \quad (0 \leq i \leq n_1 - 1). \quad (5.27)
\]
If \( f \neq 0 \) on \( I_i = [x_i^*, x_{i+1}^*] \), then \( \phi_{n_1}^*(x^*) \phi_{n_1}^*(x_{i+1}^*) > 0 \), which contradicts \( \phi_{n_1}^*(x_i^*) = \phi_{n_1}^*(x_{i+1}^*) = 0 \) and \( \phi(x) \neq 0 \) \((x \in I_i)\).

(iii) Suppose that there exist \( y_1 \) and \( y_2 \) in \( x_i^* < y_1 < y_2 < x_{i+1}^* \) such that \( f(y_1) = f(y_2) = 0 \) and \( f(x) \neq 0 \) \((y_1 < x < y_2)\). Then by (5.26), we have
\[
[\phi_{n_1}^* f' - \phi_{n_1}^* f']_{x = y_1}^{x = y_2} = [(\phi_{n_1}^* f)^2]_{x = y_1}^{x = y_2} = 0,
\]
so that
\[
\phi_{n_1}^*(y_2) f'(y_2) = \phi_{n_1}^*(y_1) f'(y_1); \quad (5.28)
\]
\( \phi_{n_1}^*(x) \neq 0 \) \((x \in I_i)\) implies \( f'(y_1) f'(y_2) > 0 \), which contradicts \( f(y_1) = f(y_2) = 0 \) and \( f(x) \neq 0 \) \((y_1 < x < y_2)\).
Let us continue the proof of Theorem 5. Assuming (5.14) and \( q \neq p \), we have shown that there exists \( f \neq 0 \) and \( h \in \mathcal{R} \) such that (5.18), (5.23), and (5.24) with \( m_1 = n_1 \). Now, consider the function

\[
K(x, y) = -\phi_n^*(x) f(y).
\]  

(5.29)

Since (5.18) and (5.24) hold, \( K \) satisfies

\[
K_{xx} - K_{yy} + q(y) K = p(x) K \quad (\bar{D})
\]  

(5.30.1)

\[
K(x, x) = \frac{1}{2} \int_0^x (p(s) - q(s)) \, ds \quad (0 \leq x \leq 1),
\]  

(5.30.2)

\[
K_{x}(x, 0) = h K(x, 0) \quad (0 \leq x \leq 1).
\]  

(5.30.3)

Therefore, \( K = K(\cdot, \cdot; p, h; q, h) \) holds, and the identity

\[
\phi(x; p, h, \lambda) = \phi(x; q, h, \lambda) + \int_0^x K(x, y) \phi(y; q, h, \lambda) \, dy
\]

\[
= \phi(x; q, h, \lambda) - \phi_n^*(x) \int_0^x f(y) \phi(y; q, h, \lambda) \, dy.
\]  

(5.31)

follows from Lemma 2. In particular, we have

\[
\phi'(1; p, h, \lambda) + h\phi(1; p, h, \lambda)
\]

\[
= [\phi'(1; q, h, \lambda) + h\phi(1; q, h, \lambda)]
\]

\[
- \phi_n^*(1) \int_0^1 f(y) \phi(y; q, h, \lambda) \, dy,
\]  

(5.32)

because of \( \phi_n^*(1) = 0 \). By means of (5.24) with \( m_1 = n_1 \), we get

\[
\phi'(1; q, h, \lambda_m(q, h)) + h\phi(1; q, h, \lambda_m(q, h))
\]

\[
= \int_0^1 f(y) \phi(y; q, h, \lambda_m(q, h)) \, dy = 0 \quad (m \neq n_1),
\]

so that

\[
\phi'(1; p, h, \lambda) + h\phi(1; p, h, \lambda) = 0 \quad (\lambda = \lambda_m(q, h), m \in \mathcal{N} \setminus \{n_1\})
\]

by (5.32), which means

\[
\lambda_m(q, h) = \lambda_{m(m)}(p, h) \quad (m \in \mathcal{N} \setminus \{n_1\})
\]  

(5.33)
for some \( n(m) \in \mathcal{N} \). We now prove

\[
n(m) = m \quad (m \in \mathcal{N} \setminus \{n_1\}) \tag{5.34}
\]

and show that (5.15) holds.

First, if \( n_1 \) is even, we have \( \phi_{n_1}'(\frac{1}{2}) = 0 \). Therefore, (5.31) gives \( \phi(\frac{1}{2}; p, h, \lambda) = \phi(\frac{1}{2}; q, h, \lambda) \) (\( \lambda \in \mathbb{R} \)), hence

\[
\phi(\frac{1}{2}; p, h, \lambda_m(q, h)) = 0 \quad (m: \text{odd}). \tag{5.35}
\]

Furthermore, (5.31) also gives

\[
\phi'(\frac{1}{2}; p, h, \lambda) = \phi'(\frac{1}{2}; q, h, \lambda) - \phi_{n_1}'(\frac{1}{2}) \int_0^{1/2} f(y) \phi(y; q, h, \lambda) \, dy.
\]

Here, we have

\[
\phi'(\frac{1}{2}; q, h, \lambda_m(q, h)) = 0 \quad (m: \text{even})
\]

by Lemma 3* and

\[
\int_0^{1/2} f(y) \phi(y; q, h, \lambda_m(q, h)) \, dy = 0 \quad (m: \text{even}, m \neq n_1)
\]

by Lemma 4* and (5.24) with \( m_1 = n_1 \). Hence,

\[
\phi'(\frac{1}{2}; p, h, \lambda_m(q, h)) = 0 \quad (m: \text{even}, m \neq n_1). \tag{5.36}
\]

Again by Lemma 3*, we get

\[
n(m) \equiv m \pmod{2; m \neq n_1} \tag{5.37}
\]

from (5.35) and (5.36). By the argument in the proof of Lemma 5, (5.33) and (5.37) yield (5.34).

Next, if \( n_1 \) is odd, we have \( f(\frac{1}{2}) = 0 \) and \( \phi_{n_1}'(\frac{1}{2}) = 0 \), hence \( \phi'(\frac{1}{2}; q, h, \lambda) = \phi'(\frac{1}{2}; p, h, \lambda) \) (\( \lambda \in \mathbb{R} \)) by (5.31). In particular,

\[
\phi'(\frac{1}{2}; p, h, \lambda_m(q, h)) = 0 \quad (m: \text{even}). \tag{5.36'}
\]

Furthermore, by means of \( \phi(\frac{1}{2}; q, h, \lambda_m(q, h)) = 0 \) (\( m: \text{odd} \)), (5.24) with \( m_1 = n_1 \) and Lemma 4*, we have

\[
\phi(\frac{1}{2}; p, h, \lambda_m(q, h)) = \phi(\frac{1}{2}; q, h, \lambda_m(q, h)) - \phi_{n_1}'(\frac{1}{2}) \int_0^{1/2} f(y) \phi(y; q, h, \lambda_m(q, h)) \, dy
\]

\[
= 0 \quad (m: \text{odd}, m \neq n_1). \tag{5.35'}
\]

Equations (5.36') and (5.35') again give (5.37), whence follows (5.34).
In this way, we have proved that if (5.14) holds, then there exists \( h \in \mathcal{R} \) such that (5.15). Furthermore, we have seen that (5.23) holds with \( m_1 = n_1 \) in the case of \( q \neq p \). We now show that conversely (5.15) implies (5.14), assuming \( q \neq p \).

Since (5.15) means \( q \in Q_{p,n_1} \), there exists \( g \in C^2([0, 1]) \) with (5.2), \( g(0) = 0 \), and \( g \neq 0 \) such that

\[
q = p + 2 \frac{d}{dx} (g \phi_{n_1}) \quad \text{(recall } \phi_{n_1} = \phi(\cdot; p, h, \lambda_{n_1}(p, h))) \tag{5.38}
\]

by means of Theorem 3. In particular, there exists some \( m_1 \in \mathcal{N}^* \) such that

\[
\lambda_{n_1} = \lambda_{m_1}(q) \quad \text{(recall, } \lambda_{n_1} = \lambda_{n_1}(p, h)) \tag{5.39}
\]

and

\[
g = \text{const.} \times \phi^*(\cdot; q, \lambda_{m_1}(q)). \tag{5.40}
\]

In the same way as in the proof of Lemma 8, the equality

\[
m_1 = n_1 \tag{5.41}
\]

can be shown.

\[
F(x, y) = -\phi_{n_1}(x) g(y) \tag{5.42}
\]

satisfies

\[
F_{xx} - F_{yy} + q(y) F - p(x) F = (B), \tag{5.43.1}
\]

\[
F(x, x) = \frac{1}{2} \int_0^x (p(s) - q(s)) \, ds \quad (0 \leq x \leq 1), \tag{5.43.2}
\]

\[
F(x, 0) = 0 \quad (0 \leq x \leq 1), \tag{5.43.3}
\]

because (5.5), (5.38), and (5.40), so that

\[
\phi^*(x; p, \lambda) = \phi^*(x; q, \lambda) - \phi_{n_1}(x) \int_0^x g(y) \phi^*(y; q, \lambda) \, dy \tag{5.44}
\]

holds by Lemma 2*.

From (5.40) with \( m_1 = n_1 \) and (5.44), we have

\[
\phi^*(1; p, \lambda_{m_1}(q)) = 0 \quad (m \in \mathcal{N} \setminus \{n_1\}). \tag{5.45}
\]

On the other hand, in the same way as in the proof of (5.35) and (5.36), we can show

\[
\phi^*(\frac{1}{2}; q, \lambda_{m_1}(q)) = 0 \quad (m: \text{even}, m \neq n_1) \tag{5.46}
\]
and
\[ \phi^*(\frac{1}{2}; q, \lambda_m^*(q)) = 0 \quad (m: \text{odd}, m \neq n_1), \]  
(5.47)

from (5.40) with \( m_1 = n_1 \) and (5.44). By the argument in the proof of (5.34), we obtain (5.14) by (5.45)–(5.47).

In this way, we have proved the first part of Theorem 5. Since Theorem 5 is "symmetric" with respect to \( p \) and \( q \), the second part follows from (5.39) with \( m_1 = n_1 \).  

We finally show the following four theorems in turn.

**Theorem 6.** The number of the elements in \( Q_{p,h,n} \), different from \( p \), is at most one.

**Theorem 7.** If \( n_1 \geq 1 \) and \( \lambda_n^*(p) = \lambda_n(p, h) \), then \( Q_{p,h,n} = \{ p \} \).

**Theorem 8.** For \( p \in C^1([0, 1], h \in \mathcal{R} \) and \( n_1 \in \mathcal{N}^* \), \( W = W(x; p, h, n_1) \) defined by (2.8) satisfies \( (d^2/dx^2) \log(W) \in C^1([0, 1]) \). \( (d^2/dx^2) \log(W) = 0 \) if and only if \( \lambda_n^*(p) = \lambda_n(p, h) \).

**Theorem 9.** We have
\[ p - 2 \frac{d^2}{dx^2} \log(W) \in Q_{p,h,n_1}. \]  
(5.48)

Obviously, these theorems imply (ii) of Theorem B and the corollary in Section 2; that is, if \( n_1 \geq 1 \), \( Q_{p,h,n_1} = \{ p, p - 2(d^2/dx^2) \log(W) \} \). Furthermore, for \( n_1 \geq 1 \), \( Q_{p,h,n_1} = \{ p \} \) if and only if \( \lambda_n^*(p) = \lambda_n(p, h) \).

It would be desirable to explain how the function \( p - 2(d^2/dx^2) \log(W) \) has been discovered. In the following section, we shall do that. We now give the proof of these theorems.

**Proof of Theorem 6.** We only consider the case \( n_1 \geq 1 \). Suppose \( q_1, q_2 \in Q_{p,h,n_1}, q_1 \neq p \), and \( q_2 \neq p \). Then, Theorem 5 yields
\[ \lambda_n^*(q_1) = \lambda_n^*(p) = \lambda_n^*(q_2) \quad (n \in \mathcal{N}^* \setminus \{ n_1 \}) \]

and
\[ \lambda_n^*(q_1) = \lambda_n(p, h) = \lambda_n^*(q_2), \]

which means \( \sigma(A_{q_1}^*) = \sigma(A_{q_2}^*) \). Now, \( q_1 \equiv q_2 \) follows from Theorem 1*.

**Proof of Theorem 7.** Let \( \lambda_n^*(p) = \lambda_n(p, h) \) be satisfied for \( n_1 \in \mathcal{N}^* \). Assume the existence of \( q \in Q_{p,h,n_1} \) different from \( p \),
\[ \lambda_n(p, h) = \lambda_n(q, h) \quad (n \neq n_1), q \neq p. \]
By Theorem 5, the equality \( \lambda_{n_1}^*(p) = \lambda_{n_1}(q, h) \) holds, hence we have

\[
\hat{\lambda}_{n_1}(p, h) - \hat{\lambda}_{n_1}(q, h)
\]

by the assumption \( \lambda_{n_1}^*(p) = \lambda_{n_1}(p, h) \). Summing up these relations, we get \( \sigma(A_{p,h,h}) = \sigma(A_{q,h,h}) \), hence \( q = p \) follows from Theorem 1, which is a contradiction.

**Proof of Theorem 8.** Recall (5.5), (5.25), and (5.49)

By Sturm–Liouville's theorem, \( \phi_{n_1} \) and \( \phi_{n_1}^* \) have \( n_1 \)-zeros and \( (n_1 + 1) \)-zeros on \([0, 1]\), respectively. Let them be \( 0 < x_1 < x_2 < \cdots < x_{n_1} < 1 \) and \( 0 = x_0^* < x_1^* < \cdots < x_{n_1 - 1}^* < x_{n_1}^* = 1 \), respectively. By the comparison theorem (see Coddington–Levinson [2, p. 208]), \( \{x_i\}_{i=0}^{n_1} \) and \( \{x_i^*\}_{i=0}^{n_1} \) are interlaced so that

\[
0 = x_0^* < x_1 < x_1^* < \cdots < x_{n_1 - 1}^* < x_{n_1}^* = 1.
\]

Theorem 8 holds. Furthermore, we see \( W > 0 \) from (5.51), so that we have \( (d^2/dx^2) \log(W) \in C^1[0, 1] \). Finally, since

\[
W(1 - x) = W(x) \quad (0 \leq x \leq 1)
\]

by Lemmas 3 and 3*, we get \( (d^2/dx^2) \log(W)(1 - x) = (d^2/dx^2) \log(W)(x) \) and so \( (d^2/dx^2) \log(W) \in C^1[0, 1] \) holds.

Let us show the second part of Theorem 8. By (5.51), \( \lambda_{n_1}^* = \lambda_{n_1} \) implies \( W' = 0 \), hence \( (d^2/dx^2) \log(W) = (W'/W)' = 0 \). Suppose, conversely, \( (d^2/dx^2) \log(W) \equiv 0 \). Then easily, we have

\[
W = ae^{\beta x} \quad (0 \leq x \leq 1)
\]
with some constants $\alpha, \beta \in \mathbb{R}$, and so $W'' = \alpha \beta e^{\beta x}$. From (5.51), we have $W'(0) = 0$, which gives $\alpha \beta = 0$, $W' \equiv 0$ and $\lambda_n^* = \lambda_n^*$, in turn.

**Proof of Theorem 9.** We set

$$q = p - 2 \frac{d^2}{dx^2} \log(W) \quad (5.53)$$

and show

$$\lambda_n(q, h) = \lambda_n(p, h) \quad (n \in \mathcal{N} \setminus \{n_1\}). \quad (5.54)$$

By virtue of Theorem 3, we have only to show that

$$g(x) = (\lambda_n^* - \lambda_n) \phi_n^*(x)/W(x) \in C^2[0, 1] \quad (5.55)$$

satisfies

$$\frac{d^2}{dx^2} g = \left(2 \frac{d}{dx} (g \phi_n) + p - \lambda_n \right) g \quad (0 \leq x \leq 1), \quad (5.56.1)$$

$$g(1 - x) = (-1)^{n_1 + 1} g(x) \quad (0 \leq x \leq 1), \quad (5.56.2)$$

$$g(0) = 0, \quad (5.56.3)$$

$$q - p = 2 \frac{d}{dx} (g \phi_n). \quad (5.56.4)$$

Equation (5.56.2) follows from Lemma 3* and (5.52). Equation (5.56.3) is obvious and (5.56.4) is verified from

$$2 \frac{d}{dx} (g \phi_n) = 2(\lambda_n^* - \lambda_n) \frac{d}{dx} \left(\phi_n^* \phi_n/W\right)$$

$$= -2 \frac{d}{dx} (W'/W) \quad \text{(in fact, (5.51))}$$

$$= -2 \frac{d^2}{dx^2} \log(W) \quad (5.57)$$

and (5.53). Finally, setting

$$\hat{g}(x) = \phi_n^*(x)/W(x),$$

we obtain

$$\hat{g}'' = (\phi_n^* W^2 - 2\phi_n^* WW' + 2\phi_n^* W^2 - \phi_n^* WW')/W^3$$

$$\hat{g}'' = \frac{\phi_n^*}{W} \left[(p - \lambda_n^*) - 2(\lambda_n^* - \lambda_n^*) \phi_n^* \phi_n/W + 2W'/W^2 - W'/W\right]$$
(in fact, $(-d^2/dx^2) + p(x)) \phi_n^* = \lambda_n^* \phi_n^*$ and $W' = (\lambda_{n_1} - \lambda_{n_1}^*) \phi_{n_1}^* \phi_{n_1}$)

$$
= \hat{g}[p - \lambda_{n_1} + (\lambda_{n_1} - \lambda_{n_1}^*)(1 - 2\phi_{n_1}^* \phi_{n_1}/W) + 2W'^2/W^2 - W''^2/W^2]
$$

$$
= \hat{g}[p - \lambda_{n_1} - (\lambda_{n_1} - \lambda_{n_1}^*)(\phi_{n_1}^* \phi_{n_1})'/W + 2W'^2/W^2 - W''/W]$

(in fact, $W = \phi_{n_1}^* \phi_{n_1} - \phi_{n_1}^* \phi_{n_1}^*$)

$$
= \hat{g}[p - \lambda_{n_1} + 2(W'^2/W^2 - W''/W)]
$$

(in fact, $(\lambda_{n_1} - \lambda_{n_1}^*) \phi_{n_1}^* \phi_{n_1} = W''$

$$
= \hat{g}[p - \lambda_{n_1} - 2\frac{d^2}{dx^2} \log(W)].
$$

Therefore, we have

$$
g'' = g(p - \lambda_{n_1} - 2\frac{d^2}{dx^2} \log(W)) = \left(2 \frac{d}{dx} (g\phi_{n_1}) + p - \lambda_{n_1}\right) g,
$$

where the last equality follows from (5.57).

6. Concluding Remarks

(1) We now explain how the function $p - 2(d^2/dx^2) \log(W)$ has been found. For this purpose, let us assume that $q \neq p$ is contained in $Q_{p,h,n_1}$ for $n_1 \geq 1$. We show that then $q = p - 2(d^2/dx^2) \log(W)$ follows.

In fact, by means of Theorem 3, there exists $g \in C^2[0, 1]$ with (4.28) and $g(0) = 0$ such that

$$
q - p = 2 \frac{d}{dx} (g\phi_{n_1}) \quad \text{(recall, } \phi_{n_1} = \phi(\cdot ; p, h, \lambda_{n_1}(p, h)).
$$

(6.1)

Also, by Theorems 5 and 3*, there exists $f \in C^2[0, 1]$ with (5.9) such that

$$
q - p = 2 \frac{d}{dx} (f\phi_{n_1}^*) \quad \text{(recall, } \phi_{n_1}^* = \phi^*(\cdot ; p, \lambda_{n_1}(p)).
$$

(6.2)

In the proof of Theorem 5, we have shown

$$
g = \text{const.} \times \phi^*(\cdot ; q, \lambda_{n_1}(q)) \neq 0
$$

(6.3)

and

$$
f = \text{const.} \times \phi(\cdot ; q, h, \lambda_{n_1}(q, h)) \neq 0.
$$

(6.4)
As in the preceding section, we set $\lambda_n = \lambda_n(p, h)$ and $\lambda_n^* = \lambda_n^*(p)$. Furthermore, we set

$$
\psi_n = \phi(\cdot; q, h, \lambda_n(q, h)), \quad \mu_n = \lambda_n(q, h) \quad (6.5)
$$
and

$$
\psi_n^* = \phi(\cdot; q, \lambda_n^*(q)), \quad \mu_n^* = \lambda_n^*(q). \quad (6.6)
$$

From Theorem 5,

$$
\lambda_n = \mu_n^*, \quad \lambda_n^* = \mu_n \quad (6.7)
$$
follows.

By means of (6.1)-(6.6), we have

$$(\psi_n^* \phi_n^*)' = \text{const.} \times (\psi_n^* \phi_n)' .$$

Setting $x = 0$, we obtain $(\psi_n^* \phi_n^*)' = (\psi_n^* \phi_n)'$, hence

$$\psi_n^* \phi_n^* = \psi_n^* \phi_n + \text{const.} .$$

Again by setting $x = 0$, we get

$$\psi_n^* \phi_n^* = \psi_n^* \phi_n .$$

By virtue of (5.50), this relation implies

$$\psi_n^* = c \phi_n^* \quad \text{and} \quad \psi_n = c \phi_n \quad (6.8)$$
for some $c \in C^2[0, 1]$. We substitute these equalities into

$$\psi_n^* \psi_n - \psi_n^* \psi_n' = (\lambda_n^* - \lambda_n) \psi_n^* \psi_n ,$$

which is derived from (6.7), and arrive at the relation

$$2c' (\phi_n^* \phi_n - \phi_n^* \phi_n') + c (\phi_n^* \phi_n - \phi_n^* \phi_n') = (\lambda_n^* - \lambda_n) c \phi_n^* \phi_n .$$

Therefore, since $\phi_n^* \phi_n - \phi_n^* \phi_n' = W = (\lambda_n^* - \lambda_n^*) \phi_n^* \phi_n$, we get

$$(cW)' = 0 .$$

By setting $x = 0$ into $cW = \text{const.}$, we obtain $cW \equiv 1$, so that we have

$$\psi_n^* - \phi_n^*/W, \quad \psi_n = \phi_n/W . \quad (6.9)$$

We now substitute $\psi_n = \phi_n/W$ into

$$q = \lambda_n + \psi_n'/\psi_n .$$
After some elementary calculations, which we omit here, we come to the relation
\[ q = p - 2 \frac{d^2}{dx^2} \log(W). \]

(2) In connection with the corollary of Section 2, we have the following theorem: We see that for \( p \in C^1_s[0,1] \) and \( n_1 \in \mathcal{N}^*, \) there exists \( s = s(x) = s(x; p, n_1) \neq 0 \) such that
\[
\left( -\frac{d^2}{dx^2} + p(x) \right) s = \lambda^*_n s \quad (0 \leq x \leq 1) \quad \text{(recall \( \lambda^*_n = \lambda^*_{n_1}(p) \), (6.10.1)}
\]
\[
s(1-x) = (-1)^{n_1} s(x) \quad (0 \leq x \leq 1). \quad \text{(6.10.2)}
\]

Since \( \phi^*_n(1-x) = (-1)^{n_1+1} \phi^*_n(x) \) (recall \( \phi^*_n = \phi^*(\cdot; p, \lambda^*_{n_1}(p)) \)), \( s \) is linearly independent of \( \phi^*_n \), and \( s(0) \neq 0 \) holds. Therefore, the real constant
\[
h_{n_1}(p) = s'(0)/s(0) \quad \text{(6.11)}
\]
is well defined. Then,

**Theorem 10.** For \( p \in C^1_s[0,1] \) and \( n_1 \in \mathcal{N}^*, \lambda^*_n(p) = \lambda_{n_1}(p, h) \) if and only if \( h = h_{n_1}(p) \).

*Proof.* If \( \lambda^*_n(p) = \lambda_{n_1}(p, h) \), then we have \( s = \text{const.} \times \phi_{n_1} (\neq \phi(\cdot; p, h, \lambda_{n_1}(p, h))) \), hence \( h = h_{n_1}(p) \) follows. If \( h = h_{n_1}(p) \), conversely \( \lambda_{n_1} = \lambda_{n_1}(p, h) \) holds for some \( m_1 \in \mathcal{N} \), and \( s \) satisfies \( s = \text{const.} \times \phi(\cdot; p, h, \lambda_{m_1}(p, h)) \). Since (6.10) gives
\[
\phi_{n_1}'s - \phi_{n_1}^*s' = \text{const.} \neq 0,
\]
m_1 = n_1 can be shown by the same argument as in the proof of Lemma 8. Therefore, \( \lambda^*_n(p) = \lambda_{n_1}(p, h) \) holds. ■

For \( p \in C^1_s[0,1] \) and \( n_1 \in \mathcal{N}, \) put
\[
\mathcal{Q}_{p,n} = \{(q, h) \in C^1_s[0,1] \times \mathcal{R} \mid \lambda_n(q, h) = \lambda_n(p, h) \ (n \in \mathcal{N} \setminus \{n_1\})\}. \quad \text{(6.12)}
\]

If \( n_1 \geq 1, \)
\[
\mathcal{Q}_{p,n_1} = \{(p, h) \mid h \in \mathcal{R}\} \cup \{(q(h), h) \mid h \in \mathcal{R}\}
\]
by Theorem B, where
\[
q(h) = p - 2 \frac{d^2}{dx^2} \log(W). \quad \text{(6.13)}
\]

From Theorems 8 and 10, we conclude with
THEOREM 11. If \( n_1 \geq 1 \), the set \( \bar{Q}_{p,n_1} \) has a bifurcation structure with a bifurcation point \( (p, h_{n_1}(p)) \) (see Fig. 4).

(3) Recently, the author has obtained

THEOREM C. For \( p, q \in C^1([0, 1], h, j \in \mathcal{R}, \text{ and } n_1 \in \mathcal{N}, \text{ if} \)

\[
\lambda_n(q, j) = \lambda_n(p, h) \quad (n \in \mathcal{N} \setminus \{n_1\})
\]

(6.14)

and

\[
\frac{1}{2}q(0) - j^2 = \frac{1}{2}p(0) - h^2,
\]

(6.15)

then

\[
q = p, \quad j = h.
\]

(6.16)

Namely, (6.15) recovers \( \lambda_n(q, j) = \lambda_n(p, h) \) for any \( n_1 \in \mathcal{N} \), in contrast with the condition \( j = h \). In a forthcoming article, we shall prove Theorem C and its generalizations.

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