LATTICE VALUED RELATIONS AND AUTOMATA

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A lattice-valued relation, lvr for short, from a set $X$ to a set $Y$ is a function from the Cartesian product of $X$ and $Y$ to a lattice. This concept is a generalization of other structures, notably tolerance spaces, nets and automata, separately investigated by the authors elsewhere. It is adequate to admit a natural definition of homogeneity and a classification of homogeneous lvr's by their isomorphism groups. The main result of the present paper is a proof of this classification. The application of this to automata, also interpretable as lvr's, is described, and an example given. We conclude with a brief discussion of the lvr theory of fuzzy and stochastic automata.

1. Introduction

1.1. This paper describes the mathematical outcome of attempts by the authors to study neural systems and their automaton equivalents. The dynamical theory of neural networks is complicated, not only by the non-linearity of the threshold conditions for the activation of a neuron, but also by the discreteness of the system, precluding the application of classical analytical techniques.

1.2. A preliminary attempt was made to overcome the problem of discreteness by employing Zeeman's [12] idea of a tolerance space. Zeeman suggested that continuity concepts might be fruitfully brought into the theory of discrete systems by formalising the notion 'nearness' as a reflexive, symmetric relation on discrete sets. Despite the lack of richness in this structure, a considerable number of familiar topological ideas may be mimicked in the simpler setting. Beginning with the analog of a continuous map as a function which preserves nearness, or tolerance, various categorical constructions can be made, creating product spaces, function spaces and so on [8].

1.3. One of our central ideas, first studied in the tolerance theory context [6], is that of homogeneity. It would seem that general neural networks might be too complicated a field to yield a coherent theory, so a physically plausible simplification was effected by looking at highly uniform structures in which every object stands in the system in an identical way. It was found that a relatively natural definition of homogeneity led to a classification of homogeneous tolerance spaces in terms of...
their symmetry groups. Furthermore, an explicit way of constructing all such homogeneous spaces resulted. The same ideas were then found to carry over, albeit less obviously, to classify homogeneous neural networks, themselves a generalisation of tolerance spaces [4].

1.4. Now there is a well-known equivalence between neural networks and finite automata. It would seem reasonable to study directly the classification and properties of such automata. A certain convergence of thinking encouraged this idea since Arbib [1] had proposed the notion of a tolerance automaton as the appropriate discrete analogy to control theory for differential dynamical systems. The idea is to equip the state space with a tolerance, permitting continuity of input action to be meaningfully employed. A natural way to do this is to impose a tolerance on the state space of any automaton by means of the input action, so that two states are within tolerance iff there exists an input taking one to the other; this is Arbib’s inertial tolerance. But why restrict ourselves to a symmetric relation? Equally well, a state may be thought of as being related to another if there exists an input, taking the first to the second. It emerged that the homogeneity classification theorem [9] still works with this relation.

1.5. At each successive stage then we have attempted to lose less information; for example, tolerance loses even the sense of direction of input action. Thus we moved on to count the number of inputs linking a couple of states, giving rise to a situation resembling a neural network. But finally it was realised that all the information could be retained by re-conceptualizing an automaton as a function from the direct product of the state space with itself into the power set of the input set.

1.6. At this point all the disparate threads came together. For the notions of tolerance space, neural network and automaton could be seen as relations on a set taking values in some lattice. A general theory of such lattice valued relations embraces all the theories hitherto described, the homogeneity treatment in particular yielding a good concept of homogeneous automata.

1.7. Such a general theory, of course, includes lattice valued functions from the product of different sets. An ordinary relation between two sets is simply a particular case of a lattice valued relation with lattice \( \{0, 1\} \). One can readily interpret an extension of this lattice to the unit interval as a theory of fuzzy relations. It is not, therefore, surprising that fuzzy automata and probabilistic automata also fit neatly into the theory, and will be cited as applications.

1.8. We start in Section 2 with a brief recapitulation of the required elements of lattice theory. Section 3 goes on to define lattice valued relations on sets, and the appropriate categorical concepts of morphism, product, subset and function spaces. Homogeneity forms the subject matter of Section 4 where the central classifica-
tion theorem in terms of a group quotient is proved for lattice valued relations. Further, it is shown how any group can be employed to construct a 'very homogeneous' object.

Section 5 is then devoted to the respecification of automata as lattice valued relations. The 'very homogeneous' requirement emerges simply as a demand that every input be bijective and that the input set be closed under conjugation. An illustrative example is given.

The deterministic automata of Section 6 are particular examples of fuzzy automata, or alternatively, of stochastic automata. The paper ends with a brief discussion of their associated lattice valued relations.

2. Lattices

2.1. Definition. A lattice is a non-empty, partially ordered set \((L, \leq)\) in which every pair of elements \(a, b\) has a greatest lower bound, \(a \wedge b\), and a least upper bound, \(a \vee b\).

Thus \(a \wedge b \leq a, b \leq a \vee b\), and \(x \leq a, b \Rightarrow x \leq a \wedge b\), while \(a, b \leq y \Rightarrow a \vee b \leq y\). These two bounds are called the meet, \(a \wedge b\), and join, \(a \vee b\), respectively of \(a\) and \(b\).

If every subset \(X\) of \(L\) has a g.l.b. and a l.u.b., then \(L\) is complete. In particular, a complete lattice has a least element, \(0\), and a greatest element, \(1\).

2.2. Definition. A morphism \(f: L \rightarrow L'\) of lattices is a function \(f\) from \(L\) to \(L'\) which preserves the binary operations meet and join; i.e.

\[ f(a \wedge b) = f(a) \wedge f(b), \quad f(a \vee b) = f(a) \vee f(b). \]

A lattice morphism preserves the partial orderings, since that ordering may be recovered from meet or join according to \(a \leq b\) iff \(a \wedge b = a\) or \(a \vee b = b\).

2.3. Definition. A sublattice \(S\) of a lattice \(L\) is a subset of \(L\) which is closed with respect to the operations meet and join of \(L\).

2.4. Definition. The product \(L \times M\) of lattices \(L, M\) is the partially ordered set \(\{(x, y); x \in L, y \in M\}\) with \((x_1, y_1) \leq (x_2, y_2)\) iff \(x_1 \leq x_2\) in \(L, y_1 \leq y_2\) in \(M\).

It is straightforward to show that the meet and join in \(L \times M\) are given respectively by

\( (x_1 \wedge x_2, y_1 \wedge y_2) = (x_1, y_1) \wedge (x_2, y_2), \quad (x_1 \vee x_2, y_1 \vee y_2) = (x_1, y_1) \vee (x_2, y_2). \)

3. Lattice valued relations

3.1. Definition. A lattice valued relation \((lvr\) henceforth) between sets \(Q, Q'\) is a function \(\lambda: Q \times Q' \rightarrow L\) where \(L\) is a lattice.
In particular, when \( Q' = Q \) we have the lvr \( \lambda \) on the set \( Q \), and the triple \((Q, L, \lambda)\) will be called an lvr space over \( L \), or an Lvr space.

### 3.2. Examples.

(i) \( L = \{0, 1\} \) yields an ordinary relation. If \( \lambda \) is reflexive and symmetric with \( Q' = Q \), \( Q \) is a tolerance space.

(ii) If \( L = [0, 1] \), \( \lambda \) is a fuzzy relation in the sense of Zadeh [15]. Here \( \lambda(q, q') \) is a measure of the belief that \( q \) is related to \( q' \).

(iii) When \( L = R \) and \( Q' = Q \) we have a net [4]. If, further, \( \lambda \) satisfies the conditions for a metric, then \((Q, \lambda)\) is a metric space.

(iv) Let \( L = B(\Omega) \), the collection (lattice) of Borel sets in a probability space \( \Omega \). Then \( \lambda \) may be interpreted as assigning, via any measure on \( \Omega \), a probability of the relation between \( q \) and \( q' \).

(v) An automaton \((X, Q, \delta)\) will be taken throughout to be a function \( \delta : X \times Q \rightarrow Q \). Here \( Q \) is the finite state set of the system, and \( X \) the finite input set causing a state transition according to \( \delta \). We shall not be concerned with outputs and shall not therefore distinguish between an automaton and a semi-automaton (Ginzburg [3]). In order to allow successive inputs of a string to be written in sequence, we adopt the convention of writing the action of \( \delta \) as right translation. Thus \( \delta(x, q) = q \cdot x \).

An automaton may be recast as an lvr \( \lambda : Q \times Q \rightarrow P(X) \), where \( P(X) \) is the lattice of subsets of \( X \), and

\[
\lambda(q, q') = \{ x \in X; q \cdot x = q' \},
\]

the set of inputs which can move the state \( q \) to the state \( q' \).

### 3.3. Henceforth, unless otherwise stated, we shall restrict ourselves to Lvr's on a set \( Q \), \( \lambda : Q \times Q \rightarrow L \).

**Definition.** A morphism from an Lvr space \((Q, L, \lambda)\) to an \( L' \)vr space \((Q', L', \lambda')\) is a pair of functions \( \beta : Q \rightarrow Q' \), \( \gamma : L \rightarrow L' \) where \( \gamma \) is a lattice morphism and

\[
\lambda'(\beta(q), \beta(q')) \geq \gamma(\lambda(q, q')) \quad \text{for all } q, q' \in Q.
\]

The square

\[
\begin{array}{ccc}
Q \times Q & \xrightarrow{\lambda} & L \\
\beta \times \beta \downarrow & & \downarrow \gamma \\
Q' \times Q' & \xrightarrow{\lambda'} & L'
\end{array}
\]

is not, then, required to commute, but to satisfy an inequality. We shall see later (Section 4) that the inequality is appropriate for yielding a good homogeneity theory, a goal not achieved by a commuting square. Further, it is this inequality rather than equality that gives straightforward definitions of induction and co-induction, thus reinforcing its claim to yield the natural morphism for the category.
If \( \beta, \gamma \) are bijective and \((\beta^{-1}, \gamma^{-1})\) is a morphism, we have an isomorphism of \(Lvr\)'s, for which

\[
\lambda'((\beta(q), \beta(q'))) = \gamma(\lambda(q, q')).
\]

When \( L = L' \) and \( \gamma = 1 \), we will refer to a morphism as \( \beta \).

3.4. Induction. Given a family \( \lambda_i: Q_i \times Q_i \to L \) of \( Lvr\)'s on the members of an indexed collection of sets \( \{Q_i\} \), together with functions \( f_j: X \to Q_i \) from \( X \) to \( Q_i \), the induced \( Lvr \) \( \lambda \) on \( X \) is defined by

\[
\lambda(x, x') = \bigwedge_i \lambda_i(f_i(x), f_i(x')) \quad \text{for all} \quad x, x' \in X.
\]

This is the 'least' requirement such that all \( f_i \) are morphisms, that is, so that

\[
\lambda_i(f_i(x), f_i(x')) \leq \lambda(x, x').
\]

Clearly, inequality rather than equality here is essential. Also, the join \( \bigwedge \) is required to exist. This can be ensured by taking the collection \( \{Q_i\} \) to be finite or the lattice \( L \) to be complete.

The definition extends to a family \( \lambda_i: Q_i \times Q_i' \to L \) and functions \( f_i: X \to Q_i \), \( g_i: X' \to Q_i' \), to give an \( Lvr \) \( (X, X', L, \lambda) \).

3.5. The subset \( Lvr \) induced on \( Q' \subseteq Q \) from \( (Q, L, \lambda) \) by the injection \( i: Q' \to Q \) is simply \( \lambda \) restricted to \( Q' \times Q' \).

3.6. Given a set \( X \) and an \( Lvr \) \( (Q, L, \lambda) \) the function space \( Lvr \) \( \mu \) on \( Q^X \) is induced by the evaluation maps \( ev_x: Q^X \to Q \), \( ev_x(f) = f(x) \) for all \( x \in X \). Thus for \( f, g \in Q^X \),

\[
\mu(f, g) = \bigwedge_{x \in X} \lambda(f(x), g(x)).
\]

Again the meet over all \( x \in X \) must exist, so we need either \( X \) finite or \( L \) complete.

If \( X \) has itself an \( Lvr \), then the set \( Q^X_m \) of lattice morphisms from \( X \) to \( Q \) is a subset of \( Q^X \) and inherits an \( Lvr \) structure from it.

4. Homogeneity

4.1. For the sake of clarity and convenience, we shall consider the set of isomorphisms \( \{((\beta, 1))\} \), or \( \{\beta\} \), from an \( Lvr \) space \( (Q, L, \lambda) \) to itself. Most of the assertions of this section can be formally stated for general isomorphisms \( (\beta, \gamma) \). Attention will be drawn to occasions when this is not the case. The set \( \{\beta\} \) of isomorphisms of \( Q \) is a group, \( G \), with respect to composition. Give \( G \) the subspace \( Lvr \) induced from the function space \( Lvr \) of Section 3.6, namely, for all \( f, g \in G \),

\[
\mu(f, g) = \bigwedge_{q \in Q} \lambda(f(q), g(q)).
\]

Henceforth denote by \( e \) the identity of \( G \).
Lemma. \( \mu \) is invariant under right and left translation by elements of \( G \).

Proof. Let \( g, g', \bar{g} \in G \). As \( q \) ranges over \( Q \), so does \( q' = \bar{g}(q) \) since \( \bar{g} \) is bijective, giving

\[
\mu(\bar{g}g, \bar{g}g') = \bigwedge_{q \in Q} \lambda(\bar{g}g(q), \bar{g}g'(q)) = \bigwedge_{q \in Q} \lambda(g(q'), g'(q')) = \mu(g, g').
\]

Thus \( \mu \) is invariant under right translation.

Secondly, since \( \bar{g} \) is an isomorphism,

\[
\lambda(\bar{g}g(q), \bar{g}g'(q)) - \lambda(g(q), g'(q)).
\]

Taking \( \bigwedge_{q \in Q} \) of both sides gives \( \mu(\bar{g}g, \bar{g}g') = \mu(g, g') \), i.e. invariance under left translation.

An Lvr space \((G, L, \mu)\), where \( G \) is a group and \( \mu \) is left and right invariant will be called an Lvr group.

Actually this concept may be generalized by considering isomorphisms \((\beta, \gamma)\) of \((Q, L, \lambda)\) as in Section 3.3. If \( g \) is an element of the group \( G \) of these generalized isomorphisms, we may denote the associated functions by \( \beta_g, \gamma_g \).

In this notation

\[
\mu(\bar{g}g, \bar{g}g') = \bigwedge_{q \in Q} \lambda(\beta_{g}(q), \beta_{g'}(q)).
\]

Right invariance is not affected, but left invariance must be replaced by

\[
\mu(\bar{g}g, \bar{g}g') = \gamma_{g'}(\mu(g, g')).
\]

4.2. Choose \( q \in Q \), and define \( \psi_q: G \to Q \) by \( \psi_q(g) = g(q) \). Consider \( q \) to be chosen and fixed throughout the ensuing discussion (4.2 to 4.6).

Lemma. \( \psi_q \) is a morphism of lvr spaces over \( L \).

Proof.

\[
\lambda(\psi_q(g), \psi_q(g')) = \lambda(g(q), g'(q)) \geq \bigwedge_{q \in Q} \lambda(g(q), g'(q)) = \mu(g, g').
\]

4.3. Let \( H = \{g \in G; g(q) = q\} \) be the subset of \( G \) which fixes \( q \). Then \( H \) is a subgroup of \( G \). Let \( F \) be the set of left cosets of \( H \) in \( G \); that is, consider an equivalence relation \( \sim \) on \( G \) defined by \( g \sim g' \) iff \( g^{-1}g' \in H \). The coinduced Lvr on \( F \) is defined by

\[
\bar{\mu}([g], [g']) = \bigvee_{g \in [g]} \mu(g, g').
\]
Lemma. $\overline{\mu}([g], [g']) = \bigvee_{h \in H} \mu(g, g'h)$.

Proof. $\overline{\mu}([g], [g']) = \bigvee_{h_1, h_2 \in H} \mu(gh_1, g'h_2) = \bigvee_{h_1, h_2 \in H} \mu(g, g'h_2h_1^{-1})$ by the right invariance of $\mu$. But as $h_1, h_2$ range over $H$, so does $h = h_2h_1^{-1}$, yielding the result.

4.4. The function $\psi_q : F \rightarrow Q$ defined by $\psi_q([g]) = \psi_q(g)$ is well-defined and injective.

Lemma. $\psi_q$ is an $Lvr$ morphism.

Proof. $\forall q' \in Q$ such that $\psi_q(g(q)) = \psi_q(g'(q))$

$\mu(g, g'h) = \psi_q(g) = \psi_q(g') = \psi_q(g'h)$ for all $h \in H$.

So

$\lambda(\psi_q([g]), \psi_q([g'])) \geq \bigvee_{h \in H} \mu(g, g'h) = \overline{\mu}([g], [g'])$.

4.5. The foregoing holds for all $Lvr$ spaces. In order to ensure that $\psi_q$ is an isomorphism, $\lambda$ needs to have some form of homogeneity.

Definition. $(Q, L, \lambda)$ is homogeneous if for all $q', q'' \in Q$ there is an isomorphism $g \in G$ such that $g(q') = q''$.

Clearly, if $Q$ is homogeneous, then $\psi_q$ is onto. For, given $q' \in Q$, choose $g \in G$ such that $q' = g(q) = \psi_q([g])$.

4.6. It remains to establish the circumstances under which $\psi_q^{-1}$ is a morphism.

Definition. $(Q, L, \lambda)$ is very homogeneous (hereafter written vh) if for all $q', q'' \in Q$ there exists $g \in G$ such that $g(q') = q''$ and

$\lambda(q', q'') \leq \bigvee_{g \in G} \mu(e, g)$.

This inequality can be replaced by equality since it is always true that

$\lambda(q', q'') \geq \bigwedge_{q \in Q} \lambda(q, g(q)) = \mu(e, g)$.

So

$\lambda(q', q'') \geq \bigvee_{g} \mu(e, g)$.

We also observe that vh implies homogeneity.

Theorem. If $(Q, L, \lambda)$ is very homogeneous, then $\psi_q^{-1}$ is a morphism, so $\psi_q$ is an isomorphism.
Proof. For general $q', q'' \in Q$ let $g'(q) = q'$, $g''(q) = q''$. Then if $g(q') = q''$, we have $gg'(q) = g''(q)$, so $gg' = g''h$ for some $h \in H$. Thus

$$\bigvee_{g \in G} \mu(e, g) = \bigvee_{g \in G} \mu(g', g''h)$$

$$= \mu([g'], [g'']) = \mu(\bar{\varphi}_q^{-1}(q'), \bar{\varphi}_q^{-1}(q'')) = \lambda(q', q'').$$

4.7. Theorem 4.6. shows how a vh Lvr can be reconstructed from its isomorphism group $Lvr$. We now strengthen this result by showing that any group can be so employed to construct an associated $Lvr$ which is automatically vh.

Continuing to restrict ourselves, for the moment, to $Lvr$ isomorphisms $(\beta, \lambda)$, we observe that the right and left invariant $Lvr$ $\mu$ on $G$ may be replaced by a function $\alpha : G \to L$, defined by $\alpha(g) = \mu(e, g)$, since every $\mu(g_1, g_2)$ can be rewritten as $\mu(e, g_1^{-1} g_2)$. The only condition on $\alpha$ is that it be a class function, that is, conjugate invariant, since

$$\alpha(g^{-1}g'g) = \mu(e, g^{-1}g'g) = \mu(g, g'g) = \mu(e, g) = \alpha(g').$$

$\alpha$ is called a lattice valued class function.

4.8. Theorem. Let $\alpha : G \to L$ be a lattice valued class function on a group $G$, and let $H$ be any subgroup of $G$. Define $\tilde{\mu} : G/H \times G/H \to L$ by

$$\tilde{\mu}([g], [g']) = \bigvee_{h \in H} \alpha(g^{-1}g'h).$$

Then $(G/H, \tilde{\mu})$ is a vh Lvr.

Proof. We need to establish that

$$\bigvee_{g \in G} \bigwedge_{[g'] \in G/H} \tilde{\mu}([g''], [g']) \succeq \tilde{\mu}([g], [g'])$$

where $\tilde{G}$ is the isomorphism group of $(G/H, \tilde{\mu})$. Now $G \subseteq \tilde{G}$ in the sense that left multiplication of cosets in $G/H$ by elements of $G$ are isomorphisms, because, for $g_1 \in G$,

$$\tilde{\mu}(g_1[g], g_1[g']) = \bigvee_{h \in H} \alpha(g^{-1}g_1^{-1}g'g'h)$$

$$= \bigvee_{h \in H} \alpha(g^{-1}g'h) = \tilde{\mu}([g], [g']).$$

It suffices, therefore, to establish the inequality

$$\bigvee_{g \in G} \bigwedge_{[g'] \in G/H} \tilde{\mu}([g''], [g']) \succeq \tilde{\mu}([g], [g']).$$

Written in terms of $\alpha$, this is

$$\bigvee_{g \in G} \bigwedge_{[g'] \in G/H} \bigvee_{h \in H} \alpha(g^{-1}gg'h) \succeq \bigvee_{h \in H} \alpha(g^{-1}g'h_2).$$
Now every $g \in G$ such that $[gg] = [g']$ has the form $g'hg^{-1}$, $h \in H$. So rewriting the left-hand side we have

$$\bigvee_{h \in H} \bigwedge_{[g'] \in G/H} \bigvee_{h_1 \in H} \alpha(g^{-1}g'hg^{-1}g''h_1).$$

But for any $g \in G$, by conjugacy invariance,

$$\bigvee_{h_1 \in H} \alpha(g^{-1}gg''h_1) \geq \alpha(g^{-1}gg'') = \alpha(g).$$

Therefore

$$\bigwedge_{[g'] \in G/H} \bigvee_{h \in H} \alpha(g^{-1}(g'hg^{-1})g''h_1) \geq \alpha(g'hg^{-1}) = \alpha(g^{-1}g'h).$$

Taking $\bigvee_{h \in H}$ of both sides establishes the result.

**4.9. Corollary.** *If the subgroup $H$ is normal, then $(G/H, \bar{\mu})$ is an Lvr group.*

**Proof.**

$$\bar{\mu}([g][g'], [g][g']) = \bar{\mu}([gg],[gg'])$$

$$= \bigvee_{h \in H} \alpha(g^{-1}g^{-1}gg'h) = \bar{\mu}([g],[g'])$$

without using the normality of $H$, and

$$\bar{\mu}([g][g'], [g'][g]) = \bar{\mu}([gg],[g'g])$$

$$= \bigvee_{h \in H} \alpha(g^{-1}g^{-1}g'gh) = \bigvee_{h \in H} \alpha(g^{-1}g'ghg^{-1})$$

$$= \bigvee_{h \in H} \alpha(g^{-1}g'h_1) \quad \text{by normality}$$

$$= \bar{\mu}([g],[g']).$$

**4.10.** The reasoning of Section 4.7 encounters difficulties when the isomorphisms are extended to pairs $(\beta, \gamma)$. Recall that $\mu$ is no longer left translation invariant, but satisfies

$$\mu(\bar{g}g, \bar{g}g') = \gamma_g \mu(g, g').$$

This in turn destroys the conjugacy invariance of $\alpha$, which must be replaced by

$$\alpha(gg^{-1}) = \gamma_g(\alpha(g)).$$

For a version of Theorem 4.8 we therefore need to choose a homomorphism $\Gamma: G \to M$, where $M$ is the group of lattice isomorphisms of $L$, and set $\gamma_g = \Gamma(g)$. Then define $\alpha$ by choosing it arbitrarily on some element in each conjugacy class, extending it to all $G$ by the modified conjugacy invariance. Then $\bar{\mu} : G/H \times G/H \to L$ is defined by

$$\bar{\mu}([g],[g']) = \gamma_g \left( \bigvee_{h \in H} \alpha(g^{-1}g'h) \right).$$
The statement and proof of the theorem may then be broadened by the formal exercises of remembering to deploy a \( g \) whenever a factor \( g \) is transferred from front to back inside \( \alpha \).

5. Automata

5.1. We saw in Section 3.2 that an automaton \( (X, Q, \delta) \) can be considered as a \( P(X) \text{vr} \) on \( Q \). We now translate some of the \( \text{lr} \) concepts into automaton terms. Recall from 3.2 that for \( q, q' \in Q \)
\[
\lambda(q, q') = \{ x \in X; q' = q \cdot x \}.
\]

**Lemma.** An automaton morphism in the sense of [3] induces a morphism of associated \( \text{lr} \)’s.

**Proof.** A morphism of the \( \text{lr} \) \( \lambda : Q \times Q \to P(X) \) to \( \lambda' : Q' \times Q' \to P(X') \) is a pair \( (\beta, \gamma) \) with \( \gamma \) an intersection and union preserving function from \( P(X) \) to \( P(X') \), \( \beta \) a function from \( Q \) to \( Q' \) such that
\[
\lambda'(\beta(q), \beta(q')) \supseteq \gamma(\lambda(q, q')).
\]

This means
\[
\gamma(\{ x \in X; q' = q \cdot x \}) \subseteq \{ x' \in X'; \beta(q') = \beta(q) \cdot x' \}.
\]

We must issue a warning here: if \( A \in P(X) \), \( \gamma(A) \) need not be obtained by the pointwise application of some function on the elements of \( A \). However, a morphism of automata is a pair \( (\beta, \gamma) \), \( \beta : Q \to Q', \gamma : X \to X' \) such that \( \beta(q \cdot x) = \beta(q) \cdot \gamma(x) \) for all \( q \in Q, x \in X \). \( \gamma \) induces a lattice morphism \( \gamma : P(X) \to P(X') \) for which
\[
x' \in \gamma(\{ x \in X; q' = q \cdot x \}) \Rightarrow x' = \gamma(x) \quad \text{with} \quad q' = q \cdot x
\]
\[
\Rightarrow \beta(q') = \beta(q \cdot x) = \beta(q) \cdot x'
\]
\[
\Rightarrow x' \in \{ x' \in X'; \beta(q') = \beta(q) \cdot x' \}
\]
and the lemma is proved.

5.2. The automata \( A, A' \) are isomorphic if \( \gamma, \beta \) are bijective and \( (\gamma^{-1}, \beta^{-1}) \) is also a morphism.

Evidently a \( P(X) \text{vr} \) morphism is more general than an automaton morphism, but in the case of self-isomorphisms the two coincide, according to

**Lemma.** If no two elements of \( X \) coincide as functions from \( Q \) to itself, an \( \text{lr} \) isomorphism is an automaton isomorphism.
Proof. For $x \in X$ define $A_q = \{ \bar{x} \in X; q \cdot x = q \cdot \bar{x} \}$, so $\bar{x} \in A_q$ has the same effect on $q$ as does $x$. By assumption $\{x\} = \bigcap_q A_q$ and since $\gamma$ is a lattice morphism

$$\gamma(\{x\}) = \bigcap_q \gamma(\{ \bar{x} \in X; q \cdot x = q \cdot \bar{x} \})$$

$$= \bigcap_q \gamma(\lambda(q, q \cdot x)) = \bigcap_q \lambda'(\beta(q), \beta(q \cdot x))$$

$$= \{ \bar{x} \in X; \beta(q \cdot x) = \beta(q) \cdot \bar{x} \text{ for all } q \in Q \}.$$

Since $\beta$ is bijective, this defines the action of any such $\bar{x}$ on the whole of $Q$ so $\gamma(\{x\})$ is either a singleton set or the empty set. But the latter would violate bijectivity of $\gamma$.

Now $\gamma$ can be considered as induced by a function $\bar{\gamma} : X \to X$. For any $x \in X$, $q \in Q$, $x \in A_q$ trivially so

$$\bar{\gamma}(x) \in \gamma(A_q) \subseteq \lambda'(\beta(q), \beta(q \cdot x))$$

and $\beta(q \cdot x) = \beta(q) \cdot \bar{\gamma}(x)$, so $(\beta, \bar{\gamma})$ is an automaton morphism.

5.3. Consider now the group $G$ of isomorphisms of $(Q, P(X), \lambda)$. For $g, g' \in G$,

$$\mu(g, g') = \bigcap_{q \in Q} \{ x \in X; \beta_g(q) = \beta_{g'}(q) \cdot x \}$$

$$= \{ x; \beta_g = x \circ \beta_{g'} \}.$$

The composite $x \circ \beta_g$ is the composite of $x$, considered as a function from $Q$ to itself, with the isomorphism $\beta_g$. Since the $\beta$'s are bijective $\mu(g, g')$ is then a singleton or empty. In the former case $x = \beta_{g' \cdot g^{-1}}$ which, being the $\beta$-map of an isomorphism, demands a corresponding $y : X \to X$ such that

$$\lambda(q \cdot x, q' \cdot x) = y(\lambda(q, q')) \quad \text{for all } q, q' \in Q$$

$$= \{ \bar{x}; q' \cdot x = q \cdot \bar{x} \} = y(\{ x_1; q' = q \cdot x_1 \})$$

$$= q' \cdot x = q \cdot x \gamma(x_1) \text{ iff } q' = q \cdot x_1$$

$$= q \cdot \gamma(x_1) = q \cdot x^{-1} x_1 x \text{ for all } q \in Q, x_1 \in X$$

where we have replaced $q \cdot x$ by $\bar{q}$ and $x^{-1}$ abbreviates the inverse of the bijective action of $x$.

Note that $x^{-1}$ is not necessarily an element of $X$. Summarizing, we have:

5.4. Lemma. A bijective input $x \in X$ is the $\beta$-map of an isomorphism iff for all $x_1 \in X$ we have $x^{-1} x_1 x \in X$. This construction determines the corresponding $\gamma$-map.

5.5. Theorem. An automaton is very homogeneous as an lvr iff every $x \in X$ is a permutation of $Q$ and $X$ is closed under conjugation.

Proof. Let $e$ be the identity $\beta$-map $\beta(q) = q$ for all $q \in Q$. From Definition 4.6 an
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automaton is vh if for all \(q, q' \in Q\) there exists \(g \in G\) whose \(\beta\)-map takes \(q\) to \(q'\) and

\[
\lambda(q, q') \subseteq \bigcup_{\beta_\xi} \mu(e, \beta_\xi) = \{x; q' = q \cdot x\} \subseteq \bigcup_{\beta_\xi} \{x; \xi = \beta_\xi\}.
\]

This is simply a demand that every \(x \in X\) be the \(\beta\)-map of an isomorphism. By the previous lemma \(X\) is then closed under conjugation.

This means that every very homogeneous automaton can be considered as a self-conjugate subset of the permutation group on \(n\) objects, where \(n\) is the number of elements of \(Q\). Clearly, if that subset is actually a subgroup the self-conjugacy condition is trivially satisfied and the automaton semigroup is just the input set itself.

5.6. Example. Let \(X = \{a, b, c, d\}, Q = \{A, B, C, D\}\) with transition function

\[
\begin{array}{c|cccc}
\delta & A & B & C & D \\
\hline
a & A & D & B & C \\
b & B & C & A & D \\
c & C & B & D & A \\
d & D & A & C & B \\
\end{array}
\]

Every input is bijective and \(X\) is closed under conjugation, defining the corresponding \(X\)-map \(\gamma\) as follows:

- Conjugation by \(a\): \((a, b, c, d) \to (a, c, d, b),\)
- Conjugation by \(b\): \((a, b, c, d) \to (d, b, a, c),\)
- Conjugation by \(c\): \((a, b, c, d) \to (b, d, c, a),\)
- Conjugation by \(d\): \((a, b, c, d) \to (c, a, b, d).\)

The group \(G\) of isomorphisms contains any composite of the above input isomorphisms and hence contains the automaton (semi)-group as a subgroup. Now \(a, b\) generate all even permutations of \(Q\). That this is all of \(G\) then follows if we demonstrate that a particular odd permutation is not the \(Q\)-map of an isomorphism.

For instance, let \(\beta: Q \to Q\) be given by \((ABCD) \to (BACD).\) If this were a \(Q\)-map, then \(\beta(A \cdot a) = \beta(A) \cdot \gamma(a) = \beta(A) \cdot \gamma(a) = B = B \cdot \gamma(a),\) so \(\gamma(a) = c.\) But then, \(\beta(B \cdot a) = \beta(B) \cdot c \Rightarrow \beta(D) = \beta(B) \cdot c \Rightarrow D = A \cdot c,\) contradicting the \(\delta\)-table.

Hence, employing \(a, b\) as generators of \(G\) we find

\[
G = \{e, a, b, a^2, ab, ba, b^2, a^2b, aba, ab^2, b^2a^2, a^2b^2\}.
\]

From 5.3, denoting by \(g\) one of the \(Q\)-maps in \(G, \mu(e, g) = \{x \in X; x = g\}.\) Thus \(\mu(e, a) = \{a\}, \mu(e, b) = \{b\}, \mu(e, a^2b^2) = \{c\}, \mu(e, b^2a^2) = \{d\},\) and \(\mu(e, g) = \emptyset\) for all other \(g \in G.\)

We proceed to verify the isomorphism theorem. Choose a base point, say \(B,\) in \(Q.\) The fix subgroup of \(B\) in \(G\) is \(H = \{e, ba, a^2b^2\},\) and has left cosets \([e] = H,\)
[a] = \{a, ab^2, ab\}, \[b\] = \{b, aba, a^2\}, \[b^2\] = \{b^2, b^2a^2, a^2b\}. To calculate these we have used the relations between the generators a, b namely \(a^3 = b^3 = e, ab^2 = ba^2, a^2b = b^2a, aba = bab\). Since input action is written on the right and read in the opposite order to the function composites in Section 4, the left coset \(bH\), for example, is obtained by applying \(b\) on the right to the elements of \(H\).

Then \(\psi: G/H \rightarrow Q\) is defined by \([e] \rightarrow B, [a] \rightarrow D, [b] \rightarrow C, [b^2] \rightarrow A\), and \(\tilde{\mu}: G/H \times G/H \rightarrow P(X)\) comes from \(\mu([e],[g']) = \bigcup_{h \in H} \mu(e, g'h)\). Then the whole table for \(\tilde{\mu}\) can be constructed and under the isomorphism \(\psi\) becomes the lvr map \(\lambda: Q \times Q \rightarrow P(X)\) which is the equivalent of the given \(\delta\).

5.7. Actually all of the above theory could be revamped into an alternative setting where the automaton \((X, Q, \delta)\) was regarded as a \(P(X^*)\)vr where \(X^*\) is the set of finite strings of elements of \(X\). On the assumption that all states are reachable from each other by some input string \(\lambda(q, q') \neq \emptyset\) for all \(q, q' \in Q\). Now, of course, we lose the uniqueness of an input considered as a map from \(Q\) to \(Q\) since many strings induce the same function. But this is precisely the relation defining the automaton semigroup \(S\) and uniqueness is restored by considering \(\lambda\) to be a \(P(S)\)vr. Again Theorem 5.5 requires each element of \(X\), and hence of \(S\), to be a permutation, so \(S\) itself is a group. Conjugation is now automatic and hence the vh condition is also.

6. Fuzzy and stochastic automata

6.1. A fuzzy automaton \((X, Q, \delta)\), as defined in [10] with input set \(X\) and state set \(Q\) has next-state function \(\delta: X \times Q \times Q \rightarrow I\) where \(I\) is the closed unit interval.

The associated lvr is \(\lambda: Q \times Q \rightarrow I^X, \lambda(q, q') = f\), where \(f: X \rightarrow I\) is given by \(f(x) = \delta(x, q, q')\). The lvr of 3.2(v) is the special case when \(I^X\) is replaced by \(\{0, 1\}^X\) representing \(P(X)\). In the lattice \(I^X\), \(f \leq g\) iff \(f(x) \leq g(x)\) for all \(x \in X\), \(f \wedge g = h\) where \(h(x) = \min_x (f(x), g(x))\), and \(f \vee g = h', h'(x) = \max_x (f(x), g(x))\).

\(I^X\) is just the set of fuzzy subsets of \(X\) in the sense of Zadeh [11] and the lattice operations are the fuzzy analogues of set inclusion, intersection and union.

6.2. It has been suggested [5] that the major deficiency of fuzzy set theory lies in the failure of the 'excluded middle law'; that is \(A \cap A'\) is not necessarily null for a fuzzy set \(A\). This, in turn, results from the fact that the lattice \(I\) is not a Boolean algebra and it seems desirable to develop an alternative fuzzy theory in which the characteristic functions of subsets of \(X\) are defined as functions from \(X\) to a Boolean algebra \(B\).

Indeed, when \(B\) is the algebra of regular open sets in \(I\) we may regard their measures as probability assignments and recover thereby the concept of a stochastic automaton [2]. In general terms this is then a function \(\lambda: Q \times Q \rightarrow L^X\) where \(L\) is the lattice of measurable sets in a probability space \(\Omega\); in other words, a Markov process with controls.
For fixed \( x \in X \) constraints are again imposed on \( \lambda \) by the requirement that transition occurs from any initial state \( q \) to one and only one final state \( q' \). Thus for any \( q \in Q, x \in X, \lambda(q, q')(x) \) yields a partition of \( Q \) as \( q' \) ranges over \( Q \).

Probability theory is a richer structure than fuzzy set theory, in that the former takes account of correlations between events. The latter, in contrast, calculates the measure of joint events merely by taking the minima of their separate measures, paying no attention to their inter-relations. In the context of the present discussion of homogeneous lattice-valued relations, this deficiency has the perhaps dubious merit of imposing no restrictions on the \( \lambda \)-map of a fuzzy automaton. So construction of homogeneous fuzzy automata is possible by the techniques of Theorem 4.8, using a freely chosen class function with values in \( I^X \).

References