JOURNAL OF COMBINATORIAL THEORY, Series B **48**, 145–154 (1990)

Map Coloring and the Vector Cross Product*

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Communicated by the Managing Editors

Received July 14, 1987; revised July 4, 1988

The Four Color Theorem is equivalent to a combinatorial problem about the three-dimensional vector cross product algebra.

I. INTRODUCTION

This paper introduces a new reformulation of the Four Color Theorem. This reformulation shows that the Four Color Theorem is equivalent to a combinatorial problem about the vector cross product algebra in three-dimensional space (Section 2).

Our technique involves a construction due to G. Spencer-Brown (the *formation*: Section 3) and a device of Roger Penrose [4] of labelling vertices with imaginary numbers. These ideas combine to give a simple proof of the equivalence of this reformulation with the original four color problem.

The paper is organized as follows. In Section 2 we state the *cross-product reformulation* of the four color problem, and illustrate it through examples. Section 3 proves the results that demonstrate the equivalence of the four color problem and this algebra problem.

I take this opportunity to thank G. Spencer-Brown, James Flagg, Lee Smolin, and Annetta Pedretti for helpful conversations.

II. AN ALGEBRA PROBLEM

Let i, j, k denote a standard unit orthogonal basis for Euclidean three-dimensional space (as a vector space over the real numbers). (See Fig. 1.

* Research partially supported by NSF Grant #DMS-8701772.

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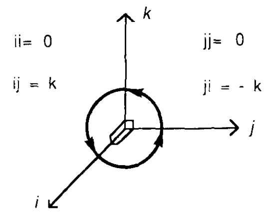


FIGURE 1

The *vector cross product algebra* is defined via the identities below (plus distributivity and scalar linearity):

$$\begin{aligned}
&00 = 0\\
&0i = 0j = 0k = 0 = i0 = j0 = k0\\
&ii = jj = kk = 0\\
&ij = k, \qquad ji = -k\\
&jk = i, \qquad kj = -i\\
&ki = j, \qquad ik = -j.
\end{aligned}$$

It is customary to write cross products via the notation $v \times w$. *Here we simply write* vw.

The cross product algebra is non-associative, as the following example shows:

$$(ii)j = 0j = 0, \qquad i(ij) = ik = -j.$$

This non-associativity leads to the problem: *locate specific values for which associated products are equal.* For example, the equation

$$x(yz) = (xy)z$$

is satisfied by $x = i$, $y = k$, $z = i$:

$$i(ki) = ij = k = -(-k) = -ji = (ik)i.$$

DEFINITION 2.1. Given any collection of variables $X_1, X_2, X_3, \ldots, X_n$, let L and R denote two specific associations of the product

$$X_1 X_2 X_3 \cdots X_n.$$

L and R will be referred to as the *left* and *right* associations for the problem. A solution to the equation $L = R$ in the cross product algebra is said to be *sharp* if *both sides are non-zero, and the values for the variables are chosen from the elements i, j, and k.*

THEOREM A. *Let n be a natural number, and let L and R be any two associations of the product $X_1 X_2 X_3 \cdots X_n$. Then the Four Color Theorem is equivalent to the existence of sharp solutions (See Definition 2.1) to the equation $L = R$ for any n and all choices of L and R.*

Remark. The Four Color Theorem states that every bridgeless plane map can be face colored with four colors so that no two faces sharing an edge receive the same color (a proper coloring). See [3], [6]. The Four Color Theorem was first conjectured in 1852. It was proved by Appel, Haken, and Koch in 1976. See [1], [2] for an account of their work.

One of the fascinations of the map coloring problem resides in the remarkable number of reformulations to which the problem is susceptible. See [6, p. 129] for a closely related reformulation due to Mycielski, and for other reformulations involving products. See also the monthly article [5].

In order to prove Theorem A we will discuss constructions related to coloring the edges of a map. I shall begin, in this section, by explaining enough background to give an algorithm for finding sharp solutions. Proofs are deferred to Section 3.

A sharp solution to $L = R$ will be found through coloring a map

$$M(L, R) = T(L) \# T(R^*),$$

where $T(L)$ and $T(R^*)$ are two planar trees tied together (tieing denoted by $\#$) at their branches and roots as described below.

First the tree construction: $T(L)$ is the tree associated to the parenthesis structure of the formal product L, obtained as shown in Fig. 2.

The branches of $T(L)$ terminate in the variables of L. $T(L)$ has a single root, and the product structure in L is encoded in $T(L)$ so that each

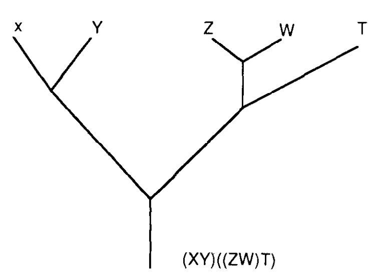
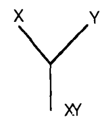


FIG. 2. The tree corresponding to an associated product.



SCHEME 1

individual multiplication in L has the diagrammatic form as shown in scheme 1.

Thus if one labels the edges of $T(L)$ with the partial products, then the root receives the full product of the variables.

By R^*, I mean the mirror image of the expression R, obtained by writing from right to left. Thus

$$[(x(y(zw)))\,t]^* = t(((wz)\,y)\,x).$$

Figure 3 illustrates how to create the planar map $T(L) \# T(R^*)$ by tieing the two trees together. Note that lines terminating in the same variable are connected with one another.

Now disregard signs, and consider a *commutative and associative* algebraic system with elements E, I, J, K where

$$EE = E$$

$$EI = I, \qquad EJ = J, \qquad EK = K$$

$$II = JJ = KK = E$$

$$IJ = K$$

(otherwise known as the Klein four group).

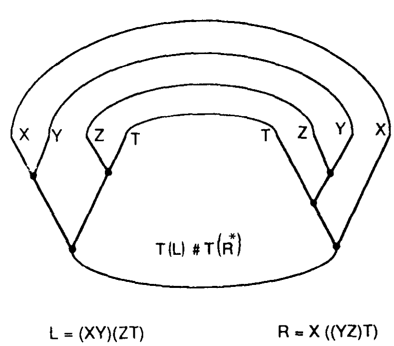


FIG. 3. $M(L, R) = T(L) \# T(R^*)$.

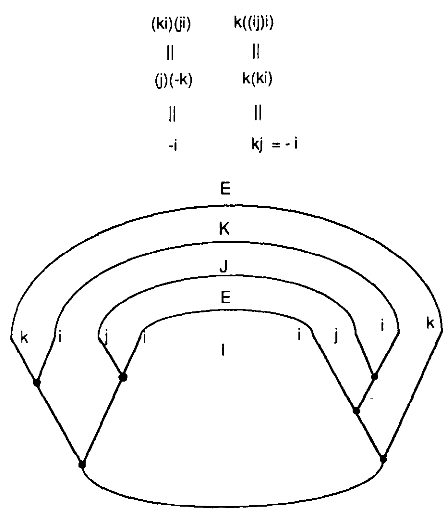
Color the faces of the map $M = M(L, R)$ with the colors E, I, J, K so that adjacent faces are colored differently (a proper coloring). Then color the *edges* of M by associating to each edge the product in the Klein four group of the colors on its two adjoining faces. The result is a coloring of the edges of M so that three distinct colors are incident to each vertex. View Fig. 4. (This equivalence between four-coloring the faces and three-coloring the edges is well known.)

Associate to each capital I, J, K labelling an edge in M the corresponding lower case i, j, k representative in the cross product algebra. In particular, assign to each variable X the element i, j, or k that labels its edge in $M(L, R)$. *This assignment of values gives a sharp solution to* $L = R$.

It is obvious from our formalism that the above method yields a non-zero solution either to the equation $L = +R$, or to the equation $L = -R$. That it actually gives the correct sign (that is, a solution to $L = +R$) is explained in Section 3.

Note that this graphical method can be used to enumerate all possible sharp solutions of the algebra problem, since these are in one-to-one correspondence with the edge-colorings of the map $M(L, R)$. In the example shown in Fig. 4 we see that there are six solutions. These correspond to permutations of i, j, k in the above solution.

Remark. Other than the subtle matter of the sign, what is needed to complete the proof of Theorem A is a theorem of Hassler Whitney [9, p. 379. Theorem II and remarks that follow, also the comments on



$(ki)(ji) = k((ij)i)$

FIG. 4. Map coloring yields sharp solution.

p. 389 of the same paper]. from which it follows that it is sufficient to prove the Four Color Theorem for a class of maps each of which has the form $M(L, R)$ (of two tied trees). The "two tied trees" form is in fact the dual of the polygonal form of Whitney.

Remark. In terms of ordinary vector algebra we have the difference formula:

$$(ab)\,c - a(bc) = (a \cdot b)\,c - a(b \cdot c),$$

where xy denotes the vector cross product of x and y, and $x \cdot y$ denotes the standard dot product of vectors. This identity can be used repeatedly to give the associativity defect $L - R$ for any pair of associations of an n-fold product. This gives an alternate (albeit more complicated) route to studying the equation $L - R = 0$.

III. Formations and Imaginary Values

I begin with a version of edge-three colorings for trivalent planar graphs.

Definition 3.1. A *formation* F is a configuration of simple closed curves in the plane with the following properties:

(1) F is a union of two disjoint collections of curves. That is, $F = R \cup B$ where R and B are each disjoint unions of simple closed curves in the plane.

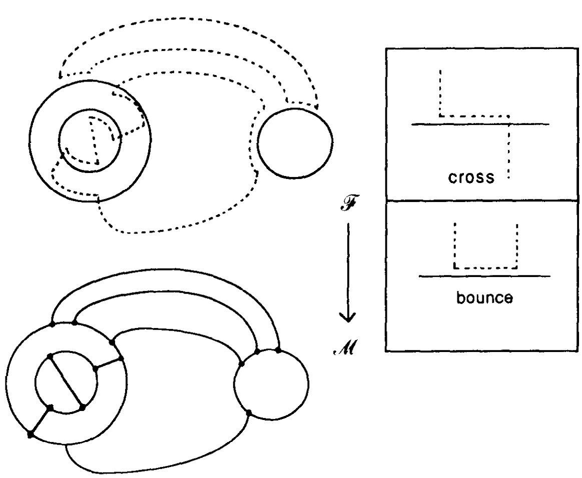


Figure 5

(2) R is called the set of *red curves* and B is called the set of *blue curves*. Curves of a given color do not intersect one another. Interactions between red and blue curves are allowed as shown in Fig. 5. Curves of different colors can share a segment. In this sharing, one curve may cross (*cross*) another, or the two curves simply touch without crossing (*bounce*).

Figure 5 illustrates a formation and the allowed interactions of curves. In this figure, and in illustrations to follow, *red* curves will be denoted by solid arcs (—), and *blue* curves by dotted arcs (---).

Figure 5 illustrates a formation in which there are three red curves and two blue curves. This figure also illustrates our interaction conventions and the relationship of a formation to a plane map. In the figure, there is given a formation F and its associated map M. M is obtained by replacing all arcs in the formation by solid color arcs. This entails superimposing nearby parallel arcs in the formation (They are drawn slightly separated for graphic clarity.).

I shall refer to the map M *underlying* the formation F. A formation F for a map M gives a three coloring of the edges of M with three distinct colors at each vertex. (red = —, blue = --- purple = ═).

Conversely, *any three coloring of the edges with three colors at each vertex gives a formation.* To see this, factorize purple as *purple* = *red* × *blue* on each purple edge, and trace the corresponding red and blue circuits. The circuits so constructed give a formation.

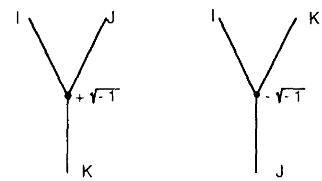
Since four coloring the faces of a trivalent plane map is equivalent to three coloring the edges, the above remarks show that *the Four Color Theorem is equivalent to the existence of formations for planar trivalent maps.* The terminology "formation" is due to G. Spencer-Brown [7].

Notation. In discussing formations it is convenient to use the colors red (r), blue (b), and purple (p). In working in the relationship with the cross-product algebra it is useful to use the colors I, J, K. Henceforth I will correspond to r, J to b, and K to p.

I now introduce a device of Roger Penrose [4], labelling each vertex of an edge coloring with either $+\sqrt{-1}$ or $-\sqrt{-1}$. A vertex is labelled with $+\sqrt{-1}$ if the clockwise cyclic order at the vertex has the form IJK (up to cyclic permutation), and it is labelled $-\sqrt{-1}$ if the order has the form IKJ. (See scheme 2.)

In the following, a plane graph denotes a graph together with a given embedding of the graph in the plane.

THEOREM 3.2. *If C is an edge coloring of a trivalent plane graph G, then the product, $P(C)$, of the imaginary values associated to the vertices of G is equal to* 1: $P(C)=1$.



SCHEME 2. Given an edge coloring C of the graph G, *let* $P(C)$ *denote the product of all the imaginary values at the vertices of* G (according to the assignment above).

Remark. This theorem is stated without proof in [4]. The result is false if we lift the restriction of planarity. Thus in Fig. 6 there is shown a graph embedded with edge crossings in the plane, and a coloring C so that $P(C) = -1$. Note that Theorem 3.2 is equivalent to the statement that the number of vertices of type $+\sqrt{-1}$ is congruent modulo 4 to the number of vertices of type $-\sqrt{-1}$ in an edge coloring of a cubic plane map. In this form, it has been known for some time. See [8] for this and related results. Our proof, using imaginary values and formations, is particularly simple. And it shows how the result fits in with the non-planar context.

Proof of Theorem 3.2. Let F be the formation associated to the edge coloring C. Note that in a formation, different color curves interact via a *bounce* or a *crossing* (see Fig. 7). Each interaction uses two vertices. Call the product of the imaginary values for these vertices the contribution of

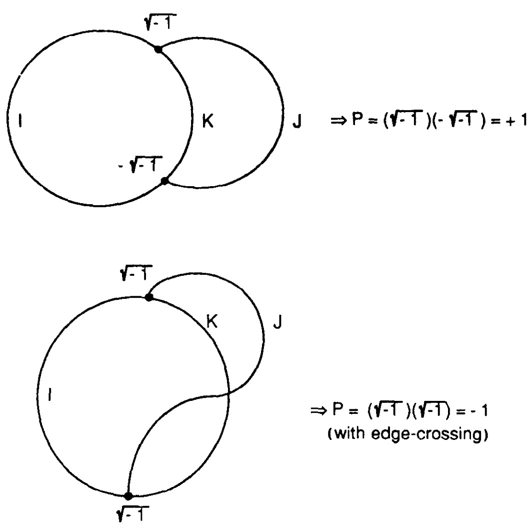


FIGURE 6

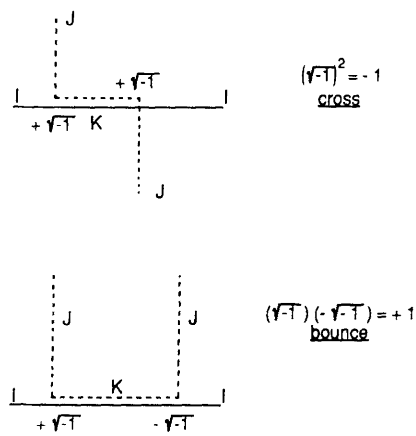


FIGURE 7

the interaction. Then (as shown in Fig. 7) each bounce contributes $+1$, while each crossing contributes -1.

The formation consists in a collection of simple closed curves in the plane. By the Jordan curve theorem *there are an even number of crossings among these curves.* Hence $P(C)$ has the value of -1 raised to an even power. This completes the proof.

We are now in a position to prove that an equation $L = R$ can be solved, including the sign, by the procedure of Section 2.

THEOREM 3.3. *Let L and R denote two associations of a product of a finite number of distinct variables. Then there exists a sharp solution (in the sense of Section 2) of the equation $L = R$, assuming the Four Color Theorem.*

Proof. Let M denote the map

$$M = T(L) \# T(R^*)$$

in the notation of Section 2. Suppose that $T(L)$ has n vertices so that $T(R)$ (hence $T(R^*)$) also has n vertices. Assume that M has been edge-three colored, and that this coloring is expressed in terms of i, j, and k. Label the vertices of M with $+\sqrt{-1}$ or $-\sqrt{-1}$ according to the convention discussed in this section. By Theorem 3.2, the product of all these imaginary values is equal to 1. Thus we may write

$$Z\bar{W} = 1,$$

where Z is the product corresponding to the tree $T(L)$ and W *is the product corresponding to the tree* $T(R)$. Here I use the crucial (and obvious) fact that *the imaginary values assigned to the mirror image tree are each the conjugates of the imaginary values assigned to the tree.* Thus we conclude that $Z = W$.

Let e denote the sign ($+1$ or -1) obtained by multiplying the assignment of i, j, and k corresponding to the edge coloring of M for L. Let e' be the corresponding sign for R. Then

$$Z = ((\sqrt{-1})^n)\, e$$
$$W = ((\sqrt{-1})^n)\, e'.$$

Hence $e = e'$ since $Z = W$. This completes the proof.

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