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Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping

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Abstract

In this paper, we are concerned with the oscillation of third order nonlinear delay differential equations of the form

$$\big(r_2(t)\big(r_1(t)y'\big)'\big)' + p(t)y' + q(t)f\big(y\big(g(t)\big)\big) = 0.$$

By using a generalized Riccati transformation and integral averaging technique, we establish some new sufficient conditions which insure that any solution of this equation oscillates or converges to zero. In particular, several examples are given to illustrate the importance of our results.

Keywords: Oscillation; Third order; Delay differential equation; Asymptotic behaviour; Riccati transformation and integral averaging

1. Introduction

In this paper we consider nonlinear third order functional differential equations of the form

$$\big(r_2(t)\big(r_1(t)y'\big)'\big)' + p(t)y' + q(t)f\big(y\big(g(t)\big)\big) = 0, \tag{1.1}$$

where $q \in C(I,\mathbb{R})$, $r_2, p \in C^1(I,\mathbb{R})$, $r_1 \in C^2(I,\mathbb{R})$, $I = [a,\infty) \subset \mathbb{R}$, $a \geqslant 0$ is a constant such that $r_1 > 0$, $r_2 > 0$, $p(t) \geqslant 0$, $q(t) > 0$, $g \in C^1(I,\mathbb{R})$ satisfies $0 < g(t) \leqslant t$, $g'(t) \geqslant 0$ and $g(t) \to \infty$ as $t \to \infty$ and $f \in C(\mathbb{R},\mathbb{R})$ satisfies $f(u)/u \geqslant K > 0$ for $u \neq 0$.

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We restrict our attention to those solutions of Eq. (1.1) which exist on I and satisfy the condition

$$\sup\{|y(t)|:\ T \leqslant t < \infty\} > 0 \quad \text{for any } T \in I.$$

Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if it has an oscillatory solution.

In recent years, the oscillation theory and asymptotic behaviour of differential equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have been written, see for example the monographs [1,10,11,17,35].

Determining oscillation criteria for particular second order differential equations has received a great deal of attention in the last few years. Compared to second order differential equations, the study of oscillation and asymptotic behaviour of third order differential equations has received considerably less attention in the literature. In the ordinary case for some recent results on third order equations the reader can refer to Bartusek [3], Cecchi and Marini [4,5], Parhi and Das [19–21,23,24], Skerlik [32–34], Tiryaki and Yaman [38].

One of the more important and useful methods of studying oscillation of nonlinear equations is the integral averaging technique, which employs weighted averages of coefficients [12,28]. As far as we know Eq. (1.1) has never been the subject of systematic investigations in this direction except for Saker's paper [31]. Recently, by using the well-known Kiguradze's lemma and the Riccati transformation Saker [31] obtained some new oscillation results of the special case of (1.1) with $p(t) \equiv 0$ and $g(t) = t - \sigma$, $\sigma \geqslant 0$ is a constant.

It is interesting to note that there are third order delay differential equations which have only oscillatory solutions or have both oscillatory and nonoscillatory solutions. For example, $y''' + 2y' + y(t - \frac{\pi}{2}) = 0$ admit an oscillatory solution $y_1(t) = \sin t$ and a nonoscillatory solution $y_2(t) = e^{\lambda t}$, where $\lambda < 0$ is a root of the characteristic equation of this equation. On the other hand, all solutions of $y''' + y(t - \tau) = 0$, $\tau > 0$, are oscillatory if and only if $\tau e > 3$ [16]. But the corresponding ordinary differential equation $y''' + y = 0$ admits a nonoscillatory solution $y_1(t) = e^{-t}$ and oscillatory solutions $y_2(t) = e^{t/2} \cos \frac{\sqrt{3}}{2} t$ and $y_3(t) = e^{t/2} \sin \frac{\sqrt{3}}{2} t$. In the literature there are some papers and books, for example Agarwal et al. [1], Dzurina [8,9], Erbe et al. [10], Grace and Lalli [6], Gyori and Ladas [11], Kartsatos and Manougian [13], Kusano and Onose [14,15], Ladde et al. [17], Parhi and Das [22,26], Parhi and Padhi [25,27], Saker [31], Tiryaki and Yaman [37] which deal with the oscillatory and asymptotic behaviour of solutions of functional differential equations. In this paper, by using a generalized Riccati transformation and an integral averaging technique, we establish some new sufficient conditions which insure that every solution of (1.1) oscillates or converges to zero. In fact, by choosing appropriate functions, we shall present several easily verifiable oscillation criteria. The results of this paper improve, extend and complement a number of existing results. Our work is different from theirs in the sense that either the conditions assumed by them are not satisfied by the equations we consider or the type of equations they consider are different from ours. The results we obtain here are different from those concerning ordinary differential equations of third order due to the presence of the delay.

The paper is organized as follows: In Section 2, we shall present some lemmas which are useful in the proof of our main results. In Section 3, we establish sufficient conditions and also condition of Philos-type for oscillation of Eq. (1.1). In Section 4, some examples are considered to illustrate our main results.

2. Some preliminary lemmas

In this section we state and prove some lemmas which we will use in the proof of our main results.

For the sake of brevity, we define

$$L_0 y(t) = y(t), \qquad L_i y(t) = r_i(t)(L_{i-1} y(t))', \quad i = 1, 2,$$
$$L_3 y(t) = (L_2 y(t))' \quad \text{for } t \in I.$$

So Eq. (1.1) can be written as

$$L_3 y(t) + p(t)y' + q(t) f(y(g(t))) = 0.$$

Remark 1. If y is a solution of (1.1), then $z = -y$ is a solution of the equation

$$L_3 z(t) + p(t)z' + q(t) f^*(z(g(t))) = 0,$$

where $f^*(z) = -f(-z)$ and $zf^*(z) > 0$ for $z \neq 0$. Thus, concerning nonoscillatory solutions of (1.1) we can restrict our attention only to the positive ones.

Definition 1. Let y be a solution of (1.1). We say that the solution y has property V_2 on $[T, \infty)$, $T \geqslant a$, if and only if

$$L_0 y(t) L_k y(t) > 0, \quad k = 0, 1, 2; \qquad L_0 y(t) L_3 y(t) \leqslant 0$$

for every $t \in [T, \infty)$.

Define the functions

$$R_1(t, T) = \int_T^t \frac{ds}{r_1(s)}, \qquad R_2(t, T) = \int_T^t \frac{ds}{r_2(s)}, \qquad R_{12}(t, T) = \int_T^t \frac{R_2(s, T)}{r_1(s)}\, ds,$$

$a \leqslant T \leqslant t < \infty$.

We assume that

$$R_1(t, a) \to \infty \quad \text{as } t \to \infty, \tag{2.1}$$

and

$$R_2(t, a) \to \infty \quad \text{as } t \to \infty. \tag{2.2}$$

Lemma 1. *Suppose that*

$$(r_2(t)z')' + \frac{p(t)}{r_1(t)} z = 0 \tag{2.3}$$

is nonoscillatory. If y is a nonoscillatory solution of (1.1) *on* $[T, \infty)$*,* $T \geqslant a$*, then there exists a* $t_0 \in [T, \infty)$ *such that* $y(t) L_1 y(t) > 0$ *or* $y(t) L_1 y(t) < 0$ *for* $t \geqslant t_0$*.*

Proof. Suppose that y is a nonoscillatory solution of (1.1) on $[T, \infty)$. Without loss of generality, we may take $y(t) > 0$ and $y(g(t)) > 0$, $t \geqslant t_0 \geqslant T$. Clearly, $x(t) = -L_1 y(t)$ is a solution of the second order nonhomogeneous delay differential equation

$$(r_2(t)x')' + \frac{p(t)}{r_1(t)} x = q(t) f(y(g(t))), \quad t \geqslant t_0. \tag{2.4}$$

We claim that, all solutions of (2.4) are nonoscillatory. Let z be a solution of (2.3), where r_2 and $p/r_1 \in C([t_0,\infty),\mathbb{R})$ such that $r_2(t) > 0$ and $p(t)/r_1(t) \geqslant 0$. Let $z(t) > 0$ for $t \geqslant t_0$. The case when $z(t)$ is ultimately negative can similarly be deal with. If possible, let x be a oscillatory solution of (2.4) with consecutive zeros at b and c ($t_0 < b < c$) such that $x'(b) \geqslant 0$ and $x'(c) \leqslant 0$. Now integrating

$$\big(r_2(t)x'(t)z(t) - r_2(t)x(t)z'(t)\big)' = z(t)q(t)f\big(y\big(g(t)\big)\big)$$

from b and c, we get a contradiction. This contradiction completes the proof. □

Lemma 2. *Let assumption* (2.2) *hold and y be a nonoscillatory solution of* (1.1) *such that $y(t)L_1y(t) \geqslant 0$ for every $t \geqslant T \geqslant a$. Then y has property V_2 for all large t.*

The proof of this lemma proceeds along the lines of that of [32, Lemma 1] given for the ordinary case and hence is omitted.

Lemma 3. *Let y be a solution of* (1.1). *If y has property V_2 for every large t, then there exists $t_1 \geqslant T$ such that*

$$L_1y\big(g(t)\big) \geqslant R_2\big(g(t),t_1\big)L_2y(t), \quad t \geqslant t_1. \tag{2.5}$$

Proof. Let y be a solution of (1.1) which has property V_2 for every large t. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$, $t \geqslant T$. Hence $L_0y(g(t)) > 0$, $L_1y(t) > 0$, $L_2y(t) > 0$ and $L_3y(t) \leqslant 0$ for $t \geqslant t_1 \geqslant T$. On the other hand, using the fact that $L_2y(t)$ is nonincreasing, we see that

$$L_1y\big(g(t)\big) \geqslant \int_{t_1}^{g(t)} \big(L_1y(s)\big)'\,ds = \int_{t_1}^{g(t)} \frac{1}{r_2(s)}L_2y(s)\,ds \geqslant L_2y\big(g(t)\big)R_2\big(g(t),t_1\big). \tag{2.6}$$

Since $L_3y(t) \leqslant 0$, we get $L_2y(g(t)) \geqslant L_2y(t)$. This and (2.6) imply that for sufficiently large t

$$L_1y\big(g(t)\big) \geqslant L_2y(t)R_2\big(g(t),t_1\big).$$

Thus the proof is complete. □

3. Main results

In this section we establish some sufficient conditions which guarantee that every solution y of (1.1) oscillates or converges to zero. Throughout this section we will impose the following conditions:

$$\left.\begin{array}{l}\limsup_{t\to\infty}\int_T^t \frac{1}{r_1(s)}\big(\int_s^t \frac{1}{r_2(u)}\big(\int_u^\infty \big(Kq(\tau)-p'(\tau)\big)\,d\tau\big)\,du\big)\,ds = \infty \quad \text{and}\\ \limsup_{t\to\infty}\int_T^t \frac{1}{r_1(s)}\big(\int_s^t \frac{p(u)}{r_2(u)}\,du\big)\,ds < \infty\end{array}\right\} \tag{3.1}$$

or

$$\limsup_{t\to\infty}\int_T^t \big(Kq(s)-p'(s)\big)\,ds = \infty, \tag{3.2}$$

where $Kq(t) - p'(t) \geqslant 0$ for $t \in I$ and not identically zero in any subinterval of I.

Theorem 1. *Assume that* (2.1), (2.2), (3.1) (*or* (3.2)) *hold, and Eq.* (2.3) *is nonoscillatory. If there exists a differentiable positive function* ρ *such that*

$$\limsup_{t\to\infty} \int_T^t \left[K\rho(s)q(s) - \frac{r_1(g(s))(\rho'(s)r_1(s) - \rho(s)p(s)R_2(g(s),T))^2}{4\rho(s)R_2(g(s),T)g'(s)r_1^2(s)} \right] ds = \infty \quad \text{for every } T, \tag{3.3}$$

then any solution y *of Eq.* (1.1) *is oscillatory or satisfies* $y(t) \to 0$ *as* $t \to \infty$.

Proof. Let y be a nonoscillatory solution of (1.1) on $[T,\infty)$, $T \geqslant a$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geqslant t_0 \geqslant T$. From Lemma 1 it follows that $L_1 y(t) > 0$ or $L_1 y(t) < 0$ for $t \geqslant t_1 \geqslant t_0$. If $L_1 y(t) > 0$ for $t \geqslant t_1$, then y has property V_2 for large t from Lemma 2. We define

$$\omega(t) = \rho(t)\frac{L_2 y(t)}{y(g(t))} \tag{3.4}$$

for $t \geqslant t_1$.

Then, $\omega(t) > 0$. By (1.1) and Lemma 3, we have

$$\omega'(t) \leqslant -K\rho(t)q(t) - \left[\omega^2(t)\left(\frac{R_2(g(t),t_1)g'(t)}{r_1(g(t))\rho(t)} \right) - \omega(t)\left(\frac{\rho'(t)}{\rho(t)} - p(t)\frac{R_2(g(t),t_1)}{r_1(t)} \right) \right] \tag{3.5}$$

and hence

$$\omega'(t) < -K\rho(t)q(t) + \frac{r_1(g(t))(r_1(t)\rho'(t) - \rho(t)p(t)R_2(g(t),t_1))^2}{4\rho(t)R_2(g(t),t_1)g'(t)r_1^2(t)}. \tag{3.6}$$

Integrating (3.6), we have, for $t \geqslant t_1$,

$$\int_{t_1}^t \left[K\rho(s)q(s) - \frac{r_1(g(s))(r_1(s)\rho'(s) - \rho(s)p(s)R_2(g(s),t_1))^2}{4\rho(s)R_2(g(s),t_1)g'(s)r_1^2(s)} \right] ds \leqslant \omega(t_1) - \omega(t) \leqslant \omega(t_1)$$

for large t, which contradicts (3.3) for $T = t_1$.

Let $y(t) > 0$, $L_1 y(t) < 0$, $t \geqslant t_1$. Assume that (3.1) holds. We consider the function $L_2 y$. The case $L_2 y(t) \leqslant 0$ cannot hold for all large t, say $t \geqslant t_2 \geqslant t_1$, since by integration of inequality $y'(t) \leqslant L_1 y(t_2)/r_1(t)$, $t \geqslant t_2$, we obtain from (2.1) $y(t) < 0$ for all large t, a contradiction.

Let $y(t) > 0$, $L_1 y(t) < 0$, $L_2 y(t) \geqslant 0$ for all large t, say $t \geqslant t_3$ and so $y'(t) < 0$ for $t \geqslant t_3$. If $\lim_{t\to\infty} y(t) = \lambda > 0$, then integrating (1.1) from s to t, we obtain

$$L_2 y(s) + p(s)y(s) = L_2 y(t) + p(t)y(t) + \int_s^t y(u)\left[\frac{f(y(g(u)))}{y(u)} q(u) - p'(u) \right] du \geqslant \lambda \int_s^\infty \left[Kq(u) - p'(u) \right] du$$

and hence

$$y(t) \leqslant y(t_3) + y(t_3)\int_{t_3}^{t} \frac{1}{r_1(s)}\left(\int_{s}^{t} \frac{p(u)}{r_2(u)}\,du\right)ds$$
$$-\lambda \int_{t_3}^{t} \frac{1}{r_1(s)}\left(\int_{s}^{t} \frac{1}{r_2(u)}\left(\int_{u}^{\infty}\big(Kq(\tau) - p'(\tau)\big)\,d\tau\right)du\right)ds,$$

so we obtain from (3.1), $y(t) < 0$ for all large t, a contradiction. Hence $\lim_{t\to\infty} y(t) = 0$.

Finally, let $y(t) > 0$, $L_1y(t) < 0$, $t \geqslant t_4 \geqslant t_1$, and suppose L_2y changes sign for arbitrarily large t. Suppose that $\lim_{t\to\infty} y(t) = \lambda > 0$ and L_2y has a t_n sequence of zeros such that $t_n \to \infty$ as $n \to \infty$. By integrating (1.1) from s to t_n, we obtain

$$L_2y(s) + p(s)y(s) = L_2y(t_n) + p(t_n)y(t_n) + \int_{s}^{t_n} y(u)\left[\frac{f(y(g(u)))}{y(u)}q(u) - p'(u)\right]du$$
$$\geqslant \lambda \int_{s}^{\infty}\big[Kq(u) - p'(u)\big]\,du$$

and hence

$$y(t_n) \leqslant y(t_4) + y(t_4)\int_{t_4}^{t_n} \frac{1}{r_1(s)}\left(\int_{s}^{t_n} \frac{p(u)}{r_2(u)}\,du\right)ds$$
$$-\lambda \int_{t_4}^{t_n} \frac{1}{r_1(s)}\left(\int_{s}^{t_n} \frac{1}{r_2(u)}\left(\int_{u}^{\infty}\big(Kq(\tau) - p'(\tau)\big)\,d\tau\right)du\right)ds.$$

Consequently, $y(t_n) < 0$ for t_n, a contradiction. Hence $\lim_{t\to\infty} y(t) = 0$.

On the other hand, assume that (3.2) holds. If $L_1y(t) < 0$ for $t \geqslant t_1$, then $y'(t) < 0$ for $t \geqslant t_1$. If $\lim_{t\to\infty} y(t) = \lambda > 0$, then by integrating (1.1) from t_1 to t $(t_1 < t)$, we obtain

$$L_2y(t) \leqslant H_1 - \int_{t_1}^{t} y(s)\left[\frac{f(y(g(s)))}{y(s)}q(s) - p'(s)\right]ds \leqslant H_1 - \lambda\int_{t_1}^{t}\big[Kq(s) - p'(s)\big]\,ds,$$

where H_1 is a constant.

From (3.2) and the above inequality, there exists $\mu < 0$ such that $L_2y(t) < \mu$ for large t and so

$$y(t) \leqslant y(t_1) + H_2R_1(t, t_1) + \mu R_{12}(t, t_1),$$

where $H_2 = L_1y(t_1) < 0$. Furthermore $R_{12}(t, t_1) \to \infty$ as $t \to \infty$ from (2.1) and (2.2). Consequently, $y(t) < 0$ for large t, a contradiction. Hence $\lim_{t\to\infty} y(t) = 0$.

This completes the proof of theorem. □

From Theorem 1, we have the following result for the equation:

$$L_3y(t) + p(t)y' + q(t)y^{\gamma}\big(g(t)\big) = 0, \tag{3.7}$$

where $\gamma > 0$ is a quotient of odd integers. We will impose the following conditions for (3.7):

$$\left.\begin{aligned}&\limsup_{t\to\infty}\int_T^t \frac{1}{r_1(s)}\Big(\int_s^t \frac{1}{r_2(u)}\Big(\int_u^\infty \big(K_1q(\tau)-p'(\tau)\big)\,d\tau\Big)\,du\Big)\,ds=\infty \quad \text{and}\\ &\limsup_{t\to\infty}\int_T^t \frac{1}{r_1(s)}\Big(\int_s^t \frac{p(u)}{r_2(u)}\,du\Big)\,ds<\infty\end{aligned}\right\} \tag{3.8}$$

or

$$\limsup_{t\to\infty}\int_T^t \big(K_1q(s)-p'(s)\big)\,ds=\infty, \tag{3.9}$$

where $K_1q(t)-p'(t)\geqslant 0$ for every $K_1>0$, $t\in I$ and not identically zero in any subinterval of I.

Corollary 1. *Assume that* $\gamma\geqslant 1$, (2.1)*,* (2.2)*,* (3.8) (*or* (3.9)) *hold and Eq.* (2.3) *is nonoscillatory. If there exists a differentiable positive function* ρ *such that*

$$\limsup_{t\to\infty}\int_T^t\left[K_1\rho(s)q(s)-\frac{r_1(g(s))(\rho'(s)r_1(s)-\rho(s)p(s)R_2(g(s),T))^2}{4\rho(s)R_2(g(s),T)g'(s)r_1^2(s)}\right]ds=\infty \tag{3.10}$$

for every T, $K_1>0$, *then any solution* y *of Eq.* (3.7) *is oscillatory or satisfies* $y(t)\to 0$ *as* $t\to\infty$.

Corollary 2. *Assume that* $0<\gamma<1$, (2.1)*,* (2.2)*,* (3.8) (*or* (3.9)) *and Eq.* (2.3) *is nonoscillatory. If there exists a differentiable positive function* ρ *such that* (3.10) *holds, then any bounded solution* y *of Eq.* (3.7) *is oscillatory or satisfies* $y(t)\to 0$ *as* $t\to\infty$.

Remark 2. When $p'(t)\leqslant 0$, we can take

$$\left.\begin{aligned}&\limsup_{t\to\infty}\int_T^t \frac{1}{r_1(s)}\Big(\int_s^t \frac{1}{r_2(u)}\Big(\int_u^\infty q(\tau)\,d\tau\Big)\,du\Big)\,ds=\infty \quad \text{and}\\ &\limsup_{t\to\infty}\int_T^t \frac{1}{r_1(s)}\Big(\int_s^t \frac{p(u)}{r_2(u)}\,du\Big)\,ds<\infty\end{aligned}\right\}$$

or

$$\limsup_{t\to\infty}\int_T^t q(s)\,ds=\infty$$

to replace (3.8) or (3.9), respectively, in Corollary 1 and Corollary 2.

Next, we present some new oscillation results for Eq. (1.1), by using an integral averages condition of Philos-type. Following Philos [28], we introduce a class of functions $\Re$. Let

$$D_0=\big\{(t,s):\ t>s\geqslant T\big\} \quad\text{and}\quad D=\big\{(t,s):\ t\geqslant s\geqslant T\big\}.$$

The function $H\in C(D,\mathbb{R})$ is said to belong to the class $\Re$ if

(i) $H(t,t)=0$ for $t\geqslant T$; $H(t,s)>0$ for $(t,s)\in D_0$;
(ii) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable such that

$$-\frac{\partial H(t,s)}{\partial s}=h(t,s)\sqrt{H(t,s)}\quad\text{for all } (t,s)\in D_0.$$

Theorem 2. *Assume that* (2.1), (2.2), (3.1) (*or* (3.2)) *hold, and Eq.* (2.3) *is nonoscillatory. If there exist functions $H \in \Re$ and $\rho \in C^1([T,\infty))$ such that $\rho(t) > 0$ and*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[K\rho(s)H(t,s)q(s) - \frac{r_1(g(s))\rho(s)Q^2(t,s)}{4R_2(g(s),T)g'(s)} \right] ds = \infty, \tag{3.11}$$

where

$$Q(t,s) = h(t,s) - \sqrt{H(t,s)} \left(\frac{\rho'(s)}{\rho(s)} - p(s)\frac{R_2(g(s),T)}{r_1(s)} \right) \quad \textit{for every } T, \tag{3.12}$$

then any solution y of Eq. (1.1) *is oscillatory or satisfies $y(t) \to 0$ as $t \to \infty$.*

Proof. Let y be a nonoscillatory solution of (1.1) on $[T,\infty)$, $T \geqslant a$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geqslant t_0 \geqslant T$. From Lemma 1 it follows that $L_1y(t) > 0$ or $L_1y(t) < 0$ for $t \geqslant t_1 \geqslant t_0$. If $L_1y(t) > 0$ for $t \geqslant t_1$, then y has property V_2 for large t from Lemma 2.

Again, defining $\omega(t)$ as in (3.4), we obtain (3.5). Let us set

$$\gamma(t) = \frac{\rho'(t)}{\rho(t)} - p(t)\frac{R_2(g(t),t_1)}{r_1(t)} \quad \text{and} \quad W(t) = \frac{R_2(g(t),t_1)g'(t)}{\rho(t)r_1(g(t))}.$$

Then from (3.5), we get

$$\begin{aligned}
&\int_{t_1}^t KH(t,s)\rho(s)q(s)\,ds \\
&\quad\leqslant \int_{t_1}^t H(t,s)\big[-\omega'(s) + \gamma(s)\omega(s) - W(s)\omega^2(s)\big]\,ds \\
&\quad= -H(t,s)\omega(s)\big|_{t_1}^t + \int_{t_1}^t \left\{ \frac{\partial H(t,s)}{\partial s}\omega(s) + H(t,s)\big[\gamma(s)\omega(s) - W(s)\omega^2(s)\big] \right\} ds \\
&\quad= H(t,t_1)\omega(t_1) - \int_{t_1}^t \big\{\omega^2(s)W(s)H(t,s) + \omega(s)\big(h(t,s)\sqrt{H(t,s)} - H(t,s)\gamma(s)\big)\big\}\,ds
\end{aligned} \tag{3.13}$$

and hence

$$\leqslant H(t,t_1)\omega(t_1) + \frac{1}{4}\int_{t_1}^t \frac{Q^2(t,s)}{W(s)}\,ds. \tag{3.14}$$

It follows that

$$\frac{1}{H(t,t_1)} \int_{t_1}^t \left(K\rho(s)H(t,s)q(s) - \frac{Q^2(t,s)}{4W(s)} \right) ds \leqslant \omega(t_1). \tag{3.15}$$

This contradicts (3.11).

The rest of the proof is the same as in Theorem 1, and hence is omitted. □

The following two results provide alternative oscillation criteria when (3.11) is difficult to verify. The notation of Theorem 2 and its proof will be used.

Theorem 3. *Let all the assumptions, except* (3.11)*, of Theorem* 2 *hold. Further, for every* T*, let*

$$0 < \inf_{s \geqslant T} \left[\liminf_{t \to \infty} \frac{H(t,s)}{H(t,T)} \right] \leqslant \infty, \tag{3.16}$$

and

$$\limsup_{t \to \infty} \frac{1}{H(t,T)} \int_T^t \frac{Q^2(t,s)}{W(s)}\, ds < \infty. \tag{3.17}$$

Let also $\Psi \in C([T,\infty),\mathbb{R})$ *be such that for* $t \geqslant T$*,*

$$\limsup_{t \to \infty} \int_T^t \Psi_+^2(s) W(s)\, ds = \infty, \tag{3.18}$$

where $\Psi_+(s) = \max\{\Psi(t), 0\}$ *and*

$$\limsup_{t \to \infty} \frac{1}{H(t,T)} \int_T^t \left[K\rho(s)H(t,s)q(s) - \frac{Q^2(t,s)}{4W(s)} \right] ds \geqslant \sup_{t \geqslant T} \Psi(t). \tag{3.19}$$

Then, any solution y *of Eq.* (1.1) *is either oscillatory or satisfies* $y(t) \to 0$ *as* $t \to \infty$*.*

Proof. As in the proof of Theorem 2, we have (3.13). It follows that

$$\begin{aligned}
&\int_{t_1}^t KH(t,s)\rho(s)q(s)\, ds \\
&\quad \leqslant H(t,t_1)\omega(t_1) - \int_{t_1}^t \left[\sqrt{H(t,s)W(s)}\, \omega(s) + \frac{1}{2} \frac{Q(t,s)}{\sqrt{W(s)}} \right]^2 ds + \int_{t_1}^t \frac{Q^2(t,s)}{4W(s)}\, ds
\end{aligned}$$

and hence

$$\begin{aligned}
&\limsup_{t \to \infty} \frac{1}{H(t,t_1)} \int_{t_1}^t \left[KH(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W(s)} \right] ds \\
&\quad \leqslant \omega(t_1) - \liminf_{t \to \infty} \frac{1}{H(t,t_1)} \int_{t_1}^t \left[\sqrt{H(t,s)W(s)}\, \omega(s) + \frac{Q(t,s)}{2\sqrt{W(s)}} \right]^2 ds.
\end{aligned}$$

By (3.19), it follows that

$$\omega(t_1) \geqslant \Psi(t_1) + \liminf_{t \to \infty} \frac{1}{H(t,t_1)} \int_{t_1}^t \left[\sqrt{H(t,s)W(s)}\, \omega(s) + \frac{Q(t,s)}{2\sqrt{W(s)}} \right]^2 ds$$

and hence

$$0 \leqslant \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W(s)}\,\omega(s) + \frac{Q(t,s)}{2\sqrt{W(s)}} \right]^2 ds$$
$$\leqslant \omega(t_1) - \Psi(t_1) < \infty. \tag{3.20}$$

Define the functions α and β by

$$\alpha(t) = \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)W(s)\omega^2(s)\,ds,$$

$$\beta(t) = \frac{1}{H(t,t_1)} \int_{t_1}^{t} \sqrt{H(t,s)}\,Q(t,s)\omega(s)\,ds.$$

Then, it follows from (3.20) that

$$\liminf_{t\to\infty} \big[\alpha(t) + \beta(t)\big] < \infty.$$

Now we claim that

$$\int_{t_1}^{\infty} W(s)\omega^2(s)\,ds < \infty. \tag{3.21}$$

Suppose to the contrary that

$$\int_{t_1}^{\infty} W(s)\omega^2(s)\,ds = \infty. \tag{3.22}$$

By (3.16), there is a positive constant ζ such that

$$\inf_{s\geqslant T} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,T)} \right] > \zeta. \tag{3.23}$$

Let μ be any arbitrary positive number, then by (3.22) there exists $t_2 \geqslant t_1$ such that

$$\int_{t_1}^{\infty} W(s)\omega^2(s)\,ds \geqslant \frac{\mu}{\zeta}, \quad t \geqslant t_2,$$

and therefore, for $t \geqslant t_2$,

$$\alpha(t) = \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) \frac{d}{ds} \left[\int_{t_1}^{s} W(u)\omega^2(u)\,du \right] ds$$
$$= \frac{1}{H(t,t_1)} \int_{t_1}^{t} -\frac{\partial H(t,s)}{\partial s} \left[\int_{t_1}^{s} W(u)\omega^2(u)\,du \right] ds$$
$$\geqslant \frac{1}{H(t,t_1)} \int_{t_2}^{t} -\frac{\partial H(t,s)}{\partial s} \left[\int_{t_1}^{s} W(u)\omega^2(u)\,du \right] ds$$

$$\geqslant \frac{\mu}{\zeta}\frac{1}{H(t,t_1)}\int_{t_2}^{t} -\frac{\partial H(t,s)}{\partial s}\,ds = \frac{\mu}{\zeta}\frac{H(t,t_2)}{H(t,t_1)}.$$

By (3.23), there exists $t_3 \geqslant t_2$ such that

$$\frac{H(t,t_2)}{H(t,t_1)} \geqslant \zeta, \quad t \geqslant t_3.$$

This implies that $\alpha(t) \geqslant \mu$ for all $t \geqslant t_3$. As μ is arbitrary, we have $\lim_{t\to\infty}\alpha(t) = \infty$.

The remainder of the proof of this case is similar to that of the proofs of similar theorems given in [2,6,7,29,30,36,39] and hence is omitted.

If the case where $y(t) > 0$ and $L_1 y(t) < 0$ holds the proof is similar to that of the proof of Theorem 1 and hence is omitted. Thus, the proof is complete. □

Theorem 4. *Let all the assumptions of Theorem* 3 *hold, except condition* (3.17)*, which is replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)q(s)\,ds < \infty \quad \textit{for every } T.$$

Then, any solution y of Eq. (1.1) *is either oscillatory or satisfies* $y(t) \to 0$ *as* $t \to \infty$.

The proof of Theorem 4 is similar to that of Theorem 3 and hence is omitted.

Remark 3. When $p(t) \equiv 0$ and $g(t) = t - \sigma$, $\sigma \geqslant 0$ a constant, Theorems 1–4 with condition (3.1) reduce to Theorems 1–4 of Saker [31] respectively.

Remark 4. For the choice $H(t,s) = (t-s)^n$ and $h(t,s) = n(t-s)^{\frac{n}{2}-1}$, the Philos-type condition reduces to the Kamenev-type condition. Other choice of H include

$$\begin{aligned}
&H(t,s) = \left(\ln\frac{t}{s}\right)^n, \qquad h(t,s) = \frac{n}{s}\left(\ln\frac{t}{s}\right)^{\frac{n}{2}-1},\\
&H(t,s) = \big((t-d)^3 - (s-d)^3\big)^n, \quad d \text{ is a constant},\\
&h(t,s) = 3n(s-d)^2\big((t-d)^3 - (s-d)^3\big)^{\frac{n}{2}-1},\\
&H(t,s) = \big(\sqrt{t} - \sqrt{s}\big)^n, \qquad h(t,s) = \frac{n}{2\sqrt{s}}\big(\sqrt{t} - \sqrt{s}\big)^{\frac{n}{2}-1},
\end{aligned}$$

and

$$H(t,s) = \big(e^{t-s} - e^{s-t}\big)^n, \qquad h(t,s) = n\big(e^{t-s} + e^{s-t}\big)\big(e^{t-s} - e^{s-t}\big)^{\frac{n}{2}-1}$$

or more generally,

$$H(t,s) = \left(\int_s^t \frac{du}{\theta(u)}\right)^n, \qquad h(t,s) = \frac{n}{\theta(s)}\left(\int_s^t \frac{du}{\theta(u)}\right)^{\frac{n}{2}-1},$$

where $n > 1$, and $\theta \in C([T,\infty),\mathbb{R}^+)$ satisfies $\lim_{t\to\infty}\int_s^t \frac{du}{\theta(u)} = \infty$.

4. Examples

In this section, we give several examples to illustrate our main results.

Example 1. Consider the third order delay differential equation

$$y''' + \frac{1}{4t^2}y' + \left(1 - \frac{1}{4t^2}\right)y\left(t - \frac{3\pi}{2}\right) = 0, \quad t > t - \frac{3\pi}{2} > \frac{1}{2}. \tag{4.1}$$

Here $r_1(t) = 1$, $r_2(t) = 1$, $p(t) = \frac{1}{4t^2}$, $q(t) = 1 - \frac{1}{4t^2}$ and $f(u) = u$ with $K = 1$. It is clear that (2.1), (2.2), and (3.2) (or (3.9)) are satisfied. The Euler equation $z'' + \frac{1}{4t^2}z = 0$ is nonoscillatory. It remains to satisfy condition (3.3).

Now, by choosing $\rho(t) = 1$ we have

$$\begin{aligned}
&\limsup_{t\to\infty} \int_T^t \left[K\rho(s)q(s) - \frac{r_1(g(s))(\rho'(s)r_1(s) - \rho(s)p(s)R_2(g(s),T))^2}{4\rho(s)R_2(g(s),T)g'(s)r_1^2(s)}\right] ds \\
&= \limsup_{t\to\infty} \int_{1/2}^t \left[1 - \frac{1}{4s^2} - \frac{\left(-\frac{1}{4s^2}\left(s - \frac{3\pi}{2} - \frac{1}{2}\right)\right)^2}{4\left(s - \frac{3\pi}{2} - \frac{1}{2}\right)}\right] ds \\
&= \limsup_{t\to\infty} \int_{1/2}^t \left[1 - \frac{1}{4s^2} - \frac{\left(s - \frac{3\pi}{2} - \frac{1}{2}\right)}{64s^4}\right] ds = \infty.
\end{aligned}$$

Consequently condition (3.3) is satisfied. Hence by Theorem 1 (or Corollary 1), any solution y of (4.1) is oscillatory or satisfies $y(t) \to 0$ as $t \to \infty$. An example of such a solution is $y(t) = -\cos t$. Note that (3.1) (and (3.8)) do not hold.

Example 2. Consider the third order delay differential equation

$$y''' + \frac{1}{t^3}y' + \left(\frac{1}{t} + \frac{1}{t^4}\right)y(t - \ln t) = 0, \quad t > t - \ln t > 1. \tag{4.2}$$

By choosing $\rho(t) = 1$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \int_T^t \left[K\rho(s)q(s) - \frac{r_1(g(s))(\rho'(s)r_1(s) - \rho(s)p(s)R_2(g(s),T))^2}{4\rho(s)R_2(g(s),T)g'(s)r_1^2(s)}\right] ds \\
&= \limsup_{t\to\infty} \int_1^t \left[\frac{1}{s} + \frac{1}{s^4} - \frac{\left(-\frac{1}{s^3}(s - \ln s - 1)\right)^2}{4(s - \ln s - 1)\left(1 - \frac{1}{s}\right)}\right] ds \\
&= \limsup_{t\to\infty} \int_1^t \left[\frac{1}{s} + \frac{1}{s^4} - \frac{(s - \ln s - 1)}{64s^5(s-1)}\right] ds \\
&\geqslant \limsup_{t\to\infty} \int_1^t \left[\frac{1}{s} + \frac{1}{s^4} - \frac{1}{64s^5}\right] ds = \infty.
\end{aligned}$$

Condition (3.3) is satisfied. It is easy to check that the other conditions of Theorem 1 (or Corollary 1) are also satisfied. Thus, any solution y of (4.2) is oscillatory or satisfies $y(t) \to 0$ as $t \to \infty$. One such solution of Eq. (4.2) is $y(t) = e^{-t}$. Note that (3.1) (and (3.8)) and (3.2) (and (3.9)) hold.

Example 3. Consider the third order delay differential equation

$$y''' + e^{-2t+2}y' + \frac{1}{e}y(t-1)\big(1 + y^2(t-1)\big) = 0, \quad t > t-1 > 1, \tag{4.3}$$

where $f(u) = u(1+u^2)$ with $K = 1$. Equation $z'' + e^{-2t+2}z = 0$ is nonoscillatory (see [18] and [35, p. 45]). Taking $H(t,s) = [\frac{(t-2)^3}{6} - \frac{(s-2)^3}{6}]^2$, $\rho(t) = 1$ for $t \geqslant s \geqslant 1$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[K\rho(s)H(t,s)q(s) - \frac{r_1(g(s))\rho(s)Q^2(t,s)}{4R_2(g(s),T)g'(s)}\right] ds$$

$$= \limsup_{t\to\infty} \frac{1}{[\frac{(t-2)^3}{6} + \frac{1}{6}]^2} \int_1^t \left\{\left[\frac{(t-2)^3}{6} - \frac{(s-2)^3}{6}\right]^2 \left(\frac{1}{e}\right) - \frac{[(s-2)^2 + (\frac{(t-2)^3}{6} - \frac{(s-2)^3}{6})e^{-2s+2}(s-2)]^2}{4(s-2)}\right\} ds = \infty.$$

Condition (3.11) is satisfied. Clearly, the other conditions of Theorem 2 are also satisfied. Hence any solution y of Eq. (4.3) is either oscillatory or satisfies $y(t) \to 0$ as $t \to \infty$. Observe that $y(t) = e^{-t}$ is a solution of Eq. (4.3). Note that (3.1) and (3.2) hold.

Example 4. Consider the third order delay differential equation

$$(ty')'' + \frac{1}{4t}y' + \left(1 - \frac{1}{t^2}\right)y(t-1)\big(\beta + e^{y(t-1)}\big) = 0, \quad t > t-1 > 1, \tag{4.4}$$

where $\beta > 0$ and $f(u) = u(\beta + e^u)$ with $K = \beta$. Taking $H(t,s) = [\frac{(t-2)^3}{6} - \frac{(s-2)^3}{6}]^2$, $\rho(t) = 1$ for $t \geqslant s \geqslant 1$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[K\rho(s)H(t,s)q(s) - \frac{r_1(g(s))\rho(s)Q^2(t,s)}{4R_2(g(s),T)g'(s)}\right] ds$$

$$= \limsup_{t\to\infty} \frac{1}{[\frac{(t-2)^3}{6} + \frac{1}{6}]^2} \int_1^t \left\{\beta\left[\frac{(t-2)^3}{6} - \frac{(s-2)^3}{6}\right]^2 \left(1 - \frac{1}{s^2}\right) - \frac{[(s-2)^2 + (\frac{(t-2)^3}{6} - \frac{(s-2)^3}{6})\frac{(s-2)}{4s^2}]^2}{4(s-2)}\right\} ds = \infty.$$

It can be easily show that all the conditions of Theorem 2 are satisfied. Thus, any solution y of Eq. (4.4) is either oscillatory or satisfies $y(t) \to 0$ as $t \to \infty$. Note that (3.2) holds, but (3.1) does not.

Example 5. Consider the third order delay differential equation

$$\left(e^{-t}\left(e^{-t}y'\right)'\right)' + \frac{e^{-2t}}{4}y' + \frac{25}{4e^3}y^3(t-1) = 0, \quad t > t-1 > 1. \tag{4.5}$$

By choosing $\rho(t) = 1$, we have

$$\limsup_{t\to\infty} \int_T^t \left[K_1\rho(s)q(s) - \frac{r_1(g(s))(\rho'(s)r_1(s) - \rho(s)p(s)R_2(g(s),T))^2}{4\rho(s)R_2(g(s),T)g'(s)r_1^2(s)}\right] ds$$

$$= \limsup_{t\to\infty} \int_1^t \left[K_1\frac{25}{4e^3} - \frac{e^{-(s-1)}\left(-\frac{e^{-2s}}{4}(e^{s-1}-e)\right)^2}{4(e^{s-1}-e)e^{-2s}}\right] ds$$

$$= \limsup_{t\to\infty} \int_1^t \left[K_1\frac{25}{4e^3} - \frac{e^{-3s+1}(e^{s-1}-e)}{64}\right] ds = \infty.$$

Condition (3.10) is satisfied. It is easy to check that the other conditions of Corollary 1 are also satisfied. Thus, any solution y of (4.5) is oscillatory or satisfies $y(t) \to 0$ as $t \to \infty$. One such solution of Eq. (4.5) is $y(t) = e^{-t}$. Note that (3.9) holds, but (3.8) does not.

We note that none of the above mentioned papers on the oscillation of third order differential equations can be applied to the delay equations (4.3)–(4.5).

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