Some Inequalities for Derivatives of Polynomials

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If \( p(z) \) is a polynomial of degree at most \( n \) having no zeros in \( |z| < 1 \), then according to a well known result conjectured by Erdős and proved by Lax
\[
\max_{|z| = 1} |p'(z)| \leq \frac{n}{2} \max_{|z| = 1} |p(z)|.
\]

On the other hand, by a result due to Turan, if \( p(z) \) has all its zeros in \( |z| \leq 1 \), then
\[
\max_{|z| = 1} |p'(z)| \geq \frac{n}{2} \max_{|z| = 1} |p(z)|.
\]

In this paper we generalize and sharpen these inequalities.

1. INTRODUCTION AND STATEMENT OF RESULTS

If \( p(z) \) is a polynomial of degree at most \( n \), then according to a famous result known as Bernstein's inequality (for references see [6])
\[
\max_{|z| = 1} |p'(z)| \leq n \max_{|z'| = 1} |p(z)|. \tag{1.1}
\]

Here equality holds if and only if \( p(z) \) has all its zeros at the origin. In case \( p(z) \) does not vanish in \( |z| < 1 \), it was conjectured by Erdős and proved by Lax [4] that (1.1) can be replaced by
\[
\max_{|z| = 1} |p'(z)| \leq \frac{n}{2} \max_{|z| = 1} |p(z)|. \tag{1.2}
\]

On the other hand it was proved by Turan [7] that if \( p(z) \) has all its zeros in \( |z| \leq 1 \), then
\[
\max_{|z| = 1} |p'(z)| \geq \frac{n}{2} \max_{|z| = 1} |p(z)|. \tag{1.3}
\]

Both the above inequalities are sharp and become equalities for \( p(z) = \lambda + \mu z^n \), \( |\lambda| = |\mu| \).
Recently Azis and Dawood [1] improved inequalities (1.2) and (1.3) by proving

**Theorem A [1, Theorem 2].** If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}.
\]  

(1.4)

**Theorem B [1, Theorem 4].** If \( p(z) \) is a polynomial of degree \( n \) which has all its zeros in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)| \right\}.
\]  

(1.5)

Here we generalize the above theorems by proving the following more general.

**Theorem 1.** If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, K \geq 1 \), then

\[
\max_{|z|=1} |p^{(s)}(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1 + K^s} \left( \max_{|z|=1} |p(z)| - \min_{|z|=1-K} |p(z)| \right).
\]  

(1.6)

**Theorem 2.** If \( p(z) \) is a polynomial of degree \( n \), having all its zeros in \( |z| \leq K \), then

\[
\max_{|z|=1} |p'(z)| \geq \left( \frac{n}{1 + K} \right) \max_{|z|=1} |p(z)| + \frac{n}{K^{n-1}(1+K)} \min_{|z|=K} |p(z)|
\]  

(1.7)

if \( K \leq 1 \), and

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{(1 + K^n)} \left( \max_{|z|=1} |p(z)| + \min_{|z|=K} |p(z)| \right)
\]  

(1.8)

if \( K \geq 1 \).

Both these inequalities are best possible. In (1.7) equality holds for \( p(z) = (z + K)^n \) and in (1.8) for \( p(z) = z^n + K^n \).

As is immediate to see, Theorem 1 sharpens a result of Govil and Rahman [3, Theorem 4]. If we take \( s = 1 \) in Theorem 1, we get the following result which sharpens a result of Malik [5].
**Corollary 1.** If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, K \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + K} (\max_{|z|=1} |p(z)| - \min_{z = K} |p(z)|).
\] (1.9)

The result is best possible and the equality holds for \( p(z) = (z + K)^n \).

Theorem A of Aziz and Dawood [1] is a special case of the above Corollary when \( K = 1 \). If we take \( K = 1 \) in Theorem 2, we get Theorem B of Aziz and Dawood [1]. In general Theorem 2 sharpens results of Govil [2] and Malik [5].

**Remark.** In all the above inequalities (1.6), (1.7), (1.8), and (1.9), it is not possible to replace the expression \( \min_{|z|=K} |p(z)| \) by \( \min_{|z|=1} |p(z)| \), as the polynomial \( p(z) = (z + K)^n \) shows for inequalities (1.6), (1.7), and (1.9) and \( p(z) = z^n + K^n \) shows for the inequality (1.8).

2. Lemmas

We need the following lemmas.

**Lemma 1.** If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, K \geq 1 \), then

\[
K^n |p^{(s)}(e^{i\theta})| \leq |q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta < 2\pi.
\] (2.1)

Here and elsewhere \( q(z) \) stands for \( z^n \{p(1/z)\} \).

This lemma is in fact implicit in the proof of Theorem 4 of Govil and Rahman [3]; however, for the sake of completeness we give here a brief outline of the proof. For this, first let us suppose that all the zeros of \( p(z) \) lie on \( |z| = K \geq 1 \). Then all the zeros of \( P_1(z) = p(Kz) \) lie on \( |z| = 1 \) and so do the zeros of \( Q_1(z) = z^n \{P_1(1/z)\} = K^n q(z/K) \). For every \( \lambda \) with \( |\lambda| > 1 \), the polynomial \( P_1(z) - \lambda Q_1(z) \) has all its zeros on \( |z| = 1 \); hence by the Gauss Lucas Theorem all the zeros of the \( s \)th derivative \( P_1^{(s)}(z) - \lambda Q_1^{(s)}(z) \) lie in \( |z| \leq 1 \). This implies that

\[
K^n |p^{(s)}(Kz)| = |P_1^{(s)}(z)| \leq |Q_1^{(s)}(z)| = K^{n-s} |q^{(s)}(z/K)|,
\]

for \( |z| \geq 1 \). In particular we have

\[
|p^{(s)}(K^2 e^{i\theta})| \leq K^{n-2s} |q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta < 2\pi.
\] (2.2)
The polynomial $p^{(s)}(Kz)$ is a polynomial of degree $n - s$ having all its zeros in $|z| \leq 1$; hence on considering the quotient
\[ z^{n-s} \left\{ \frac{\overline{p^{(s)}(K/\bar{z})}}{p^{(s)}(Kz)} \right\} \]
in $|z| \geq 1$ one gets easily as a consequence of the maximum modulus principle that
\[ |z^{n-s} \left\{ \frac{\overline{p^{(s)}(K/\bar{z})}}{p^{(s)}(Kz)} \right\} | \leq | p^{(s)}(Kz) | \quad \text{for} \quad |z| \geq 1, \]
which gives
\[ K^{n-s} | p^{(s)}(e^{i\theta}) | \leq | p^{(s)}(K^2 e^{i\theta}) |, \quad 0 \leq \theta < 2\pi. \quad (2.3) \]
Combining this with (2.2) we get (2.1) for polynomials having all their zeros on $|z| = K \geq 1$.

If the zeros of $p(z)$ lie in $|z| \geq K \geq 1$ but not necessarily on $|z| = K$, then for every real $\gamma$, the polynomial $p(z) + e^{\gamma Q(z/K)}$ has all its zeros on $|z| = K \geq 1$ and applying (2.1), which has been proved for polynomials having all the zeros on $|z| = K \geq 1$, to the polynomial $p(z) + e^{\gamma Q(z/K)}$, Lemma 1 will follow.

**Lemma 2.** If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < K$, $K \geq 1$, and $q(z) = z^{n-1} \{ \overline{p(1/\bar{z})} \}$, then for $|z| \geq 1/K$,
\[ |q^{(s)}(z)| \geq mn(n-1) \cdots (n-s+1) |z|^{n-s}, \quad (2.4) \]
where $m = \min_{|z| = K} |p(z)|$.

**Proof of Lemma 2.** Because the polynomial $p(z)$ has no zeros in $|z| < K$, $K \geq 1$, the polynomial $q(z) = z^{n-1} \{ \overline{p(1/\bar{z})} \}$ has all its zeros in $|z| \leq 1/K \leq 1$. Therefore for every $z$, $|z| < 1$, the polynomial $q(z) - zmz^n$ has all its zeros in $|z| \leq 1/K$, which implies by the Gauss–Lucas theorem that $q^{(s)}(z) - zm(n-1) \cdots (n-s+1) z^{n-s}$ has all its zeros in $|z| \leq 1/K$ and from which (2.4) will follow.

**Lemma 3.** If $p(z)$ is a polynomial of degree $n$ having all its zeros in $|z| < K$, $K \geq 1$, then
\[ \max_{|z|=1} |p'(z)| \geq \frac{n}{1 + K^{n-1}} \max_{|z|=1} |p(z)|. \]
The result is best possible with equality for $p(z) = z^n + K^n$.

The above result is due to Govil [2].
3. PROOFS OF THEOREMS

Proof of Theorem 1. Let \( p(z) \) be a polynomial of degree \( n \) having all its zeros in \( |z| \leq 1 \). Then \( q(z) = z^n \{ p(1/z) \} \) has no zeros in \( |z| < 1 \); hence by Lemma 1,

\[
|q^{(s)}(e^{i\theta})| \leq |p^{(s)}(e^{i\theta})|, \quad 0 \leq \theta < 2\pi. \tag{3.1}
\]

If \( p(z) \) is a polynomial of degree \( n \), \( \| p \| = \max_{|z|=1} |p(z)| \), then by Rouché's theorem for every \( \lambda \) with \( |\lambda| > 1 \), the polynomial \( p(z) - \lambda \| p \| z^n \) has all its zeros in \( |z| < 1 \); hence applying (3.1) to the polynomial \( p(z) - \lambda \| p \| z^n \) we conclude that if \( q(z) = z^n \{ p(1/z) \} \), then

\[
|p^{(s)}(e^{i\theta})| + |q^{(s)}(e^{i\theta})| \leq \| p \| n(n-1)\ldots(n-s+1). \tag{3.2}
\]

If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, K \geq 1 \), and if \( m = \min_{|z|=K} |p(z)| \), then for every \( z \) with \( |z| < 1 \) the polynomial \( p(z) - zm \) has no zeros in \( |z| < K, K \geq 1 \). This result is clear if \( p(z) \) has a zero on \( |z| = K \) for then \( m = 0 \) and hence \( p(z) - zm = p(z) \). In case \( p(z) \) has no zeros on \( |z| = K \), then, for every \( z \) with \( |z| < 1 \), we have \( |p(z)| > |z| m \) on \( |z| = K \) and the result follows from Rouché's theorem. Thus in any case \( p(z) - zm \) has no zeros in \( |z| < K, K \geq 1 \), and therefore applying Lemma 1 to the polynomial \( p(z) - zm \), we get

\[
K^s |p^{(s)}(e^{i\theta})| \leq |q^{(s)}(e^{i\theta})| \frac{zm(n-1)\ldots(n-s+1)}{me^{\xi n^{-1}}}. \tag{3.3}
\]

Choosing argument of \( z \) suitably, making \( |z| \to 1 \), and noting that by Lemma 2, \( |q^{(s)}(e^{i\theta})| \geq mn(n-1)\ldots(n-s+1) \), we get from (3.3)

\[
K^s |p^{(s)}(e^{i\theta})| \leq |q^{(s)}(e^{i\theta})| - mn(n-1)\ldots(n-s+1),
\]

which is clearly equivalent to

\[
|q^{(s)}(e^{i\theta})| \geq K^s |p^{(s)}(e^{i\theta})| + mn(n-1)\ldots(n-s+1). \tag{3.4}
\]

Now combining (3.4) with (3.2), Theorem 1 follows.

Proof of Theorem 2. First we prove (1.7). Since the polynomial \( p(z) \) has all its zeros in \( |z| \leq K \leq 1 \), the polynomial \( q(z) = z^n \{ p(1/z) \} \) has no zeros in \( |z| < 1/K, 1/K \geq 1 \); hence applying Theorem 1, with \( s = 1 \), to \( q(z) \) we get

\[
|q'(z)| = \frac{n}{(1 + 1/K)} \left( \max_{|z|=1} |q(z)| - \min_{|z|-1} |q(z)| \right),
\]
which gives that on $|z| = 1$,

$$|np(z) - zp'(z)| \leq \frac{nK}{1 + K} \max_{|z| = 1} |p(z)| - \frac{nK}{1 + K} \min_{|z| = 1} |q(z)|$$

$$\leq \frac{nK}{1 + K} \max_{|z| = 1} |p(z)| - \frac{nK}{(1 + K) K^n} \min_{|z| = K} |p(z)|,$$

which implies that for $|z| - 1$,

$$n |p(z)| - |p'(z)| \leq \frac{nK}{1 + K} \max_{|z| = 1} |p(z)| - \frac{nK}{(1 + K) K^n} \min_{|z| = K} |p(z)|. \tag{3.5}$$

Now choosing $z_0$ such that $|p(z_0)| = \max_{|z| = 1} |p(z)|$, we get from (3.5)

$$|p'(z_0)| \geq \left( \frac{n}{1 + K} \right) \max_{|z| = 1} |p(z)| + \frac{nK}{(1 + K) K^n} \min_{|z| = K} |p(z)|,$$

from which (1.7) follows.

To prove (1.8), note that if $m = \min_{|z| = K} |p(z)|$, then for every $z$ with $|z| < 1$, the polynomial $p(z) + zm$ has all its zeros in $|z| < K$. This is clear if $p(z)$ has a zero on $|z| = K$, because in that case $m = 0$ and therefore $p(z) + zm = p(z)$. In case $p(z)$ has not zero on $|z| = K$, then, for every $z$ with $|z| < 1$, we have $|p(z)| > m |z|$ on $|z| = K$ and on applying Rouché's theorem the result will follow. Thus $p(z) + zm$ has all its zeros in $|z| < K$, $K > 1$ and hence, applying Lemma 3 to $p(z) + zm$, we get

$$\max_{|z| = 1} |p'(z)| \geq \frac{n}{1 + K} \max_{|z| = 1} |p(z) + zm|. \tag{3.6}$$

If we choose $z_0$ such that $|p(z_0)| = \max_{|z| = 1} |p(z)|$, (3.6) in particular gives

$$\max_{|z| = 1} |p'(z)| \geq \frac{n}{1 + K} (|p(z_0)| + zm). \tag{3.7}$$

Now choosing $z$ so that the right hand side of (3.5) is

$$\frac{n}{1 + K} (|p(z_0)| + zm)$$

and making $|z| \to 1$, we get (1.8).

The proof of Theorem 2 is thus complete.
REFERENCES