Optimal birth control of age-dependent competitive species
II. Free horizon problems

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Received 22 May 2004
Available online 19 November 2004
Submitted by M. Iannelli

Abstract

We study optimal birth policies for two age-dependent populations in a competing system, which is controlled by fertilities. New results on problems with free final time and integral phase constraints are presented, and the approximate controllability of system is discussed. © 2004 Elsevier Inc. All rights reserved.

Keywords: Population model; Age-dependence; Maximum principle; Optimal control; Controllability

We continue the study initiated in [1]. This paper presents further new results on several optimal birth control problems. We first investigate the problem with fixed final state and free final time, of which the time-optimal problem is a special case. Then we examine problems with integral phase constraints. Finally, we study the approximate controllability of controlled system. It is supposed that the reader is familiar with the terminology and notation in [1]. More research work on the control problems of age-dependent population models can be found in [4–22].

✩ This work is supported by the scientific research project (no. 20040467) of Educational Bureau of Zhejiang province and the science foundation of Hangzhou Dianzi University under grant no. ZX0202Y43.

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7. Problems with fixed terminal

Consider the optimal control problem:

$$
\text{minimize } J(p, \beta) = \int_0^{t_1} \int_0^A L\left(p_1(a, t), p_2(a, t), \beta_1(t), \beta_2(t)\right) da dt,
$$

where \((p(a, t), \beta(t))\), \(p(a, t) = (p_1(a, t), p_2(a, t))\), \(\beta(t) = (\beta_1(t), \beta_2(t))\), is subject to

$$
\begin{align*}
\frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} &= -\mu_1(a, t)p_1 - \lambda_1(a, t)P_2(t)p_1, \\
\frac{\partial p_2}{\partial t} + \frac{\partial p_2}{\partial a} &= -\mu_2(a, t)p_2 - \lambda_2(a, t)P_1(t)p_2, \\
p_i(0, t) &= \beta_i(t) \int_{a_1}^{a_2} m_i(a, t)p_i(a, t) da, \\
p_i(a, 0) &= p_{i0}(a), \\
P_i(t) &= \int_0^{a} p_i(a, t) da, \quad i = 1, 2, \quad (a, t) \in Q,
\end{align*}
$$

and

$$
p_i(a, t_1) = p_{i0}(a), \quad i = 1, 2.
$$

Here \(t_1 > 0\) is not fixed, \(p_{i0}(a)\) is prescribed nonnegative function.

For each \(t_1 > 0\), choose a measurable function \(v \geq 0\), define the time transformation

$$
t(\tau) = \int_0^\tau v(s) ds, \quad t(1) = t_1,
$$

and

$$
p_i(a, \tau) = p_i(a, t(\tau)), \quad \beta_i(\tau) = \begin{cases} \beta_i(t(\tau)), & \tau \in S_1, \\
\text{arbitrary}, & \tau \in S_2, \end{cases} \quad i = 1, 2,
$$

where

$$
S_1 = \{\tau \in [0, 1]: t(\tau) > 0\}, \quad S_2 = \{\tau \in [0, 1]: t(\tau) = 0\}.
$$

If we define similarly \(\mu_i(a, \tau), \lambda_i(a, \tau), m_i(a, \tau)\), then \((p(a, \tau), \beta(\tau))\) satisfies

$$
\begin{align*}
\frac{\partial p_1}{\partial \tau} + v(\tau) \frac{\partial p_1}{\partial a} &= -[\mu_1(a, \tau) + \lambda_1(a, \tau)P_2(\tau)]p_1 v(\tau), \\
\frac{\partial p_2}{\partial \tau} + v(\tau) \frac{\partial p_2}{\partial a} &= -[\mu_2(a, \tau) + \lambda_2(a, \tau)P_1(\tau)]p_2 v(\tau), \\
v(\tau)p_i(0, \tau) &= v(\tau)\beta_i(\tau) \int_{a_1}^{a_2} m_i(a, \tau)p_i(a, \tau) da, \\
p_i(a, 0) &= p_{i0}(a), \\
P_i(\tau) &= \int_0^{a} p_i(a, \tau) da, \quad i = 1, 2, \quad (a, \tau) \in [0, A] \times [0, 1],
\end{align*}
$$

and

$$
p_i(a, 1) = p_{i0}^0(a), \quad i = 1, 2.
$$
Consequently, if \((p^*(a, t), \beta^*(t), t^*_1)\) is a solution of problem (1)–(3), and \(v^*(\tau)\) is a measurable function corresponding to \(t^*_1\), then \((p^*(a, \tau), \beta^*(\tau), v^*(\tau))\) must be a solution to the following problem:

\[
\text{minimize } J(p, \beta) = \int_0^A \int_0^A v(\tau) L(p_1(a, \tau), p_2(a, \tau), \beta_1(\tau), \beta_2(\tau)) \, da \, d\tau,
\]

where \((p(a, \tau), \beta(\tau), v(\tau))\) is subject to (7)–(8).

Let \(\beta(\tau)\) be fixed as \(\beta^*(\tau)\), then \((p^*(a, \tau), v^*(\tau))\) solves the problem

\[
\text{minimize } J(p, \beta^*, v) = \int_0^A \int_0^A v(\tau) L(p_1(a, \tau), p_2(a, \tau), \beta^*_1(\tau), \beta^*_2(\tau)) \, da \, d\tau,
\]

where \((p(a, \tau), \beta^*(\tau), v(\tau))\) satisfies (7)–(8).

Suppose that \((p^*, v^*)\) is a solution of the problem (10), we seek the optimality conditions via Dubovitskii–Milyutin general extremal theory.

Let \(X = C(0, 1; L^2(0, A; R^2)) \times L^\infty(0, 1)\) define the inequality constraint

\[
\Omega_1 = \{(p, v) \in X: v(\tau) \geq 0, \forall \tau \in [0, 1]\}
\]

and the equality constraint

\[
\Omega_2 = \{(p, v) \in X: (p, \beta^*, v) \text{ is subject to } (7)–(8)\}.
\]

It is clear that the problem (10) is equivalent to the problem

\[
\begin{align*}
\text{minimize} & \quad J(p, \beta^*, v) = \int_0^A \int_0^A v(\tau) L(p_1(a, \tau), p_2(a, \tau), \beta^*_1(\tau), \beta^*_2(\tau)) \, da \, d\tau, \\
(p, v) & \in \Omega_1 \cap \Omega_2 \subset X.
\end{align*}
\]

It is easy to see that the functional \(J\) is differentiable at every \((\bar{p}, \bar{v}),\) and

\[
J'(\bar{p}, \bar{v})(p, v) = \int_0^A \int_0^A \left[ v(\tau) \left[ p_1(a, \tau) \frac{\partial L}{\partial p_1}(\bar{p}, \beta^*) + p_2(a, \tau) \frac{\partial L}{\partial p_2}(\bar{p}, \beta^*) \right] \\
+ v(\tau) L(\bar{p}, \beta^*) \right] da \, d\tau.
\]

Hence \(J\) is regularly decreasing at \((p^*, v^*)\) and its cone of directions of decrease is characterized by

\[
K_0 = \{(p, v) \in X: J'(p^*, v^*)(p, v) < 0\}.
\]

If \(K_0 \neq \emptyset\), then [2, Proposition 6.3.5] for any \(f_0 \in K_0^*\), there exists \(\lambda_0 \geq 0\) such that

\[
f_0(p, v) = -\lambda_0 \int_0^A \int_0^A \left[ v(\tau) L(p^*, \beta^*) + v^*(\tau) \sum_{i=1}^2 p_i(a, \tau) \frac{\partial L}{\partial p_i}(p^*, \beta^*) \right] da \, d\tau.
\]
For the closed convex inequality constraint $\Omega_1$, its interior is given by
\[
\text{int}(\Omega_1) = C(0, 1; L^2(0, A; R^2)) \times \text{int}(\hat{\Omega}_1) \neq \emptyset,
\]
where $\hat{\Omega}_1 = \{v \in L^\infty(0, 1): v(\tau) > 0, \forall \tau \in [0, 1]\}$. Consequently [2, Proposition 6.3.6] the cone of feasible directions of $\Omega_1$ at $(p^*, v^*)$ is
\[
K_1 = \{\lambda [(p, v) - (p^*, v^*)]: (p, v) \in \text{int}(\Omega_1), \lambda > 0\}.
\]
For every $f_1 \in K_1^*$, if there exists $c(\tau) \in L^1(0, 1)$ such that
\[
f_1(p, v) = \int_0^1 c(\tau)v(\tau)d\tau,
\]
then [3, p. 76, Example 10.5]
\[
c(\tau)[v - v^*(\tau)] \geq 0, \quad \forall v \in [0, +\infty), \quad \tau \in [0, 1] \text{ a.e.} \quad (13)
\]
Next we determine the cone of tangent directions of $\Omega_2$ at $(p^*, v^*)$. Note that the solution of system (7) corresponding to $\beta = \beta^*$ satisfies
\[
\begin{cases}
  u_1(a, \tau) := \int_0^\tau p_1(\theta, \tau)d\theta - \int_0^\tau p_{10}(\theta)d\theta \\
  &+ \int_0^\tau v(\sigma)[p_1(a, \sigma) - \beta_1^*(\sigma)\int_{a_i}^\tau m_1(\theta, \sigma)p_1(\theta, \sigma)d\theta]d\sigma \\
  &+ \int_0^\tau \int_0^\tau [\mu_1(\theta, \sigma) + \lambda_1(\theta, \sigma)P_1(\sigma)]v(\sigma)p_1(\theta, \sigma)d\sigma d\sigma = 0,
  \\
  u_2(a, \tau) := \int_0^\tau p_2(\theta, \tau)d\theta - \int_0^\tau p_{20}(\theta)d\theta \\
  &+ \int_0^\tau v(\sigma)[p_2(a, \sigma) - \beta_2^*(\sigma)\int_{a_i}^\tau m_2(\theta, \sigma)p_2(\theta, \sigma)d\theta]d\sigma \\
  &+ \int_0^\tau \int_0^\tau [\mu_2(\theta, \sigma) + \lambda_2(\theta, \sigma)P_1(\sigma)]v(\sigma)p_2(\theta, \sigma)d\sigma d\sigma = 0.
\end{cases}
\]
Define the operator $G : X \to C(0, 1; L^2(0, A; R^2)) \times L^\infty(0, A; R^2)$,
\[
G(p_1(a, \tau), p_2(a, \tau), v(\tau)) = (u_1(a, \tau), u_2(a, \tau), p_1(a, 1) - p_{10}(a), p_2(a, 1) - p_{20}(a)).
\]
So, $\Omega_2 = \{(p, v) \in X: G(p, v) = 0\}$.
It is easy to get that
\[
G'(p^*, v^*)(p_1, p_2, v) = (v_1(a, \tau), v_2(a, \tau), p_1(a, 1), p_2(a, 1)),
\]
where
\[
v_1(a, \tau) = \int_0^\tau p_1(\theta, \tau)d\theta + \int_0^\tau [v^*(\sigma)p_1(a, \sigma) + v(\sigma)p_1^*(a, \sigma)] d\sigma
\]
\[
- \int_0^a \beta_1^*(\sigma) \int_{\theta}^a m_1(\theta, \sigma) \left[ v^*(\sigma) p_1(\theta, \sigma) + v(\sigma) p_1^*(\theta, \sigma) \right] d\theta d\sigma \\
+ \int_0^a \int_0^a \mu_1(\theta, \sigma) \left[ v^*(\sigma) p_1(\theta, \sigma) + v(\sigma) p_1^*(\theta, \sigma) \right] d\theta d\sigma \\
+ \int_0^a \int_0^a \lambda_1(\theta, \sigma) p_1^*(\theta, \sigma) P_2(\sigma) v^*(\sigma) d\theta d\sigma \\
+ \int_0^a \int_0^a \lambda_1(\theta, \sigma) P_2^*(\sigma) \left[ v^*(\sigma) p_1(\theta, \sigma) + v(\sigma) p_1^*(\theta, \sigma) \right] d\theta d\sigma, \tag{14}
\]

\[
v_2(a, \tau) = \int_0^a p_2(\theta, \tau) d\theta + \int_0^a \left[ v^*(\sigma) p_2(a, \sigma) + v(\sigma) p_2^*(a, \sigma) \right] d\sigma \\
- \int_0^a \beta_2^*(\sigma) \int_{\theta}^a m_2(\theta, \sigma) \left[ v^*(\sigma) p_2(\theta, \sigma) + v(\sigma) p_2^*(\theta, \sigma) \right] d\theta d\sigma \\
+ \int_0^a \int_0^a \mu_2(\theta, \sigma) \left[ v^*(\sigma) p_2(\theta, \sigma) + v(\sigma) p_2^*(\theta, \sigma) \right] d\theta d\sigma \\
+ \int_0^a \int_0^a \lambda_2(\theta, \sigma) p_2^*(\theta, \sigma) P_1(\sigma) v^*(\sigma) d\theta d\sigma \\
+ \int_0^a \int_0^a \lambda_2(\theta, \sigma) P_1^*(\sigma) \left[ v^*(\sigma) p_2(\theta, \sigma) + v(\sigma) p_2^*(\theta, \sigma) \right] d\theta d\sigma. \tag{15}
\]

To show that \( G'(p^*, v^*) \) is an onto mapping, we solve the equation

\[
G'(p^*, v^*)(p_1, p_2, v) = (w_1, w_2, w_3, w_4),
\]

that is, finding \((p_1, p_2, v)\) such that

\[
\begin{align*}
\begin{cases}
  v_i(a, \tau) = w_i(a, \tau), & i = 1, 2, \\
  p_1(a, 1) = w_3(a), & p_2(a, 1) = w_4(a),
\end{cases}
\end{align*} \tag{16}
\]

in which \(v_i\) is given by (14)–(15).
It can be proved that if the linearized system around \((p^*, v^*)\) of system (7) corresponding to \(\beta = \beta^*\),
\[
\begin{aligned}
\frac{\partial p}{\partial t} + v^*(\tau) \frac{\partial p}{\partial x} &= - [\mu_1(a, \tau) + \lambda_1(a, \tau) P^*_x(\tau)] \\
&\quad \times [v^*(\tau) p_1(a, \tau) + v(\tau) p^*_1(a, \tau)] \\
- \lambda_1(a, \tau) P_2(\tau) v^*(\tau) p^*_2(a, \tau) - v(\tau) \frac{\partial p}{\partial x}, \\
\frac{\partial p}{\partial t} + v(\tau) \frac{\partial p}{\partial x} &= - [\mu_2(a, \tau) + \lambda_2(a, \tau) P^*_y(\tau)] \\
&\quad \times [v(\tau) p_2(a, \tau) + v(\tau) p^*_2(a, \tau)] \\
- \lambda_2(a, \tau) P_1(\tau) v^*(\tau) p^*_1(a, \tau) - v(\tau) \frac{\partial p}{\partial x},
\end{aligned}
\]
(17)
\[
\begin{align*}
\left. v(\tau) p_1(0, \tau) + v(\tau) p_2^*(0, \tau) = \beta^* \right|_{t=0} \int_0^\ell \gamma_1(a, \tau) \, da + P^*_x(\tau) \gamma_1(a, \tau)
\end{align*}
\]
\[
\begin{align*}
\left. v(\tau) p_2(0, \tau) + v(\tau) p_1^*(0, \tau) = \beta^* \right|_{t=0} \int_0^\ell \gamma_2(a, \tau) \, da + P^*_y(\tau) \gamma_2(a, \tau)
\end{align*}
\]
is exactly controllable at \(\tau = 1\), then there must be a solution to the system (16). In fact, there exists \(\hat{\nu}(\tau)\) such that the solution of the system (17) satisfies
\[
\hat{\nu}_1(1) = w_3(a) - \gamma_1(1), \quad \hat{\nu}_2(1) = w_4(a) - \gamma_2(1),
\]
where \(\gamma_i, i = 1, 2\), is the unique solution to the system of the following integral equations:
\[
\begin{aligned}
\int_0^\ell \gamma_1(\theta, \tau) \, d\theta + \int_0^\ell v^*(\sigma) \left[ \gamma_1(a, \sigma) - \beta^*_x(\sigma) \int_0^\ell (m_1 \gamma_1)(\theta, \sigma) \, d\theta \right] \, d\sigma \\
+ \int_0^\ell \int_0^\ell \mu_1(\theta, \sigma) \gamma_1(\theta, \sigma) v^*(\sigma) \, d\sigma \, d\theta \\
+ \int_0^\ell \int_0^\ell \gamma_1(a, \sigma) v(\sigma) \, d\sigma \, d\theta \\
= w_1(a, \tau),
\end{aligned}
\]
(18)
\[
\begin{aligned}
\int_0^\ell \gamma_2(\theta, \tau) \, d\theta + \int_0^\ell v^*(\sigma) \left[ \gamma_2(a, \sigma) - \beta^*_y(\sigma) \int_0^\ell (m_2 \gamma_2)(\theta, \sigma) \, d\theta \right] \, d\sigma \\
+ \int_0^\ell \int_0^\ell \mu_2(\theta, \sigma) \gamma_2(\theta, \sigma) v^*(\sigma) \, d\sigma \, d\theta \\
+ \int_0^\ell \int_0^\ell \gamma_1(a, \sigma) v(\sigma) \, d\sigma \, d\theta \\
= w_2(a, \tau).
\end{aligned}
\]
Note that the solution of the system (17) satisfies \(v_1(a, \tau) = v_2(a, \tau) = 0\). From (18) it is easy to show that \((\hat{\nu}_1 + \gamma_1, \hat{\nu}_2 + \gamma_2, \hat{\nu})\) is a solution to the system (16). Thus, the tangent directions cone of \(\Omega_2\) at \((p^*, v^*)\) is given by
\[
K_2 = \{(p, v) \in X: G^t(p^*, v^*)(p, v) = 0\}.
\]
Let
\[
\begin{align*}
K_{11} &= \{(p, v) \in X: (p, v) \text{ is subject to (17)}\}, \\
K_{12} &= \{(p, v) \in X: p_i(a, 1) = 0, \ i = 1, 2\},
\end{align*}
\]
(19)
Then \(K_2 = K_{11} \cap K_{12}\). Since \(K_{11}\) and \(K_{12}\) are subspaces, so \(K^*_2 = K^*_{11} + K^*_{12}\).
For any \( f_2 \in K_*^2 \), \( f_2 = f_{11} + f_{12} \), \( f_{1i} \in K_*^i \), \( i = 1, 2 \), there exists
\[
\alpha(a) = (\alpha_1(a), \alpha_2(a)) \in L^2(0, A; R^2),
\]
such that
\[
f_{12}(p, v) = \int_0^A \alpha(a) \cdot p(a, 1) \, da = \sum_{i=1}^2 \int_0^A \alpha_i(a) p_i(a, 1) \, da.
\]  
(20)

According to Dubovitskii–Milyutin theorem [3], there are functionals \( f_i \in K_*^i \), \( i = 0, 1, 2 \), not all zero and such that
\[
f_0 + f_1 + f_{11} + f_{12} = 0.
\]  
(21)

In particular, \( f_{11}(p, v) = 0 \) if \( (p, v) \) satisfies (17). From (12) and (20)–(21) it follows that
\[
f_1(p, v) = -f_0(p, v) - f_{12}(p, v)
\]
\[
= \int_0^A \int_0^A \lambda_0 v(\tau)L(p^*(a, \tau), \beta^*(\tau)) \, da \, d\tau
\]
\[
+ \sum_{i=1}^2 \int_0^A \int_0^A \lambda_0 v^*(\tau)p_i(a, \tau) \frac{\partial L}{\partial p_i}(p^*(a, \tau), \beta^*(\tau)) \, da \, d\tau
\]
\[
- \int_0^A \alpha_i(a) p_i(a, 1) \, da.
\]  
(22)

Define the adjoint system
\[
\frac{\partial q_1}{\partial \tau} + v^*(\tau) \frac{\partial q_1}{\partial a} = [\mu_1(a, \tau) + \lambda_1(a, \tau) P_2^*(\tau)]q_1(a, \tau) v^*(\tau)
\]
\[
+ \left[ \lambda_0 \frac{\partial L}{\partial p_1}(p^*(a, \tau), \beta^*(\tau)) - \beta^*_1(a, \tau) m_1(a, \tau) q_1(0, \tau) \right]
\]
\[
+ f_0^A (\lambda_2 p_2 q_2)(\theta, \tau) d\theta v^*(\tau),
\]
\[
\frac{\partial q_2}{\partial \tau} + v^*(\tau) \frac{\partial q_2}{\partial a} = [\mu_2(a, \tau) + \lambda_2(a, \tau) P_1^*(\tau)]q_2(a, \tau) v^*(\tau)
\]
\[
+ \left[ \lambda_0 \frac{\partial L}{\partial p_2}(p^*(a, \tau), \beta^*(\tau)) - \beta^*_2(a, \tau) m_2(a, \tau) q_2(0, \tau) \right]
\]
\[
+ f_0^A (\lambda_1 p_1 q_1)(\theta, \tau) d\theta v^*(\tau),
\]
\[
q_i(a, 1) = \alpha_i(a), \quad q_i(A, \tau) = 0, \quad i = 1, 2.
\]  
(23)

After some calculations by means of (23), we can obtain that the solution of (17) and the solution of (23) have the following relation:
\[
\sum_{i=1}^2 \int_0^A \int_0^A \lambda_0 v^*(\tau)p_i(a, \tau) \frac{\partial L}{\partial p_i}(p^*(a, \tau), \beta^*(\tau)) \, da \, d\tau - \int_0^A \alpha_i(a) p_i(a, 1) \, da
\]
\[
\begin{align*}
&= \int_0^A \int_0^A v(\tau) \left\{ (q_1 p_1^*)(a, \tau) \left[ \mu_1(a, \tau) + \lambda_1(a, \tau) P_2^*(\tau) \right] \\
&\quad - p_1^*(a, \tau) \left[ \frac{\partial q_1}{\partial a} + \beta_1^*(\tau)m_1(a, \tau)q_1(0, \tau) \right] \\
&\quad + (q_2 p_2^*)(a, \tau) \left[ \mu_2(a, \tau) + \lambda_2(a, \tau) P_1^*(\tau) \right] \\
&\quad - p_2^*(a, \tau) \left[ \frac{\partial q_2}{\partial a} + \beta_2^*(\tau)m_2(a, \tau)q_2(0, \tau) \right] \right\} \, da \, d\tau,
\end{align*}
\]

which holds for every \( \lambda_0, \alpha(a) \).

Combining (22) with (24) derives
\[
 f_1(p, v) = \int_0^A \int_0^A v(\tau) \left\{ (q_1 p_1^*)(a, \tau) \left[ \mu_1(a, \tau) + \lambda_1(a, \tau) P_2^*(\tau) \right] \\
\quad - p_1^*(a, \tau) \left[ \frac{\partial q_1}{\partial a} + \beta_1^*(\tau)m_1(a, \tau)q_1(0, \tau) \right] \\
\quad + (q_2 p_2^*)(a, \tau) \left[ \mu_2(a, \tau) + \lambda_2(a, \tau) P_1^*(\tau) \right] \\
\quad - p_2^*(a, \tau) \left[ \frac{\partial q_2}{\partial a} + \beta_2^*(\tau)m_2(a, \tau)q_2(0, \tau) \right] \right\} \, da \, d\tau. \tag{25}
\]

Let
\[
S(\tau) = \int_0^A \left\{ (q_1 p_1^*)(a, \tau) \left[ \mu_1(a, \tau) + \lambda_1(a, \tau) P_2^*(\tau) \right] \\
\quad - p_1^*(a, \tau) \left[ \frac{\partial q_1}{\partial a} + \beta_1^*(\tau)m_1(a, \tau)q_1(0, \tau) \right] \\
\quad + (q_2 p_2^*)(a, \tau) \left[ \mu_2(a, \tau) + \lambda_2(a, \tau) P_1^*(\tau) \right] \\
\quad - p_2^*(a, \tau) \left[ \frac{\partial q_2}{\partial a} + \beta_2^*(\tau)m_2(a, \tau)q_2(0, \tau) \right] \right\} \, da. \tag{26}
\]

From (25) and (13) it follows that
\[
S(\tau) \left[ v - v^*(\tau) \right] \geq 0, \quad \forall v \in [0, \infty), \quad \tau \in [0, 1] \text{ a.e.} \tag{27}
\]

Define the sets
\[
S_1 = \{ \tau \in [0, 1]: v^*(\tau) > 0 \}, \quad S_2 = \{ \tau \in [0, 1]: v^*(\tau) = 0 \}.
\]
We can see from (27) that

\[ S(\tau) = 0, \quad \text{if} \quad \tau \in S_1, \quad S(\tau) \geq 0, \quad \text{if} \quad \tau \in S_2. \quad (28) \]

We claim that \( \lambda_0 \) and \( \alpha(\cdot) \) both are not zero. Otherwise, it follows from (12) and (20) that \( f_0 = 0, f_{12} = 0 \). Then (23) implies \( q_i = 0, \ i = 1, 2; \) consequently (25) and (21) lead to \( f_1 = 0, f_{11} = 0 \), which is a contradiction.

Besides, if \( K_0 = \emptyset \), that is, for any \( (p, v) \in X \),

\[ \int_0^1 \int_0^A v^*(\tau) \left[ p_1(a, \tau) \frac{\partial L}{\partial p_1}(p^*(a, \tau), \beta^*(\tau)) + p_2(a, \tau) \frac{\partial L}{\partial p_2}(p^*(a, \tau), \beta^*(\tau)) \right] + v(\tau) L(p^*(a, \tau), \beta^*(\tau)) \] \[ da \ dt \geq 0, \quad (29) \]

choosing \( \lambda_0 = 1, \alpha(a) = 0 \) in (24) gives

\[ \sum_{i=1}^2 \int_0^1 \int_0^A v^*(\tau) p_i(a, \tau) \frac{\partial L}{\partial p_i}(p^*(a, \tau), \beta^*(\tau)) da \ dt \]

\[ = \int_0^1 \int_0^A v(\tau) \left[ (q_1 p_1^*)(a, \tau) \left[ \mu_1(a, \tau) + \lambda_1(a, \tau) P_2^*(\tau) \right] \right. \]

\[ - p_1^*(a, \tau) \left[ \frac{\partial q_1}{\partial a} + \beta_1^*(\tau) m_1(a, \tau) q_1(0, \tau) \right] \]

\[ + (q_2 p_2^*)(a, \tau) \left[ \mu_2(a, \tau) + \lambda_2(a, \tau) P_1^*(\tau) \right] \]

\[ - p_2^*(a, \tau) \left[ \frac{\partial q_2}{\partial a} + \beta_2^*(\tau) m_2(a, \tau) q_2(0, \tau) \right] \] \[ da \ dt. \quad (30) \]

From (29)–(30) we know that \( \int_0^1 S(\tau)v(\tau) \, d\tau \geq 0 \) for any \( (p, v) \in X \), in which \( S(\tau) \) is given by (26). So \( S(\tau) \in K_1^* \). Again from (13) we see that (27) and (28) still hold.

Finally, if the adjoint system (23) has a nonzero solution \( q_i, \ i = 1, 2 \), such that

\[ \int_0^A \left[ (q_1 p_1^*)(a, \tau) \left[ \mu_1(a, \tau) + \lambda_1(a, \tau) P_2^*(\tau) \right] \right. \]

\[ - p_1^*(a, \tau) \left[ \frac{\partial q_1}{\partial a} + \beta_1^*(\tau) m_1(a, \tau) q_1(0, \tau) \right] \]

\[ + (q_2 p_2^*)(a, \tau) \left[ \mu_2(a, \tau) + \lambda_2(a, \tau) P_1^*(\tau) \right] \]

\[ - p_2^*(a, \tau) \left[ \frac{\partial q_2}{\partial a} + \beta_2^*(\tau) m_2(a, \tau) q_2(0, \tau) \right] \] \[ da = 0. \quad (31) \]
then choosing $\lambda_0 = 0$ in (26) enables (27) to be correct. If for every nonzero solution of the adjoint system the following relation holds:

$$
\int_0^A \left\{ (q_1 p_1^a)(a, \tau) \left[ \mu_1(a, \tau) + \lambda_1(a, \tau) P_2^a(\tau) \right] \\
- p_1^a(a, \tau) \left[ \frac{\partial q_1}{\partial a} + \beta_1^a(\tau)m_1(a, \tau)q_1(0, \tau) \right] \\
+ (q_2 p_2^a)(a, \tau) \left[ \mu_2(a, \tau) + \lambda_2(a, \tau) P_1^a(\tau) \right] \\
- p_2^a(a, \tau) \left[ \frac{\partial q_2}{\partial a} + \beta_2^a(\tau)m_2(a, \tau)q_2(0, \tau) \right] \right\} da \neq 0,
$$

then the linearized system (17) must be exactly controllable at $\tau = 1$. Otherwise there exists $\alpha(a) \in L^2(0, A; \mathbb{R}^2)$, $\alpha \neq 0$ such that

$$
\int_0^A \alpha(a) \cdot p(a, 1) \, da = 0.
$$

Taking $\lambda_0 = 0$ in (24), we arrive at (31), a contradiction.

In all cases, (28) is always true.

Define the time transformation

$$
\tau(t) = \inf \{ \tau \in [0, 1]: t(\tau) = t \},
$$

and

$$
q_i(a, t) = q_i(a, \tau(t)), \quad q_i(0, t) = q_i(0, \tau(t)), \quad i = 1, 2, \quad S(t) = S(\tau(t)),
$$

where $S(\tau)$ is given by (26).

Because $\{ t: t = t(\tau), \tau \in S_1 \}$ is at most measurable [3, p. 99], it follows from the first part of (28) that

$$
S(t) := \int_0^A \left\{ (q_1 p_1^a)(a, t) \left[ \mu_1(a, t) + \lambda_1(a, t) P_2^a(t) \right] \\
- p_1^a(a, t) \left[ \frac{\partial q_1}{\partial a} + \beta_1^a(t)m_1(a, t)q_1(0, t) \right] \\
+ (q_2 p_2^a)(a, t) \left[ \mu_2(a, t) + \lambda_2(a, t) P_1^a(t) \right] \\
- p_2^a(a, t) \left[ \frac{\partial q_2}{\partial a} + \beta_2^a(t)m_2(a, t)q_2(0, t) \right] \\
+ \lambda_0 L(p^a(a, t), \beta^a(t)) \right\} da = 0
$$

holds for almost every $t \in [0, t^*_1]$.

Let $S_1$ be a perfect nowhere dense subset of $[0, 1]$, define

$$
v^*(\tau) = \begin{cases} 
\frac{t^*_1}{\mu(S_1)}, & \tau \in S_1, \\
0, & \tau \in S_2 := [0, 1] - S_1.
\end{cases}
$$
In a similar manner as that in [3], we can define \( \beta^*(\tau) \) on \( S_2 \), and an analysis of the second part of (28) shows that
\[
\int_0^A \left\{ (q_1 p_1^*)(a, \tau) \left[ \mu_1(a, \tau) + \lambda_1(a, \tau) P_2^*(\tau) \right] \\
- p_1^*(a, \tau) \left[ \frac{\partial q_1}{\partial a} + \beta_1^*(\tau) m_1(a, \tau) q_1(0, \tau) \right] \\
+ (q_2 p_2^*)(a, \tau) \left[ \mu_2(a, \tau) + \lambda_2(a, \tau) P_1^*(\tau) \right] \\
- p_2^*(a, \tau) \left[ \frac{\partial q_2}{\partial a} + \beta_2^*(\tau) m_2(a, \tau) q_2(0, \tau) \right] \\
+ \lambda_0 L \left( p^*(a, \tau), \beta \right) \right\} da \geq 0 (33)
\]
hold for every \( \beta \in [\beta_0, \beta_0] \) and every \( t \in [0, t^*_1] \).

We have so far proved that

**Theorem 1.** If \( (p^*, \beta^*, t^*_1) \) is a solution of problem (1)–(3), then there exist a number \( \lambda_0 \geq 0 \) and a function \( \alpha(a) \in L^2(0, A; \mathbb{R}^2) \) such that (32) and (33) hold, in which \( q_i \), \( i = 1, 2 \), solves the following adjoint system:
\[
\begin{align*}
\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial a} &= [\mu_1(a, t) + \lambda_1(a, t) P_2^*(t)] q_1(a, t) - \beta_1^*(t) m_1(a, t) q_1(0, t) \\
&+ \lambda_0 \frac{\partial L}{\partial p_1}(p^*(a, t), \beta^*(t)) + \int_0^A (\lambda_2 p_1^* q_2)(a, t) da, \\
\frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial a} &= [\mu_2(a, t) + \lambda_2(a, t) P_1^*(t)] q_2(a, t) - \beta_2^*(t) m_2(a, t) q_2(0, t) \\
&+ \lambda_0 \frac{\partial L}{\partial p_2}(p^*(a, t), \beta^*(t)) + \int_0^A (\lambda_1 p_2^* q_1)(a, t) da, \\
q_1(a, t^*_1) &= \alpha_1(a), \quad q_2(A, t) = 0, \quad i = 1, 2.
\end{align*}
\]

**Remark 1.** If the phase constraint (3) is replaced with
\( p(a, t_1) \in \{ h(a) : \| h - p_0 \| < \varepsilon \} \),
then the corresponding optimality conditions can be obtained by choosing \( \lambda_0 = 1, \alpha(a) = p^*(a, t^*_1) - p^0(a) \) in Theorem 1.

**Remark 2 (Time-optimal control).** Let \( L(p, \beta) \equiv 1 \), one can readily deduce the maximum principle for the time-optimal problem.

8. Problems with integral phase constraint

Consider first the control problem
\[
\text{minimize} \quad J(p, \beta) = \int_0^T \int_0^A L(p_1(a, t), p_2(a, t), \beta_1(t), \beta_2(t)) da dt, (34)
\]
where \( T > 0 \) is fixed, \((p, \beta)\) is subject to (2) and the following two constraints:

\[
p_i(\cdot, T) = p_i^0, \quad i = 1, 2, \tag{35}
\]

\[
\int_0^A G(p_1(a, t), p_2(a, t), t) \, da \leq 0, \quad \forall t \in [0, T]. \tag{36}
\]

Let the state space be \( X = C(0, T; L^2(0, A; R^2)) \times L^\infty(0, T; R^2); \) define

\[
\Omega_1 = \{(p, \beta) \in X: \beta_i(t) \in [\beta_0^i, \beta_0^i], t \in [0, T] \text{ a.e., } i = 1, 2\},
\]

\[
\Omega_2 = \{(p, \beta) \in X: (p, \beta) \text{ is subject to (2) and (35)}\},
\]

\[
\Omega_3 = \{(p, \beta) \in X: (p, \beta) \text{ is subject to (36)}\}.
\]

So the problem (34)–(36) is equivalent to the problem: finding \((p^*, \beta^*)\) \(\in\) \(\Omega_1 \cap \Omega_2 \cap \Omega_3\), such that

\[
\begin{align*}
J(p^*, \beta^*) &= \min \{ J(p, \beta), \quad (p, \beta) \in \Omega_1 \cap \Omega_2 \cap \Omega_3 \}. \tag{37}
\end{align*}
\]

We have discussed the cones corresponding to the functional \(J\), inequality constraint \(\Omega_1\) and equality constraint \(\Omega_2\). Now we need only to analyze the inequality constraint \(\Omega_3\).

It is clear that \(\Omega_3\) can be rewritten as

\[
\Omega_3 = \{(p, \beta) \in X: F(p) \leq 0\},
\]

where

\[
F(p) = \max_{0 \leq t \leq T} \int_0^A G(p_1(a, t), p_2(a, t), t) \, da. \tag{38}
\]

We assume that the following conditions hold:

\begin{enumerate}
  \item \(\int_0^A G(p_1(a), p_2(a), t) \, da\) is a continuous functional on \(L^2(0, A; R^2) \times [0, \infty)\);
  \item \(\int_0^A G(p_{10}(a), p_{20}(a), 0) \, da < 0, \int_0^A G(p_1^0(a), p_2^0(a), T) \, da < 0\);
  \item \(\int_0^A G_{p_1}(p_1(a), p_2(a), t) \, da\) is continuous on \(L^2(0, A; R^2) \times [0, \infty)\), and \(\int_0^A G_{p_1}(p_1(a), p_2(a), t) \, da \neq 0, i = 1, 2\) if \(\int_0^A G(p_1(a), p_2(a), t) \, da = 0\).
\end{enumerate}

Let \((p^*, \beta^*)\) be a solution of the problem (34)–(36). Without loss of generality, we need only to consider the case of \(F(p^*) = 0\). In fact, if \(F(p^*) < 0\), then it follows from (1) that the cone of feasible directions of \(\Omega_3\) at \((p^*, \beta^*)\) is \(K_3 = X\), which implies \(K_3^* = \{0\}\). This situation is equivalent to the absence of the constraint \(\Omega_3\). Therefore

\[
\Omega_3 = \{(p, \beta) \in X: F(p) \leq F(p^*)\}.
\]

By means of [3, Example 7.5, p. 52], we can state
Lemma 1. $F(p)$ is differentiable at every $\hat{p}$ in every direction $p$, and

$$F'(\hat{p}, p) = \max_{t \in S} \sum_{i=1}^{2} \int_{0}^{A} G'_{p_{i}} (\hat{p}_1(a,t), \hat{p}_2(a,t), t) p_{i}(a,t) da,$$

where

$$S = \left\{ t \in [0,T]: \int_{0}^{A} G(\hat{p}_1(a,t), \hat{p}_2(a,t), t) da = F(\hat{p}) \right\}.$$  \hfill (39)

In addition, $F(p)$ is of Lipschitz in any ball.

Notice that

$$F'(p^*, -G'(p^*, t)) < 0,$$

where $G'_p (p^*, t) = (G'_{p_{1}} (p^*, t), G'_{p_{2}} (p^*, t))$.

According to the lemma in [3, p. 59], we have

$$K_3 = \{(p, \beta) \in X: F'(p^*, p) < 0\}.$$

Define the linear operator $B: X \rightarrow C[0, T]$,

$$B(p, \beta) = -\sum_{i=1}^{2} \int_{0}^{A} G'_{p_{i}} (p^*(a,t), t) p_{i}(a,t) da,$$

and the set

$$K = \{ y \in C[0,T]: y(t) \geq 0, \forall t \in S \},$$

where $S$ is given by (39) corresponding to $\hat{p} = p^*$. It is easy to see

$$K_3 = \{(p, \beta) \in X: B(p, \beta) \in K\}.$$

Since $B(-G'_p (p^*, t)) \in \text{int}(K)$, it follows from [3, Theorem 10.4] that $K^*_3 = B^* K^*$, in which $B^*$ denotes the adjoint operator of $B$. Thus, Riesz’s theorem implies that for any $f_3 \in K^*_3$, there exists a measure $dm(t)$, which is supporting on $S$ and

$$f_3(p, \beta) = \int_{0}^{T} B(p(a,t), \beta(t)) dm(t)$$

$$= \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{A} G'_{p_{i}} (p^*(a,t), t) p_{i}(a,t) da dm(t).$$  \hfill (40)
Combining (40) with the discussions for \( J, \Omega_1, \Omega_2 \), we assert that there exist \( \lambda_0 \geq 0, \alpha(a) = (\alpha_1(a), \alpha_2(a)) \in L^2(0, A; R^2) \) such that

\[
\begin{align*}
&f_1(p, \beta) = \sum_{i=1}^{2} \left\{ \int_{0}^{T} \int_{0}^{A} \lambda_0 \left[ p_i(a, t) \frac{\partial L}{\partial p_i}(p^*(a, t), \beta^*(t)) \right. \right. \\
&\quad \left. \left. + \beta_i(t) \frac{\partial L}{\partial \beta_i}(p^*(a, t), \beta^*(t)) \right] \, da \, dt - \int_{0}^{A} p_i(a, T) \alpha(a) \, da \right\} \\
&\quad + \sum_{i=1}^{2} \int_{0}^{T} \int_{0}^{A} G'_{p_i}(p^*(a, t), t) p_i(a, t) \, da \, dm(t),
\end{align*}
\]

(41)

where \((p, \beta)\) satisfies the following linearized system:

\[
\begin{align*}
\frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} &= -\mu_1(a, t) p_1(a, t) - \lambda_1(a, t) \{ P^*_2(t) p_1(a, t) + P_2(t) p^*_1(a, t) \}, \\
\frac{\partial p_2}{\partial t} + \frac{\partial p_2}{\partial a} &= -\mu_2(a, t) p_2(a, t) - \lambda_2(a, t) \{ P^*_1(t) p_2(a, t) + P_1(t) p^*_2(a, t) \}, \\
p_i(0, t) &= f_{q_i(a)}^{a_2} m_i(a, t) \{ \beta^*_i(t) p_i(a, t) + \beta_i(t) p_i^*(a, t) \} \, da, \quad \beta_i(a, 0) = 0, \quad P^*_i(t) = \int_{0}^{A} p^*_i(a, t) \, da, \quad i = 1, 2, \quad (a, t) \in Q.
\end{align*}
\]

(42)

Equality (41) must be true as long as the cone of decrease directions of \( J \) is not empty and the system (42) is exactly controllable at \( T \).

Define the adjoint system:

\[
\begin{align*}
\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial a} &= -\mu_1(a, t) q_1(a, t) + \lambda_1(a, t) P^*_1(t) q_1(a, t) q_1(a, t) q_1(0, t) \\
&\quad + \lambda_0 \frac{\partial L}{\partial p_1}(p^*(a, t), \beta^*(t)) + \int_{0}^{A} (\lambda_2 p^*_2 q_2)(a, t) \, da \\
&\quad + G'_{p_1}(p^*(a, t), t) \frac{dm(t)}{dt}, \\
\frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial a} &= -\mu_2(a, t) q_2(a, t) + \lambda_2(a, t) P^*_2(t) q_2(a, t) q_2(a, t) q_2(0, t) \\
&\quad + \lambda_0 \frac{\partial L}{\partial p_2}(p^*(a, t), \beta^*(t)) + \int_{0}^{A} (\lambda_1 p^*_1 q_1)(a, t) \, da \\
&\quad + G'_{p_2}(p^*(a, t), t) \frac{dm(t)}{dt}, \\
q_i(a, T) &= \alpha_i(a), \quad q_i(a, 0) = 0, \quad i = 1, 2, \quad (a, t) \in (0, A) \times (0, T).
\end{align*}
\]

(43)

Some computations show the following

**Lemma 2.** The solutions of the system (42) and of the system (43) are connected with the following relation:
Finally, a similar analysis leads to

**Theorem 2.** If \((p^*, \beta^*)\) is a solution of the problem (34)–(36), then there exist \(\lambda_0 \geq 0\) and a function \(q_i, i = 1, 2\), both not zero and such that

\[
\sum_{i=1}^{2} \int_0^T \left[ \int_0^A \lambda_0 \left( \frac{\partial L}{\partial p_i} (p^*(a,t), \beta^*(t)) \right) \right] da \cdot \beta_i(t) dt + \int_0^T \left[ \int_0^A G_{p_i} (p^*(a,t), \beta^*(t)) p_i(a,t) \right] da \cdot \beta_i(t) dt \\
\sum_{i=1}^{2} \int_0^T \left[ \int_0^A \lambda_0 \left( \frac{\partial L}{\partial \beta_i} (p^*(a,t), \beta^*(t)) \right) \right] da \cdot \beta_i(t) dt \\
\geq \sum_{i=1}^{2} \int_0^T \left[ \int_0^A \lambda_0 \left( \frac{\partial L}{\partial \beta_i} (p^*(a,t), \beta^*(t)) \right) \right] da \cdot \beta_i(t) dt.
\]

holds for every \(\beta_i \in [\beta_0, \beta^0]\) and for every \(t \in [0, T]\), in which \(q_i\) is the solution of system (43).

Next we consider further the optimal control problem

\[
\text{minimize} \quad J(p, \beta) = \int_0^{t_1} \int_0^A L(p_1(a,t), p_2(a,t), \beta_1(t), \beta_2(t)) \right) da \cdot dt,
\]

where \(t_1 > 0\) is not fixed and \((p, \beta)\) is subject to (2) and

\[
\int_0^A G(p_1(a,t), p_2(a,t), t) da \leq 0, \quad \forall t \in [0, t_1],
\]

\[
p_1(a,t_1) = p_1^0(a), \quad a \in (0, A), \quad i = 1, 2.
\]

Applying the approaches in the preceding two sections to the above problem, we obtain that

**Theorem 3.** If \((p^*, \beta^*, t_1^*)\) is a solution of the above problem, then there exist \(\lambda_0 \geq 0\), a function \(\alpha(a) \in L^2(0, A; \mathbb{R}^2)\), which is supporting on

\[
S = \left\{ t \in [0, t_1^*]: \int_0^A G(p^*(a,t), t) da = F(p^*), \quad F \text{ is given by (38)} \right\}.
\]
and a measure \( dm(t) \), such that (32)–(33) hold, but \( q_i \), \( i = 1, 2 \), is the solution of (43) corresponding to \( T = t_1^* \).

9. Approximate controllability of the state system

In what follows, we seek conditions for the approximate controllability of the state system.

**Definition 1.** The system (2) is said to be approximately controllable if for any \( \varepsilon > 0 \) and a prescribed age distribution \( \bar{p}(a) \in L^\infty((0, A)) \) (i.e., the space of all of the 2-dimensional functions essentially bounded on \((0, A)\)), there exist a finite time \( T > 0 \) and a continuous function \( \beta(t) \in L^\infty_2((0, T)) \), \( 0 \leq \beta_0 \leq \beta_i(t) \leq \beta^0 \), \( i = 1, 2 \); \( t \in [0, T] \), such that the corresponding solution of system (2) satisfies

\[
\|p(\cdot, T) - \bar{p}\|_\infty \leq \varepsilon.
\]

For given \( v = (v_1, v_2) \in L^\infty((0, A) \times (0, \infty)) \), \( v_i(a, t) \geq 0 \), \( i = 1, 2 \), consider the linear system

\[
\begin{aligned}
\frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} &= - \left[ \mu_1(a, t) + \lambda_1(a, t) \int_0^A v_2(a, t) \, da \right] p_1, \\
\frac{\partial p_2}{\partial t} + \frac{\partial p_2}{\partial a} &= - \left[ \mu_2(a, t) + \lambda_2(a, t) \int_0^A v_1(a, t) \, da \right] p_2, \\
p_1(0, t) &= \beta_1(t) \int_{a_1}^{a_2} m_1(a) p_1(a, t) \, da, \\
p_1(a, 0) &= p_{10}(a), \quad (a, t) \in (0, A) \times (0, \infty).
\end{aligned}
\]

(44)

It follows from [2, Theorem 6.2.5] that the following result is true.

**Theorem 4.** If the conditions below hold, then the system (44) is approximately controllable:

1. \( p_{10}(a) \geq c_i > 0 \), \( \forall a \in [0, a_2] \), \( c_i \) are constants, \( i = 1, 2 \);
2. for any \( \varepsilon > 0 \), \( \exp\left[ \int_0^{a_2} \mu_i(\rho, t+\rho) \, d\rho \right] = O(e^{\varepsilon t}) \), \( i = 1, 2 \);
3. \( \beta^0 > \liminf_{t \to \infty} \int_{a_1}^{a_2} m_1(s, t) \exp\left[ - \int_0^s \mu_1(\rho, \rho - s + t) \, d\rho \right] \, ds \), and

\[
\beta^0 > \liminf_{t \to \infty} \int_{a_1}^{a_2} m_2(s, t) \exp\left\{ \int_0^s \left[ \lambda_2(\rho, \rho - s + t) \int_0^A \gamma_1(a, \rho - s + t) \, da \\ - \mu_1(\rho, \rho - s + t) \right] \, d\rho \right\} \, ds,
\]
where \( y_1 \) is the solution of the following system:

\[
\begin{align*}
\frac{\partial y_1}{\partial t} + \frac{\partial y_1}{\partial a} &= -\mu_1(a,t)y_1, \\
y_1(0, t) &= \beta_0 \int_{a_2}^{a_1} m_1(a, t)y_1(a, t) \, da, \\
y_1(a, 0) &= p_{10}(a);
\end{align*}
\]

(4) for any given \( \delta > 0 \), there exists \( m_0(\delta) > 0 \) such that

\[
\int_{a_2 - \delta}^{a_2} m_i(a, t) \, da \geq m_0(\delta), \quad i = 1, 2,
\]

holds whenever \( t > 0 \).

Define the operator

\[
D : L^2_2((0, A) \times (0, \infty)) \to L^2_2((0, A) \times (0, \infty)), \quad Dv = p^v,
\]

where \( p^v \) is the solution of (2) corresponding to \( \beta_i = \beta^v_i \), \( \beta^v_i \) is the control function determined by the approximate controllability of the system (44).

Treating in a similar manner as that in the analysis of well-posedness, we are able to prove that

**Theorem 5.** If the assumptions in Theorem 4 are satisfied, then the system (2) is approximately controllable.

References