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# An algorithm for determining copositive matrices ☆

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ABSTRACT

In this paper, we present an algorithm of simple exponential growth called COPOMATRIX for determining the copositivity of a real symmetric matrix. The core of this algorithm is a decomposition theorem, which is used to deal with simplicial subdivision of $\widehat{T}^{-} = \{y \in \Delta_m | \beta^T y \leqslant 0\}$ on the standard simplex $\Delta_m$, where each component of the vector $\beta$ is $-1$, 0 or 1.



## 1. Introduction

**Question 1.** Let $A$ be a given $n \times n$ real symmetric matrix, $\mathbb{R}_+$ be the set of nonnegative real numbers, and

$$Q(X) = X^T AX, \quad X \neq 0$$

be a quadratic form. What conditions should $A$ satisfy for $[\forall X \in \mathbb{R}_+^n,\ Q(X) \geqslant 0\ (>0)]$?

If $[\forall X \in \mathbb{R}_+^n,\ Q(X) \geqslant 0\ (>0)]$, then the quadratic form $Q(X)$ is called a (strictly) copositive quadratic form and the corresponding matrix $A$ is called a (strictly) copositive matrix.

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Copositive matrices have numerous applications in diverse fields of applied mathematics, especially in mathematical programming and graph theory (see [3,5,7,11,13,15,16,25,26,29,39]). Therefore copositivity has been studied thoroughly since 1950s (see [1,6,14,17,20,22,23,27,28,33,34,36,41–43,48,49]).

In general, it is an NP-complete problem to determine whether a given $n \times n$ symmetric matrix is not copositive [37,38]. This means that every algorithm that solves the problem, in the worst case, will require at least an exponential number of operations, unless P $=$ NP. For that reason, it is still valuable for the existence of so many incomplete algorithms discussing some special kinds of matrices (see [3,4,10,12,18,19,21,24,30,38]). For small values of $n(\leqslant 6)$, some necessary and sufficient conditions have been constructed (see [1,14,30,49]). From another viewpoint, Question 1 is a typical real quantifier elimination problem [2,8,9,32,35,40,44,45], which can be solved by standard tools of real quantifier elimination (e.g., using CAD) [2,8,9,46,47]. Thus, there is a complete algorithm for determining copositive matrices theoretically. Unfortunately, this algorithm is not efficient in practice for the CAD algorithm is of doubly exponential time complexity (see [2,8,9]). In this paper, we will construct a complete algorithm with singly exponential time bound.

The standard simplex $\Delta_m (m \geqslant 2)$ is defined as the following set

$$\Delta_m = \left\{(y_1, \ldots, y_m)^T \,|\, y_1 + \cdots + y_m = 1, y_1 \geqslant 0, \ldots, y_m \geqslant 0\right\}.$$

It is well known that the dimension of $\Delta_m$ is $m - 1$. Denote the vertices of $\Delta_m$ as $e_1, \ldots, e_m$, namely, $e_1 = (1, 0, \ldots, 0)^T, \ldots, e_m = (0, 0, \ldots, 1)^T$.

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and be partitioned as

$$A = [\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & \alpha^T \\ \alpha & A_2 \end{bmatrix}.$$

Define $B = \alpha_{11}A_2 - \alpha\alpha^T$. It is easy to see the following facts (cf. [1])

1. If $\alpha_{1i} \geqslant 0,\ i = 2, \ldots, n$, then $A$ is (strictly) copositive $\Longleftrightarrow \alpha_{11} \geqslant 0\ (> 0)$ and $A_2$ is (strictly) copositive.
2. If at least one of $\alpha_{1i}$ is negative, then we need only to focus on the set of points $T^- = \{y \in \Delta_{n-1} |\, \alpha^T y \leqslant 0\}$. It is well known that $T^-$ is a convex polytope on $\Delta_{n-1}$ (see [1]). The polytope $T^-$ can be subdivided into the simplices $S_1, \ldots, S_p$, that is,

$$T^- = \bigcup_{i=1}^{p} S_i, \quad \mathrm{int}(S_i) \bigcap \mathrm{int}(S_j) = \emptyset, \quad \text{for } i \neq j,$$

   where $\mathrm{int}(S_i)$ denotes the interior of simplex $S_i$. The coordinates of the vertices that span the simplex $S_i$ constitute a matrix denoted as $W_i$. Andersson et al. [1, p. 23] proved the following results.

**Lemma 1.1**

(a) *A is copositive iff* $\alpha_{11} \geqslant 0$ *and* $A_2,\ W_1^T BW_1, \ldots, W_p^T BW_p$ *are all copositive.*
(b) *A is strictly copositive iff* $\alpha_{11} > 0$ *and* $A_2,\ W_1^T BW_1, \ldots, W_p^T BW_p$ *are all strictly copositive.*

In order to formulate the algorithm of Lemma 1.1, we first consider how to obtain the simplicial subdivision of the polytope $T^- = \{y \in \Delta_{n-1} |\, \alpha^T y \leqslant 0\}$. For small values of $n(\leqslant 6)$, Andersson et al. [1] and Yang and Li [49] give the simplicial subdivision of $T^-$. However, they do not provide a procedure for a simplicial subdivision of $T^-$ for arbitrary values of $n$. We propose a simplicial subdivision of $T^-$ for all values of $n$, and consequently construct a complete algorithm for determining the copositivity of an $n \times n$ matrix.

We will adopt a flexible approach. Rather than subdivide $T^-$ into simplices (of course our method is also valid for subdividing $T^-$ into simplices), we first transform the matrix $A$ into the following matrix called $\widehat{A}$.

Let $\alpha = (\alpha_{12}, \ldots, \alpha_{1n})^T$ and $D = \mathrm{diag}(d_2, \ldots, d_n)$, where

$$d_i = \begin{cases} 1, & \text{if } \alpha_{1i} = 0; \\ 1/|\alpha_{1i}|, & \text{if } a_{1i} \neq 0. \end{cases}$$

Then

$$\widehat{A} = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \widehat{\alpha}^T \\ \widehat{\alpha} & DA_2D \end{bmatrix}. \tag{1}$$

where $\widehat{\alpha} = (\mathrm{sign}(\alpha_{12}), \ldots, \mathrm{sign}(\alpha_{1n}))^T$.

Obviously, $A$ is (strictly) copositive $\Longleftrightarrow$ $\widehat{A}$ is (strictly) copositive. Apply Lemma 1.1 to $\widehat{A}$. Let

$$\beta_1 = \mathrm{sign}(\alpha_{12}), \ldots, \ \beta_{n-1} = \mathrm{sign}(\alpha_{1n}).$$

Thus we just need to subdivide $\widehat{T}^-$ into simplices, where

$$\widehat{T}^- = \{y \in \Delta_{n-1} | (\beta_1, \ldots, \beta_{n-1})y \leqslant 0, \quad \beta_i \in \{-1, 0, 1\}\}.$$

Next we make further simplification: separate $-1, 0, 1$ from $\beta_1, \ldots, \beta_{n-1}$, namely let

$$\begin{aligned} &\beta_{a_1} = \cdots = \beta_{a_s} = 1, \quad \beta_{b_1} = \cdots = \beta_{b_t} = -1, \quad \beta_{c_1} = \cdots = \beta_{c_r} = 0. \\ &\{a_1, \ldots, a_s, \ b_1, \ldots, b_t, \ c_1, \ldots, \ c_r\} = \{1, \ldots, n-1\}, \\ &r, \ s, \ t \geqslant 0, \ t \geqslant 1, \ r + s + t = n - 1. \end{aligned}$$

In geometry it is easy to see that the convex polytope $\widehat{T}^-$ is the convex hull of its surface $S^-$ and its vertices $e_{c_1}, \ldots, e_{c_r}$, that is,

$$\widehat{T}^- = \mathrm{conv}\{e_{c_1}, \ldots, e_{c_r}, S^-\}. \tag{2}$$

$$S^- = \Big\{(y_1, \ldots, y_{n-1})^T \in \Delta_{n-1} | y_{a_1} + \cdots + y_{a_s} - y_{b_1} - \cdots - y_{b_t} \leqslant 0, \\ (y_{a_1}, \ldots, y_{a_s}, y_{b_1}, \ldots, y_{b_t})^T \in \Delta_{s+t}\Big\}. \tag{3}$$

If the simplicial subdivision of $S^-$ is known, the simplicial subdivision of $\widehat{T}^-$ is directly obtained by (2). So we just need to study the simplicial subdivision of the polytope $S^-$.

## 2. A simplicial subdivision algorithm for the convex polytope $S^-$

### 2.1. Fundamental notations

The notation $S^-$ is simple, but it cannot reveal the information of convex polytopes. In order to simplify the descriptions, we will introduce a new notation, which is fundamental to our study.

**Definition 2.1.** Suppose that two sequences of positive integers $[a_1, a_2, \ldots, a_s]$, $[b_1, b_2, \ldots, b_t]$ satisfy

$$\{a_1, \ldots, a_s, \ b_1, \ldots, b_t\} \subseteq \{1, 2, \ldots, m\}, \quad s \geqslant 0, \ t \geqslant 1, \ m \geqslant s + t \geqslant 2,$$

where all of $s + t$ elements of $\{a_1, \dots, a_s,\ b_1, \dots, b_t\}$ are distinct. Then the notation $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$ is defined as the polytope $S^-$ in (3).

For example, let us compare the polytope $[[2, 3], [5]]_5$ and the polytope $[[2, 3], [5]]_6$. $[[2, 3], [5]]_5$ denotes the polytope

$$\{(y_1, \dots, y_5)^T \in \Delta_5 | y_2 + y_3 - y_5 \leqslant 0, (y_2, y_3, y_5)^T \in \Delta_3\}.$$

Here $(y_2, y_3, y_5)^T \in \Delta_3$ implies that $y_1 = 0, y_4 = 0$. $[[2, 3], [5]]_6$ indicates the polytope

$$\{(y_1, \dots, y_6)^T \in \Delta_6 | y_2 + y_3 - y_5 \leqslant 0, (y_2, y_3, y_5)^T \in \Delta_3\}.$$

Here $(y_2, y_3, y_5)^T \in \Delta_3$ implies that $y_1 = 0, y_4 = 0, y_6 = 0$. It is clear that $[[2, 3], [5]]_5$ and $[[2, 3], [5]]_6$ are congruent, although they are sets in simplices of different dimensions.

For $0 \leqslant k \leqslant m - 1$, the polytope $L_k^-$ is defined as

$$L_k^- = \{(y_1, \dots, y_m)^T \in \Delta_m | y_1 + \dots + y_k - y_{k+1} - \dots - y_m \leqslant 0\}.$$

$L_k^-$ is written as $[[1, \dots, k], [k+1, \dots, m]]_m$ by the notation of Definition 2.1. $L_k^-$ is a special case of $S^-$, but this notation is more convenient for our analysis.

In the following, we will study the basic geometric properties of convex polytope $[[a_1, a_2, \dots, a_s], [b_1, b_2, \dots, b_t]]_m$.

### 2.2. Geometric properties of $S^-$

Let $e_1, \dots, e_m$ be vertices of the standard simplex $\Delta_m$, and $M_{i,j} = (e_i + e_j)/2$ be the midpoint of the line segment $\overline{e_i e_j}$.

The following result is stated in [1] without proof. For completeness, we give a proof.

**Lemma 2.1** [1]. *Given a convex polytope $L_k^-$, then all of its vertices are*

$$V = \{e_{k+1}, \dots, e_m,\ M_{i,j}, i = 1, 2, \dots, k,\ j = k+1, \dots, m\}.$$

*The number of the vertices is $|V| = (k+1)(m-k)$.*

**Proof.** Note that the convex polytope $L_k^-$ is obtained by cutting the standard simplex $\Delta_m$ with the hyperplane

$$L_{=0} : \ y_1 + \dots + y_k - y_{k+1} - \dots - y_m = 0.$$

Therefore the vertices of the polytope $L_k^-$ come from two parts: one part is vertices of $\Delta_m$, that is, $\{e_{k+1}, \dots, e_m\}$; while the other part is the intersection points of the hyperplane $L_{=0}$ and the edges of standard simplex $\Delta_m$.

First consider the intersection point of $L_{=0}$ and the edge $ae_1 + be_{k+1}$ $(a, b \geqslant 0,\ a + b = 1)$. Substitute $ae_1 + be_{k+1}$ into the following equations,

$$y_1 + \dots + y_k - y_{k+1} - \dots - y_m = 0, \quad y_1 + \dots + y_m = 1$$

Therefore, the solutions are $a = 1/2, b = 1/2$, namely, the intersection point is $M_{1,k+1}$.

In the same way, we get all intersection points of $L_{=0}$ and the edges of $\Delta_m$. They are $\{M_{i,j}, i = 1, 2, \dots, k,\ j = k+1, \dots, m.\}$.

Hence the number of vertices of $L_k^-$ is $|V| = m - k + k(m-k) = (k+1)(m-k)$. □

Likewise, we can prove the following lemma.

**Lemma 2.2.** *Given a convex polytope* $[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$*, then all of its vertices are*

$$V = \{e_{b_1}, \ldots, e_{b_t},\ M_{a_i,b_j}, i = 1, 2, \ldots, s,\ j = 1, \ldots, t\}.$$

*The number of the vertices is* $|V| = (s+1)t$*.*

We see that the polytope $[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$ and the polytope $L_k^-$ are similar in many respects, which will be further discussed.

**Lemma 2.3.** *The convex polytope* $L_k^-$ *is simplicial iff* $k = 0$*, or* $k = m-1$*.*

**Proof.** When $k = 0$, $L_0^- = \Delta_m$ is simplicial.
When $k = m-1$, consider the convex polytope

$$L_{m-1}^- := \{(y_1, \ldots, y_m)^T \in \Delta_m |\, y_1 + \cdots + y_{m-1} - y_m \leqslant 0\}.$$

By Lemma 2.1, we know that all vertices of $L_{m-1}^-$ are $\{e_m,\ M_{i,m}, i = 1, 2, \ldots, m-1\}$. Obviously all the vectors of $\{M_{i,m} - e_m, i = 1, 2, \ldots, m-1\}$ are linearly independent, so $L_{m-1}^-$ is simplicial.

Conversely, we know that the dimension of the polytope $L_k^-$ is $m-1$. If $k \neq 0, m-1$, then by Lemma 2.1, the number of the vertices of $L_k^-$ is $(k+1)(m-k) \neq m$, so $L_k^-$ is not simplicial. □

**Lemma 2.4.** *The convex polytope* $[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$ *(here the vertices are obtained by Lemma 2.2) is simplicial iff* $s = 0$*, or* $t = 1$*.*

**Lemma 2.5.** *The dimension of the polytope* $[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$ *is* $(s + t - 1)$*.*

If the polytope $[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$ is not a simplex, we will subdivide it into simplices.

**Lemma 2.6.** *If the polytope* $[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$ *is not a simplex, then there are only two* $(s+t-2)$*-dimensional surfaces that do not include the vertex* $M_{a_1,b_1}$*. They are*

$$[[a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m,\ [[a_1, \ldots, a_s], [b_2, \ldots, b_t]]_m.$$

*(obtained by deleting* $a_1, b_1$ *from array* $[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$*, respectively)*

**Proof.** All the $(s+t-2)$-dimensional surfaces of the convex polytope $[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$ are obviously

$$[[\widehat{a}_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m,$$
$$[[a_1, \widehat{a}_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m,$$
$$\ldots,$$
$$[[a_1, a_3, \ldots, a_s], [b_1, b_2, \ldots, \widehat{b}_t]]_m$$

(where the notation $[[\widehat{a}_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$ is the polytope with $a_1$ removed) and

$$\{(y_1, \ldots, y_m)^T \in \Delta_m | y_{a_1} + \cdots + y_{a_s} - y_{b_1} - \cdots - y_{b_t} = 0,\ (y_{a_1}, \ldots, y_{a_s}, y_{b_1}, \ldots, y_{b_t})^T \in \Delta_{s+t}\}.$$

That makes $s+t+1$ $(s+t-2)$-dimensional surfaces in all. We can verify that only

$$[[a_2,\ldots,a_s],[b_1,b_2,\ldots,b_t]]_m,\ [[a_1,\ldots,a_s],[b_2,\ldots,b_t]]_m$$

do not include the vertex $M_{a_1,b_1}$. □

Lemma 2.6 leads to the following decomposition theorem.

### 2.3. The decomposition process for the polytope $S^-$

**Theorem 2.1** Decomposition theorem. *If the polytope* $[[a_1,a_2,\ldots,a_s],$ $[b_1,b_2,\ldots,b_t]]_m$ *is not simplicial, then it can be decomposed into the union of two convex polytopes (not always simplicial). The expression is*

$$\begin{aligned}[[a_1,a_2,\ldots,a_s],[b_1,b_2,\ldots,b_t]]_m &= \mathrm{conv}\{M_{a_1,b_1},[[a_2,\ldots,a_s],[b_1,b_2,\ldots,b_t]]_m\}\\ &\quad\bigcup \mathrm{conv}\{M_{a_1,b_1},[[a_1,a_2,\ldots,a_s],[b_2,\ldots,b_t]]_m\}.\end{aligned}$$

*Here* conv$\{S\}$ *denotes the convex hull of the set $S$ of points .*

**Proof.** This follows from Lemma 2.6. □

Based on Theorem 2.1, the polytope $S^-$ can be easily subdivided into simplices.

**Example 1.** Show the simplicial subdivision of the following convex polytope

$$L_2^- := \{(y_1,\ldots,y_5)\,|\,y_1+y_2-y_3-y_4-y_5\leqslant 0,\ (y_1,\ldots,y_5)^T\in\Delta_5\}.$$

**Solution.** Denote $L_2^-$ as $[[1,2],[3,4,5]]_5$. We know that $[[1,2],[3,4,5]]_5$ is not simplicial by Lemma 2.4. Using Theorem 2.1 we have

$$[[1,2],[3,4,5]]_5 = \mathrm{conv}\{M_{1,3},[[2],[3,4,5]]_5\}\bigcup \mathrm{conv}\{M_{1,3},[[1,2],[4,5]]_5\}.$$

By Lemma 2.4 we know that both $[[2],[3,4,5]]_5$ and $[[1,2],[4,5]]_5$ are not simplicial. Therefore we repeatedly apply Theorem 2.1 to them and have

$$\begin{aligned}[[2],[3,4,5]]_5 &= \mathrm{conv}\{M_{2,3},[[\ ],[3,4,5]]_5\}\bigcup \mathrm{conv}\{M_{2,3},[[2],[4,5]]_5\}\\ &= \mathrm{conv}\{M_{2,3},[[\ ],[3,4,5]]_5\}\bigcup \mathrm{conv}\{M_{2,3},M_{2,4},[[\ ],[4,5]]_5\}\\ &\bigcup \mathrm{conv}\{M_{2,3},M_{2,4},[[2],[5]]_5\} = \mathrm{conv}\{M_{2,3},e_3,e_4,e_5\}\\ &\bigcup \mathrm{conv}\{M_{2,3},M_{2,4},e_4,e_5\}\bigcup \mathrm{conv}\{M_{2,3},M_{2,4},M_{2,5},e_5\}.\end{aligned}$$

$$\begin{aligned}[[1,2],[4,5]]_5 &= \mathrm{conv}\{M_{1,4},[[2],[4,5]]_5\}\bigcup \mathrm{conv}\{M_{1,4},[[1,2],[5]]_5\}.\\ &= \mathrm{conv}\{M_{1,4},M_{2,4},[[\ ],[4,5]]_5\}\bigcup \mathrm{conv}\{M_{1,4},M_{2,4},[[2],[5]]_5\}\\ &\bigcup \mathrm{conv}\{M_{1,4},[[1,2],[5]]_5\}\\ &= \mathrm{conv}\{M_{1,4},M_{2,4},e_4,e_5\}\bigcup \mathrm{conv}\{M_{1,4},M_{2,4},M_{2,5},e_5\}\\ &\bigcup \mathrm{conv}\{M_{1,4},M_{1,5},M_{2,5},e_5\}.\end{aligned}$$

Finally we get the expression of simplicial subdivision of $[[1,2],[3,4,5]]_5$,

$$\begin{aligned}[[1,2],[3,4,5]]_5 = \ & \mathrm{conv}\{M_{1,3}, M_{2,3}, e_3, e_4, e_5\} \bigcup \mathrm{conv}\{M_{1,3}, M_{2,3}, M_{2,4}, e_4, e_5\} \\ & \bigcup \ \mathrm{conv}\{M_{1,3}, M_{2,3}, M_{2,4}, M_{2,5}, e_5\} \bigcup \mathrm{conv}\{M_{1,3}, M_{1,4}, M_{2,4}, e_4, e_5\} \\ & \bigcup \ \mathrm{conv}\{M_{1,3}, M_{1,4}, M_{2,4}, M_{2,5}, e_5\} \bigcup \mathrm{conv}\{M_{1,3}, M_{1,4}, M_{1,5}, M_{2,5}, e_5\}.\end{aligned}$$

So $[[1,2],[3,4,5]]_5$ is a union of six 4-dimensional simplices.

We summarize the decomposition process of Example 1 into the following algorithm.

**Algorithm 1 (Vmatrix)**

Input: The expression of polytope $[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m$.
Output: Simplices $D_1, D_2, \ldots, D_p$(denoted by matrices) such that

$$[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m = \bigcup_{i=1}^{p} D_i, \ \mathrm{int}(D_i) \bigcap \mathrm{int}(D_j) = \emptyset, \ \text{for } i \neq j.$$

V1: Let $F := \{[[a_1, a_2, \ldots, a_s], [b_1, b_2, \ldots, b_t]]_m\}$, temp $:= \emptyset$.
V2: When $F \neq \emptyset$, repeat the following procedures
 V21: Choose a polytope $N \in F$. If $N$ is simplicial, then
  temp := temp $\cup \{N\}$, $F := F \setminus \{N\}$.
 V22: If the polytope $N$ is not simplicial, then by Theorem 2.1 decompose it into two convex polytope $B_1, B_2$.
  $F := F \setminus \{N\} \cup \{B_1, B_2\}$. Go to step V2.
V3: Return temp.

We have written a function in Maple [31] to implement the above algorithm.

Lastly, we will present a formula for computing the number of simplices given by the polytope $L_k^-$ subdivision.

**Lemma 2.7.** *According to algorithm Vmatrix, the convex polytope* $[[1, \ldots, k], [k+1, \ldots, m]]_m$ $(0 \leqslant k \leqslant m-1, m \geqslant 2)$ *can be subdivided just into* $f(k,m)$ *simplices, where*

$$f(k,m) = \binom{m-1}{k} = \frac{(m-1)!}{k!(m-1-k)!}.$$

We know that $f(k,m)$ has the same recurrence formula as binomial coefficients by Theorem 2.1. Thus the proof of Lemma 2.7 is easy via an induction argument. This formula will be used to estimate the cost of Algorithm 2 in the next section.

## 3. Determining algorithm for copositive matrices

In this section, we will present the complete determining algorithm of a copositive matrix.

Given an $n \times n$ symmetric matrix

$$A = [\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & \alpha^T \\ \alpha & A_2 \end{bmatrix},$$

compute $\widehat{A}$ (see (1))

$$\widehat{A} = \begin{bmatrix} \alpha_{11} & \widehat{\alpha}^T \\ \widehat{\alpha} & DA_2D \end{bmatrix}.$$

Let $B = \alpha_{11} D A_2 D - \widehat{\alpha}\widehat{\alpha}^T$, and let

$$\widehat{\alpha} = (\text{sign}(\alpha_{12}), \ldots, \text{sign}(\alpha_{1n}))^T = (\beta_1, \ldots, \beta_{n-1})^T.$$

Define the projection operator **Proj** of the matrix $A$ as follows,

- If $\beta_i \geqslant 0,\ i = 1, \ldots, n-1$, then

$$\text{Proj(A)} = \{DA_2D\}.$$

- If there is at least one $-1$ in $\beta_i$, then

$$\text{Proj(A)} = \{DA_2D,\ W_1^T B W_1, \ldots, W_p^T B W_P\}.$$

Here the matrices $W_1, \ldots, W_p$ is fixed by the simplicial subdivision of the convex polytope $\widehat{T}^-$ (see (2)).

**Algorithm 2 (COPOMATRIX)**

Input: Symmetric matrix $A \in \mathbf{R}^{n\times n} (n \geqslant 2)$.
Output: $A$ is copositive, or $A$ is not copositive.
C1: Let $F := \{A\}$.
C2: Repeat the following steps for the set $F$.
  C21: If the set $F$ is empty, then return "$A$ is copositive".
  C22: Check the $(1,1)^{th}$ entry of every matrix $K$ in set $F$. If at least one of them is negative, then return "$A$ is not copositive".
  C23: Compute the projective set $P := \bigcup_{K \in F} \text{Proj}(K)$ of set $F$.
    $F := P \setminus \{$the nonnegative matrices of $P\}$. Go to step C21.

Note that the above algorithm is also valid for $2 \times 2$ matrices. Furthermore, for strictly copositive matrices we can also formulate a similar algorithm.

The correctness of the algorithm COPOMATRIX is guaranteed by Lemma 1.1, and the algorithm obviously terminates. The cost of the algorithm mainly depends on the number of simplicial subdivisions of the polytope. According to Lemma 2.7, we can estimate that in the worst case it is at most:

$$\begin{aligned}
&\left(\binom{n-2}{\left[\frac{n-2}{2}\right]} + 1\right)\left(\binom{n-3}{\left[\frac{n-3}{2}\right]} + 1\right)\cdots\left(\binom{2}{1} + 1\right) \\
&\leqslant (2^{n-3})(2^{n-4})\cdots(2)(2) \\
&= 2^{(n-2)(n-3)/2+1}.
\end{aligned}$$

The bound $2^{(n-2)(n-3)/2+1}$ is already much lower than doubly-exponential cost of CAD [2,9]. We have written a function in Maple to implement the algorithm COPOMATRIX. For free non-commercial request contact yaoyong@casit.ac.cn or j.jia.xu@gmail.com.

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