

Bessel Integrals and Fundamental Solutions for a Generalized Tricomi Operator

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The method of partial Fourier transform is used to find explicit formulas for two remarkable fundamental solutions for a generalized Tricomi operator. These fundamental solutions reflect clearly the mixed type of the Tricomi operator. In proving these results, we establish explicit formulas for Fourier transforms of some functions involving Bessel functions. © 2001 Academic Press

1. INTRODUCTION

Consider the generalized Tricomi operator

$$P = y\Delta + \frac{\partial^2}{\partial y^2}, \quad (1.1)$$

where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator. Our aim is to find, via partial Fourier transform with respect to $x = (x_1, \dots, x_n)$, fundamental solutions relative to an arbitrary point $(a, 0)$ on the hyperplane $y = 0$ in \mathbb{R}^{n+1} , that is, distributions that are solutions to

$$Pu = \delta(x - a, y),$$

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where $\delta(x-a, y)$ is the Dirac measure concentrated at $(a, 0)$. Since P is invariant under translations parallel to that hyperplane, it suffices to consider the case when the Dirac measure is concentrated at the origin.

If $n = 1$, then (1.1) is the classical Tricomi operator,

$$\mathcal{T} = y \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (1.2)$$

It is known that, for this operator, the equation $9x^2 + 4y^3 = 0$ defines the two characteristic curves that originate at the origin. They divide the plane in two disjoint regions, namely,

$$D_+ = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^3 > 0\},$$

the region "outside" the characteristics and

$$D_- = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^3 < 0\},$$

the region "inside" the characteristics. Note that D_- is entirely contained in the hyperbolic region $y < 0$.

In the paper [2] it was shown the existence of the following two fundamental solutions relative to the origin:

$$F_+(x, y) = \begin{cases} \frac{\Gamma(1/6)}{3 \cdot 2^{2/3} \pi^{1/2} \Gamma(2/3)} (9x^2 + 4y^3)^{-1/4} & \text{in } D_+ \\ 0 & \text{elsewhere} \end{cases} \quad (1.3)$$

and

$$F_-(x, y) = \begin{cases} \frac{3\Gamma(4/3)}{2^{2/3} \pi^{1/2} \Gamma(5/6)} |9x^2 + 4y^3|^{-1/6} & \text{in } D_- \\ 0 & \text{elsewhere.} \end{cases} \quad (1.4)$$

The support of F_- , the closure of D_- , is, except for the origin, entirely contained in the hyperbolic region ($y < 0$), while the support of F_+ , the closure of D_+ , consists of the whole elliptic region ($y > 0$), the parabolic region (x -axis), and extends to the hyperbolic region up to and included the characteristic curves. The method used in [2] to prove these results was based upon the property that solutions to the equation $\mathcal{T} = 0$ are invariant under the dilation $d_t(x, y) = (t^3x, t^2y)$ in \mathbb{R}^2 .

In [1] we proved formulas (1.3) and (1.4) by using partial Fourier transform with respect to the x variable. In this paper we extend the results of [1] to the generalized Tricomi operator (1.1). Since the dimension of the space variable x is now $n > 1$, technical difficulties in evaluating Fourier

transforms involving Bessel functions do occur. We circumvent them by calculating integrals of the type

$$I_\varepsilon(a, b) = \int_0^\infty e^{-\varepsilon t} t^{-\lambda} J_\mu(at) J_\nu(bt) dt$$

(Section 3). As a result, we obtain for the operator (1.1) the following fundamental solutions:

$$F_+(x, y) = \begin{cases} -\frac{3^{n-2}}{2^{2/3}\pi^{n/2}} \frac{\Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{\Gamma(2/3)} (9|x|^2 + 4y^3)^{1/3-n/2} & \text{in } D_+^n \\ 0 & \text{elsewhere,} \end{cases}$$

whose support is the closure of the region $D_+^n = \{(x, y) \in \mathbb{R}^{n+1} : 9|x|^2 + 4y^3 > 0\}$ and

$$F_-(x, y) = \begin{cases} \frac{3^n \Gamma\left(\frac{4}{3}\right)}{2^{2/3}\pi^{n/2} \Gamma\left(\frac{4}{3} - \frac{n}{2}\right)} |9|x|^2 + 4y^3|^{1/3-n/2} & \text{in } D_-^n \\ 0 & \text{elsewhere} \end{cases}$$

supported by the closure of the region $D_-^n = \{(x, y) \in \mathbb{R}^{n+1} : 9|x|^2 + 4y^3 < 0\}$.

These fundamental solutions clearly generalize formulas (1.3) and (1.4). In addition, we also show that no matter the parity of the dimension n , $F_-(x, y)$ **does not** have support on the boundary of the region D_- in the hyperbolic region and hence, in contrast with what happens for strictly hyperbolic operators—like the wave operator—the Huyghens principle does not hold for the generalized Tricomi operator.

It is well known [9] that, for the wave operator, the regularity of the fundamental solutions degenerates as the dimension n increases: a locally constant function, when $n = 1$; an absolutely continuous measure relative to the Lebesgue measure, when $n = 2$; a measure carried by the surface of the forward light-cone, when $n = 3$; and so on. However, for the generalized Tricomi operator they always remain locally integrable.

The plan of this paper is as follows. In Section 2 we reduce, via partial Fourier transform, the original problem to an equivalent one of finding fundamental solutions for a second order ordinary differential equation and show how these can be represented in terms of Airy functions or Bessel functions. In Section 3 we obtain explicit formulas for Fourier transforms of the functions $|\xi|^{\pm\nu} J_\nu(|\xi|)$, $|\xi|^\nu K_\nu(|\xi|)$, and $|\xi|^\nu N_\nu(|\xi|)$, $\xi = (\xi_1, \dots, \xi_n)$, $n \geq 1$. These formulas are used to construct the fundamental solution

supported by $\overline{D_-^n}$ (Section 4) and the fundamental solution supported by $\overline{D_+^n}$ (Section 5). In the Appendix the reader will find the definitions of the Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $N_\nu(z)$, and a list of properties that are needed throughout this work. We also recall the definition of hypergeometric functions and some of their main properties.

The method of finding solutions, via partial Fourier transform, to the equation

$$Pu = y \Delta u + \frac{\partial^2 u}{\partial y^2} = f \quad (1.5)$$

with, say, $f \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ is a natural one although not particularly new. In the monograph [5], R. J. P. Groothuizen exhibits a solution u in terms of a Fourier integral operator whose symbol is obtained by the partial Fourier transform analysis employed in this paper. He also obtains fundamental solutions given by Fourier integral operators. Our results are more precise since we give explicit formulas for the fundamental solutions and these are tempered distributions. Consequently, we can obtain more transparent representations for the solution u to the Eq. (1.5) as a convolution of f with any of the fundamental solutions here described.

2. PRELIMINARIES

Consider the more general problem of finding fundamental solutions for the operator (1.1) relative to an arbitrary point $(0, b)$ on the y -axis. That is, one wishes to find distributional solutions to the equation

$$y \Delta F + \frac{\partial^2}{\partial y^2} F = \delta(x) \otimes \delta(y - b). \quad (2.1)$$

Partial Fourier transform with respect to x reduces this problem into finding fundamental solutions to the ordinary differential equation

$$\tilde{F}_{yy} - y |\xi|^2 \tilde{F} = \delta(y - b), \quad (2.2)$$

where \tilde{F} denotes the partial Fourier transform of F , $|\xi|^2 = \sum_{j=1}^n \xi_j^2$, and $\delta(y - b)$ is the Dirac measure concentrated at b .

One way of solving (2.2) (see [9]) consists in selecting two linearly independent solutions $U_1(\xi, y)$ and $U_2(\xi, y)$ to the homogeneous equation

$$\tilde{u}_{yy} - y |\xi|^2 \tilde{u} = 0 \quad (2.3)$$

so that their Wronskian at $y=b$ (which in the case under consideration is the same as the Wronskian at $y=0$) is equal to -1 and in defining

$$\tilde{F}(\xi, y; b) = \begin{cases} U_2(\xi, b) U_1(\xi, y) & \text{if } y \geq b \\ U_1(\xi, b) U_2(\xi, y) & \text{if } y \leq b. \end{cases} \quad (2.4)$$

It is a matter of verification that $\tilde{F}(\xi, y; b)$ is a solution to (2.2).

Equivalently [4], one can obtain a fundamental solution to the Eq. (2.2) by finding two linearly independent solutions to the homogeneous Eq. (2.3): $u_1(\xi, y; b)$, defined for $y > b$, and $u_2(\xi, y; b)$, defined for $y < b$, so that

- (i) the limit of u_1 as $y \rightarrow b+$ equals the limit of u_2 as $y \rightarrow b-$ and
- (ii) the limit of $u_{1,y}$ as $y \rightarrow b+$ minus the limit of $u_{2,y}$ as $y \rightarrow b-$ is equal to 1. The function $\tilde{F}(\xi, y; b)$ is now defined by

$$\tilde{F}(\xi, y; b) = \begin{cases} u_1(\xi, y; b) & \text{if } y \geq b \\ u_2(\xi, y; b) & \text{if } y \leq b. \end{cases} \quad (2.5)$$

In what follows we will use at our convenience either one of these two expressions for $\tilde{F}(\xi, y; b)$.

Returning to the original problem (2.1), if we can choose $U_1(\xi, y)$ and $U_2(\xi, y)$, in formula (2.4), or $u_1(\xi, y; b)$ and $u_2(\xi, y; b)$, in formula (2.5), so that $\tilde{F}(\xi, y; b)$ is a tempered distribution with respect to $\xi = (\xi_1, \dots, \xi_n)$, then its inverse Fourier transform of $\tilde{F}(\xi, y; b)$ defines a fundamental solution to the operator (1.1).

We now proceed to find linearly independent solutions to the homogeneous Eq. (2.3). The change of variables $z = |\xi|^{2/3} y$ transforms (2.3) into the classical Airy's equation $u'' - zu = 0$. Two linearly independent solutions to that equation are $Ai(z)$ and $Bi(z)$ respectively called *the Airy functions of the first and second kinds*. It is known [6] that these two functions can be represented in terms of Bessel functions of order $\pm 1/3$ as follows. If $|\arg z| < (2\pi/3)$, then

$$Ai(z) = \frac{z^{1/2}}{3} \left[I_{-1/3} \left(\frac{3}{3} z^{2/3} \right) - I_{1/3} \left(\frac{2}{3} z^{3/2} \right) \right] = \frac{1}{\pi} \left(\frac{z}{3} \right)^{1/2} K_{1/3} \left(\frac{2}{3} z^{3/2} \right), \quad (2.6)$$

$$Bi(z) = \left(\frac{z}{3} \right)^{1/2} \left[I_{-1/3} \left(\frac{2}{3} z^{3/2} \right) + I_{1/3} \left(\frac{2}{3} z^{3/2} \right) \right].$$

If $|\arg z| < (2\pi/3)$, then

$$\begin{aligned} Ai(-z) &= \frac{z^{1/2}}{3} \left[J_{-1/3} \left(\frac{2}{3} z^{3/2} \right) + J_{1/3} \left(\frac{2}{3} z^{3/2} \right) \right], \\ Bi(-z) &= \left(\frac{z}{3} \right)^{1/2} \left[J_{-1/3} \left(\frac{2}{3} z^{3/2} \right) - J_{1/3} \left(\frac{2}{3} z^{3/2} \right) \right]. \end{aligned} \quad (2.7)$$

In the Appendix I, where a brief review of Bessel functions is presented, we show that the following relations hold:

$$Ai(0) = \frac{3^{-2/3}}{\Gamma(2/3)}, \quad Ai'(0) = -\frac{3^{-4/3}}{\Gamma(4/3)} \quad (2.8)$$

and

$$Bi(0) = \frac{3^{-1/6}}{\Gamma(2/3)}, \quad Bi'(0) = \frac{3^{-5/6}}{\Gamma(4/3)}. \quad (2.9)$$

As a consequence, the Wronskian of $Ai(z)$ and $Bi(z)$ evaluated at $z=0$ is

$$W(Ai(z), Bi(z))|_{z=0} = 1/\pi. \quad (2.10)$$

Indeed, we have

$$W(Ai(z), Bi(z))|_{z=0} = \begin{vmatrix} \frac{3^{-2/3}}{\Gamma(2/3)} & \frac{3^{-1/6}}{\Gamma(2/3)} \\ \frac{3^{-4/3}}{-\Gamma(4/3)} & \frac{3^{-5/6}}{\Gamma(4/3)} \end{vmatrix} = \frac{2 \cdot 3^{-3/2}}{\Gamma(2/3) \Gamma(4/3)} = \frac{1}{\pi},$$

because

$$\Gamma(2/3) \Gamma(4/3) = \frac{2\pi}{3^{3/2}}. \quad (2.11)$$

We now choose the following two linearly independent solutions to the Eq. (2.3):

$$U_1(\xi, y) = \sqrt{\pi} |\xi|^{-1/3} Ai(|\xi|^{2/3} y)$$

and

$$U_2(\xi, y) = -\sqrt{\pi} |\xi|^{-1/3} Bi(|\xi|^{2/3} y),$$

and note that, by virtue of (2.10), the Wronskian of $U_1(\xi, y)$ and $U_2(\xi, y)$ is equal to -1 . Next, according to (2.4), the distribution

$$\tilde{F}(\xi, y; b) = \begin{cases} -\pi |\xi|^{-2/3} Bi(|\xi|^{2/3} b) Ai(|\xi|^{2/3} y) & \text{if } y \geq b \\ -\pi |\xi|^{-2/3} Ai(|\xi|^{2/3} b) Bi(|\xi|^{2/3} y) & \text{if } y \leq b \end{cases} \quad (2.12)$$

is a fundamental solution to the ordinary differential Eq. (2.2). A fundamental solution $F(x, y; b)$ to the generalized Tricomi operator (1.1) and relative to the point $(0, b)$ would then be the inverse Fourier transform of $\tilde{F}(\xi, y; b)$ whenever that Fourier transform exists.

We do not know how to obtain an explicit formula (or formulas) for the inverse Fourier transform of $\tilde{F}(\xi, y; b)$ when $b \neq 0$, a problem that merits to be investigated. We conjecture that when $b < 0$, that is, the point $(0, b)$ is in the hyperbolic region, there exists two fundamental solutions that converge, as $b \rightarrow 0$, to the fundamental solutions $F_+(x, y)$ and $F_-(x, y)$ defined, respectively, by the formulas (5.6) and (4.2). The two fundamental solutions described in the monograph [5] do not seem to satisfy these requirements.

However, when $b = 0$, we will show in the following sections how to obtain from formula (2.12), as well as formulas similar to it, explicit expressions for fundamental solution to (1.1).

The case $n = 1$. For sake of completeness and in order to motivate our work in the forthcoming sections we briefly present the results of the paper [1], for the Tricomi operator (1.2), that is the case when $n = 1$. If $b = 0$ and we take into account the values of $Ai(0)$ and $Bi(0)$ as given by (2.8) and (2.9), then formula (2.12) simplifies as follows:

$$\tilde{F}(\xi, y) = \begin{cases} -\frac{\pi |\xi|^{-2/3}}{3^{1/6} \Gamma(2/3)} Ai(|\xi|^{2/3} y) & \text{if } y \geq 0 \\ -\frac{\pi |\xi|^{-2/3}}{3^{2/3} \Gamma(2/3)} Bi(|\xi|^{2/3} y) & \text{if } y \leq 0, \end{cases} \quad (2.13)$$

where, for simplicity, we wrote $\tilde{F}(\xi, y)$ for $\tilde{F}(\xi, y; 0)$. Now the inverse Fourier transform of both expressions on the right-hand side of (2.13) can be explicitly calculated. To see this, first consider the representations of $Ai(z)$ and $Bi(z)$ in terms of Bessel functions as given by formulas (2.6) and (2.7):

$$Ai(|\xi|^{2/3} y) = \frac{1}{\pi} \left(\frac{|\xi|^{2/3} y}{3} \right)^{1/2} K_{1/3} \left(\frac{2}{3} |\xi|^{3/2} \right)$$

and

$$Bi(|\xi|^{2/3} y) = \left(\frac{|\xi|^{2/3} (-y)}{3}\right)^{1/2} \left[J_{-1/3} \left(\frac{2}{3} |\xi| (-y)^{3/2}\right) - J_{1/3} \left(\frac{2}{3} |\xi| (-y)^{3/2}\right) \right].$$

Next by introducing the change of variables $s = (2/3) y^{3/2}$, whenever $y \geq 0$, and $t = (2/3)(-y)^{3/2}$, whenever $y \leq 0$, rewrite formula (2.13) as

$$\tilde{F}(\xi, y) = \begin{cases} \alpha \cdot \left(\frac{s}{|\xi|}\right)^{1/3} K_{1/3}(x|\xi|) & \text{if } y \geq 0 \\ \beta \cdot \left(\frac{t}{|\xi|}\right)^{1/3} [J_{-1/3}(t|\xi|) - J_{1/3}(t|\xi|)] & \text{if } y \leq 0, \end{cases} \tag{2.14}$$

where α and β are constants given by

$$\alpha = -\frac{1}{2^{1/3} 3^{1/3} \Gamma(2/3)} \quad \text{and} \quad \beta = -\frac{\pi}{2^{1/3} 3^{5/6} \Gamma(2/3)}. \tag{2.15}$$

Theorem 3.1 of [1] proves that $F(x, y)$, the inverse Fourier transform of $\tilde{F}(\xi, y)$, is then

$$F = \frac{3}{2}F_+ - \frac{1}{2}F_-,$$

a linear combination of the two fundamental solutions F_+ and F_- respectively defined by formulas (1.3) and (1.4). To prove this theorem, one relies on known formulas (see [4, 7]) for the inverse Fourier transforms of the functions $|\xi|^{-1/3} J_{1/3}(|\xi|)$, $|\xi|^{-1/3} J_{-1/3}(|\xi|)$, and $|\xi|^{-1/3} K_{1/3}(|\xi|)$, formulas that need be generalized to the case $n > 1$.

3. FOURIER TRANSFORMS INVOLVING BESSEL FUNCTIONS

As we have indicated in the previous section, we are going to need explicit formulas for the inverse Fourier transforms of the following functions: $|\xi|^{\pm\nu} J_\nu(|\xi|)$, $|\xi|^\nu K_\nu(|\xi|)$, and $|\xi|^\nu N_\nu(|\xi|)$, where $\xi = (\xi_1, \dots, \xi_n)$, $n \geq 1$. Since, when $n = 1$, explicit formulas for the Fourier transforms are known and can be found, for example, in [4] and [7], we concentrate on the case $n > 1$.

1. The Inverse Fourier Transform of $|\xi|^\nu K_\nu(|\xi|)$

Assume that $|\text{Re } \nu| < 1/2$ and define

$$\mathcal{F}^{-1}[|\xi|^\nu K_\nu(|\xi|)](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_n} e^{i\langle x, \xi \rangle} |\xi|^\nu K_\nu(|\xi|) d\xi. \tag{3.1}$$

The assumption on the real part of ν secures convergence of the integral at the origin. On the other hand, in view of the asymptotic behavior of $K_\nu(|\xi|)$ for large values of $|\xi|$, the integral converges absolutely. By introducing spherical coordinates, rewrite the integral on the right-hand of (3.1) as

$$\int_0^\infty r^{n+\nu-1} K_\nu(r) \left\{ \int_{S_{n-1}} e^{i\langle rx, \omega \rangle} d\omega \right\} dr.$$

Since

$$\int_{S_{n-1}} e^{i\langle rx, \omega \rangle} d\omega = \frac{(2\pi)^{n/2}}{|rx|^{(n/2)-1}} J_{(n/2)-1}(r|x|), \quad (3.2)$$

it follows that

$$\begin{aligned} \mathcal{F}^{-1}[|\xi|^\nu K_\nu(|\xi|)](x) \\ = \frac{|x|^{1-(n/2)}}{(2\pi)^{n/2}} \int_0^\infty r^{(n/2)+\nu} J_{(n/2)-1}(r|x|) K_\nu(r) dr. \end{aligned} \quad (3.3)$$

We now quote the following result found in Watson's treatise [11]:

LEMMA 3.1. *If $\operatorname{Re}(\mu+1) > |\operatorname{Re} \nu|$ and $\operatorname{Re} b > |\operatorname{Im} a|$ then*

$$\int_0^\infty t^{\mu+\nu+1} J_\mu(at) K_\nu(bt) dt = \frac{(2a)^\mu (2b)^\nu \Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}}. \quad (3.4)$$

From this lemma, we obtain the following result

THEOREM 3.1. *If $|\operatorname{Re} \nu| < 1/2$ then*

$$\mathcal{F}^{-1}[|\xi|^\nu K_\nu(|\xi|)](x) = \frac{2^{\nu-1} \Gamma\left(\frac{n}{2} + \nu\right)}{\pi^{n/2}} (1+|x|^2)^{-(n/2)-\nu}. \quad (3.5)$$

Proof. The integral in (3.3) is the same as the integral in (3.4), where $\mu = n/2 - 1$, $a = |x|$, and $b = 1$. Clearly the conditions of the lemma are satisfied and Theorem 3.1 follows at once. ■

We remark that, when $n = 1$, (3.5) becomes

$$\mathcal{F}^{-1}[|\xi|^\nu K_\nu(|\xi|)](x) = \frac{2^{\nu-1} \Gamma\left(\frac{1}{2} + \nu\right)}{\pi^{1/2}} (1+|x|^2)^{-(1/2)-\nu},$$

a formula found in [4] and [7].

2. *Inverse Fourier Transforms of $|\xi|^\nu J_\nu(|\xi|)$ and $|\xi|^{-\nu} J_\nu(|\xi|)$*

As before, assume that $|\operatorname{Re} \nu| < 1/2$. Formally, the inverse Fourier transform of $|\xi|^\nu J_\nu(|\xi|)$ is

$$\mathcal{F}^{-1}[|\xi|^\nu J_\nu(|\xi|)](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_n} e^{i\langle x, \xi \rangle} |\xi|^\nu J_\nu(|\xi|) d\xi.$$

In general, the integral diverges at ∞ and so we introduce a converging factor $e^{-\varepsilon|\xi|}$ and take a limit as $\varepsilon \rightarrow 0$. Since $|\xi|^\nu J_\nu(|\xi|)$ is locally integrable, it defines a tempered distribution and the limit exists in $\mathcal{S}'(\mathbb{R}_n)$, the space of tempered distributions on \mathbb{R}_n . Thus the precise meaning of the inverse Fourier transform is

$$\overline{\mathcal{F}}^{-1}[|\xi|^\nu J_\nu(|\xi|)](x) = \frac{1}{(2\pi)^n} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}_n} e^{i\langle x, \xi \rangle - \varepsilon|\xi|} |\xi|^\nu J_\nu(|\xi|) d\xi. \quad (3.6)$$

By introducing spherical coordinates, one can see that the integral on the right-hand can be written as

$$\int_0^\infty e^{-\varepsilon r} r^{n+\nu-1} J_\nu(r) \left\{ \int_{S_{n-1}} e^{i\langle rx, \omega \rangle} d\omega \right\} dr.$$

Since

$$\int_{S_{n-1}} e^{i\langle rx, \omega \rangle} d\omega = \frac{(2\pi)^{n/2}}{|rx|^{(n/2)-1}} J_{(n/2)-1}(r|x|), \quad (3.7)$$

it follows that

$$\begin{aligned} &\mathcal{F}^{-1}[|\xi|^\nu J_\nu(|\xi|)](x) \\ &= \frac{|x|^{1-(n/2)}}{(2\pi)^{n/2}} \lim_{\varepsilon \rightarrow 0} \int_0^\infty e^{-\varepsilon r} r^{(n/2)+\nu} J_{(n/2)-1}(r|x|) J_\nu(r) dr. \end{aligned} \quad (3.8)$$

In a similar manner we also have

$$\begin{aligned} &\overline{\mathcal{F}}^{-1}[|\xi|^{-\nu} J_\nu(|\xi|)](x) \\ &= \frac{|x|^{1-(n/2)}}{(2\pi)^{n/2}} \lim_{\varepsilon \rightarrow 0} \int_0^\infty e^{-\varepsilon r} r^{(n/2)-\nu} J_{(n/2)-1}(r|x|) J_\nu(r) dr. \end{aligned} \quad (3.9)$$

Both integrals appearing in formulas (3.8) and (3.9) are particular cases of the following integral

$$I_\varepsilon(a, b) = \int_0^\infty e^{-\varepsilon t} t^{-\lambda} J_\mu(at) J_\nu(bt) dt, \quad (3.10)$$

where a and b are positive real numbers, λ , μ , and ν , complex numbers such that $\operatorname{Re}(\mu + \nu + 1) > \operatorname{Re}(\lambda)$. This integral is a variant of the discontinuous integral of Weber and Schafheitlin studied by Watson in his treatise on Bessel functions [11].

LEMMA 3.2. *As $\varepsilon \rightarrow 0$ $I_\varepsilon(a, b)$ tends, in the sense of distributions, to either one of the following limits:*

$$\begin{aligned} & \frac{b^\nu \Gamma\left(\frac{\mu + \nu - \lambda + 1}{2}\right)}{2^\lambda a^{\nu - \lambda + 1} \Gamma(\nu + 1) \Gamma\left(\frac{\lambda + \mu - \nu + 1}{2}\right)} \\ & \times F\left(\frac{\mu + \nu - \lambda + 1}{2}, \frac{\nu - \lambda - \mu + 1}{2}; \nu + 1; \frac{b^2}{a^2}\right), \end{aligned} \quad (3.11)$$

if $0 < b < a$, or

$$\begin{aligned} & \frac{a^\mu \Gamma\left(\frac{\mu + \nu - \lambda + 1}{2}\right)}{2^\lambda b^{\mu - \lambda + 1} \Gamma(\mu + 1) \Gamma\left(\frac{\lambda + \nu - \mu + 1}{2}\right)} \\ & \times F\left(\frac{\mu + \nu - \lambda + 1}{2}, \frac{\mu - \lambda - \nu + 1}{2}; \mu + 1; \frac{a^2}{b^2}\right), \end{aligned} \quad (3.12)$$

if $0 < a < b$.

Proof. We adapt to our situation the proof of the Weber–Schafheitlin theorem as found in Section 13.4 of Watson's treatise [11]. It consists of expanding the integrand in (3.10) in power series of b , and passing to the limit as $\varepsilon \rightarrow 0$.

1. Consider the case when $0 < b < a$. If we replace b by z , the integral (3.10) is an analytic function of z when $\operatorname{Re} z > 0$ and $|\operatorname{Im} z| < \varepsilon$. Introduce new constants α , β , and γ defined by

$$2\alpha = \mu + \nu - \lambda + 1, \quad 2\beta = \nu - \lambda - \mu + 1, \quad \gamma = \nu + 1$$

or, equivalently,

$$\lambda = \gamma - \alpha - \beta, \quad \mu = \alpha - \beta, \quad \nu = \gamma - 1,$$

and rewrite (3.10) as follows:

$$\begin{aligned} I_\varepsilon(a, z) &= \int_0^\infty e^{-\varepsilon t} t^{\alpha + \beta - \gamma} J_{\alpha - \beta}(at) J_{\gamma - 1}(zt) dt \\ &= \int_0^\infty e^{-\varepsilon t} J_{\alpha - \beta}(at) \left\{ \sum_{m=0}^\infty \frac{(-1)^m (z/2)^{\gamma + 2m - 1} t^{\alpha + \beta + 2m - 1}}{m! \Gamma(\gamma + m)} \right\} dt \\ &= \sum_{m=0}^\infty \frac{(-1)^m (z/2)^{\gamma + 2m - 1}}{m! \Gamma(\gamma + m)} \int_0^\infty e^{-\varepsilon t} J_{\alpha - \beta}(at) t^{\alpha + \beta + 2m - 1} dt. \end{aligned} \tag{3.13}$$

The interchange between the integrating and summation signs is justified because, when $|z| < \varepsilon$, the series

$$\sum_{m=0}^\infty \frac{(-1)^m (z/2)^{\gamma + 2m - 1}}{m! \Gamma(\gamma + m)} \int_0^\infty e^{-\varepsilon t} |J_{\alpha - \beta}(at) t^{\alpha + \beta + 2m - 1}| dt$$

is absolutely convergent.

2. We now evaluate the last integral on the right-hand side of (3.13). By expanding $J_{\alpha - \beta}(at)$ in power series and integrating term by term we obtain

$$\begin{aligned} &\int_0^\infty e^{-\varepsilon t} J_{\alpha - \beta}(at) t^{\alpha + \beta + 2m - 1} dt \\ &= \sum_{k=0}^\infty \frac{(-1)^k (a/2)^{\alpha - \beta + 2k}}{k! \Gamma(\alpha - \beta + k + 1)} \int_0^\infty e^{-\varepsilon t} t^{2\alpha + 2(m+k) - 1} dt \\ &= \sum_{k=0}^\infty \frac{(-1)^k (a/2)^{\alpha - \beta + 2k}}{k! \Gamma(\alpha - \beta + k + 1)} \frac{\Gamma(2\alpha + 2m + 2k)}{\varepsilon^{2\alpha + 2m + 2k}}. \end{aligned}$$

By using the duplication formula $\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + \frac{1}{2})$ together with the notation $(z)_k = z(z + 1) \dots (z + k - 1) = \Gamma(z + k)/\Gamma(z)$ we rewrite the last expression as:

$$\begin{aligned} &\int_0^\infty e^{-\varepsilon t} J_{\alpha - \beta}(at) t^{\alpha + \beta + 2m - 1} dt \\ &= \frac{(a/2)^{\alpha - \beta}}{(\varepsilon^2)^{\alpha + m}} \frac{\Gamma(2\alpha + 2m)}{\Gamma(\alpha - \beta + 1)} \sum_{k=0}^\infty \frac{(\alpha + m)_k (\alpha + m + \frac{1}{2})_k}{k! (\alpha - \beta + 1)_k} \left(-\frac{a^2}{c^2} \right)^k \\ &= \frac{(a/2)^{\alpha - \beta}}{(\varepsilon^2)^{\alpha + m}} \frac{\Gamma(2\alpha + 2m)}{\Gamma(\alpha - \beta + 1)} F\left(\alpha + m, \alpha + m + \frac{1}{2}; \alpha - \beta + 1; -\frac{a^2}{c^2} \right). \end{aligned}$$

By using formula (A.16) in Appendix Section II:

$$F(a, b; c; z) = (1 - z)^{-a} F\left(a, c - b; c; \frac{z}{z - 1}\right),$$

we obtain that

$$\begin{aligned} & \int_0^\infty e^{-\varepsilon t} J_{\alpha-\beta}(at) t^{\alpha+\beta+2m-1} dt \\ &= \frac{(a/2)^{\alpha-\beta}}{(a^2 + \varepsilon^2)^{\alpha+m}} \frac{\Gamma(2\alpha + 2m)}{\Gamma(\alpha - \beta + 1)} F\left(\alpha + m, \alpha + m + \frac{1}{2}; \alpha - \beta + 1; \frac{a^2}{a^2 + \varepsilon^2}\right). \end{aligned} \quad (3.14)$$

Substituting (3.14) into (3.13) we get

$$\begin{aligned} I_\varepsilon(a, z) &= \sum_{m=0}^\infty \frac{(-1)^m (z/2)^{\gamma+2m-1} (a/2)^{\alpha-\beta} \Gamma(2\alpha + 2m)}{m! \Gamma(\gamma + m) (a^2 + \varepsilon^2)^{\alpha+m} \Gamma(\alpha - \beta + 1)} \\ &\quad \times F\left(\alpha + m, \alpha + m + \frac{1}{2}; \alpha - \beta + 1; \frac{a^2}{a^2 + \varepsilon^2}\right), \end{aligned} \quad (3.15)$$

whenever $|z| < \varepsilon$.

3. Following [11], one can show that (3.15) is valid provided that z satisfies the conditions

$$\operatorname{Re}(z) > 0, \quad |\operatorname{Im}(z)| < \varepsilon, \quad |z| < \sqrt{a^2 + \varepsilon^2} - \varepsilon.$$

Let $\delta > 0$ be small enough so that $b < \sqrt{a^2 + \delta^2} - \delta$, and take $0 < \varepsilon \leq \delta$ so that we also have $b < \sqrt{a^2 + \varepsilon^2} - \varepsilon$. In (3.15) we may now let $z = b$ and, when this is done, one can show, by the method of majorants, that the resulting series converges uniformly with respect to ε , $0 < \varepsilon \leq \delta$, and therefore, as $\varepsilon \rightarrow 0$, the limit of the series is equal to its value at $\varepsilon = 0$. Thus

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_\varepsilon(a, b) &= \sum_{m=0}^\infty \frac{(-1)^m (b/2)^{\gamma+2m-1} (a/2)^{\alpha-\beta} \Gamma(2\alpha + 2m)}{m! \Gamma(\gamma + m) a^{2\alpha+2m} \Gamma(\alpha - \beta + 1)} \\ &\quad \times F\left(\alpha + m, \alpha + m + \frac{1}{2}; \alpha - \beta + 1; 1\right). \end{aligned} \quad (3.16)$$

By formula (A.15) in the Appendix Section II, we have

$$F\left(\alpha + m, \frac{1}{2} - \beta - m; \alpha - \beta + 1; 1\right) = \frac{\Gamma(\alpha - \beta + 1) \Gamma(1/2)}{\Gamma(1 - \beta - m) \Gamma(\alpha + m + \frac{1}{2})}.$$

On the other hand,

$$\Gamma(1/2) \Gamma(2\alpha + 2m) = 2^{2\alpha + 2m - 1} \Gamma(\alpha + m) \Gamma\left(\alpha + m + \frac{1}{2}\right),$$

$$\Gamma(1 - \beta - m) = \frac{(-1)^m \pi \csc(\pi\beta)}{\Gamma(\beta + m)},$$

$$\Gamma(1 - \beta) \Gamma(\beta) = \pi \csc(\pi\beta);$$

therefore

$$\frac{\Gamma(2\alpha + 2m) \Gamma(1/2)}{\Gamma(\gamma + m) \Gamma(1 - \beta - m) \Gamma(\alpha + m + \frac{1}{2})} = \frac{2^{2\alpha + 2m - 1} \Gamma(\alpha + m) \Gamma(\beta + m)}{(-1)^m \Gamma(1 - \beta) \Gamma(\beta) \Gamma(\gamma + m)}.$$

Substituting these formulas into (3.16), we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_\varepsilon(a, b) &= \sum_{m=0}^{\infty} \frac{b^{\gamma-1} \Gamma(\alpha + m) \Gamma(\beta + m)}{2^{\gamma-\alpha-\beta} a^{\alpha+\beta} m! \Gamma(1 - \beta) \Gamma(\beta) \Gamma(\gamma + m)} \left(\frac{b^2}{a^2}\right)^m \\ &= \frac{b^{\gamma-1} \Gamma(\alpha)}{2^{\gamma-\alpha-\beta} a^{\alpha+\beta} \Gamma(\gamma) \Gamma(1 - \beta)} F\left(\alpha, \beta; \gamma; \frac{b^2}{a^2}\right) \end{aligned} \tag{3.17}$$

Finally, returning to the constants $\mu, \nu,$ and $\lambda,$ we obtain the expression (3.11) in the first part of the lemma.

4. In the case $0 < a < b,$ we proceed in an analogous manner to obtain the expression (3.12) in the second part of the lemma. ■

We now use Lemma 3.2 to evaluate the inverse Fourier transforms of $|\xi|^\nu J_\nu(|\xi|)$ and $|\xi|^{-\nu} J_\nu(|\xi|)$ respectively defined by formulas (3.8) and (3.9).

THEOREM 3.2. *The inverse Fourier transform of $|\xi|^\nu J_\nu(|\xi|)$ is the distribution defined by*

$$\begin{aligned} &\mathcal{F}^{-1}[|\xi|^\nu J_\nu(|\xi|)](x) \\ &= \begin{cases} \sin\left(\frac{n\pi}{2}\right) \frac{2^\nu \Gamma\left(\frac{n}{2} + \nu\right)}{\pi^{(n/2)+1}} (1 - |x|^2)^{-(n/2)-\nu}, & 0 < |x| < 1, \\ -\sin(\nu\pi) \frac{2^\nu \Gamma\left(\frac{n}{2} + \nu\right)}{\pi^{(n/2)+1}} (|x|^2 - 1)^{-(n/2)-\nu}, & 1 < |x|. \end{cases} \end{aligned} \tag{3.18}$$

Proof. The integral on the right-hand side of (3.8) corresponds to the integral in (3.10) where

$$\mu = \frac{n}{2} - 1, \quad \nu = \nu, \quad \lambda = -\frac{n}{2} - \nu, \quad a = |x|, \quad b = 1.$$

If $1 < |x|$, then from formula (3.11) we obtain for the limit in (3.8):

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^{\infty} e^{-\varepsilon r} r^{(n/2)+\nu} J_{(n/2)-1}(r|x|) J_{\nu}(r) dr \\ &= \frac{\Gamma\left(\frac{n}{2} + \nu\right)}{2^{-(n/2)-\nu} |x|^{(n/2)+2\nu+1} \Gamma(\nu+1) \Gamma(-\nu)} F\left(\frac{n}{2} + \nu, \nu + 1; \nu + 1; \frac{1}{|x|^2}\right). \end{aligned}$$

From the known relation for hypergeometric series

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z) \quad (3.19)$$

we have that

$$F\left(\frac{n}{2} + \nu, \nu + 1; \nu + 1; \frac{1}{|x|^2}\right) = \left(\frac{|x|^2 - 1}{|x|^2}\right)^{-(n/2)-\nu}.$$

On the other hand

$$\Gamma(\nu+1) \Gamma(-\nu) = \frac{-\pi}{\sin(\pi\nu)}.$$

Thus, for $1 < |x|$, the above limit is equal to

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^{\infty} e^{-\varepsilon r} r^{(n/2)+\nu} J_{(n/2)-1}(r|x|) J_{\nu}(r) dr \\ &= -\sin(\pi\nu) \frac{|x|^{(n/2)-1} \Gamma\left(\frac{n}{2} + \nu\right)}{2^{-(n/2)-\nu} \pi} (|x|^2 - 1)^{-(n/2)-\nu}, \quad (3.20) \end{aligned}$$

If $0 < |x| < 1$, then formula (3.12) yields

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^\infty e^{-\varepsilon r} r^{(n/2)+v} J_{(n/2)-1}(r|x|) J_\nu(r) dr \\ &= \frac{|x|^{(n/2)-1} \Gamma\left(\frac{n}{2} + v\right)}{2^{-(n/2)-v} \Gamma\left(\frac{n}{2}\right) \Gamma\left(1 - \frac{n}{2}\right)} F\left(\frac{n}{2} + v, \frac{n}{2}; \frac{n}{2}; |x|^2\right) \\ &= \sin\left(\frac{n\pi}{2}\right) \frac{|x|^{(n/2)-1} \Gamma\left(\frac{n}{2} + v\right)}{2^{-(n/2)-v}\pi} (1 - |x|^2)^{-(n/2)-v}, \end{aligned} \tag{3.21}$$

after making the replacements

$$\Gamma\left(\frac{n}{2}\right) \Gamma\left(1 - \frac{n}{2}\right) = \frac{\pi}{\sin(n\pi/2)}$$

and

$$F\left(\frac{n}{2} + v, \frac{n}{2}; \frac{n}{2}; |x|^2\right) = (1 - |x|^2)^{-(n/2)-v}.$$

The last expression results from (3.19). Substituting (3.20) and (3.21) into (3.8) we obtain (3.18) and the theorem is proved. ■

By reasoning in the same manner the following result, whose proof is left to the reader, holds.

THEOREM 3.3. *The inverse Fourier transform of $|\zeta|^{-v} J_\nu(|\zeta|)$ is the distribution defined by*

$$\mathcal{F}^{-1}[|\zeta|^{-v} J_\nu(|\zeta|)](x) = \begin{cases} \frac{(1 - |x|^2)^{v-(n/2)}}{2^v \pi^{(n/2)} \Gamma\left(v - \frac{n}{2} + 1\right)}, & 0 < |x| < 1, \\ 0, & 1 < |x|. \end{cases} \tag{3.22}$$

Remark. From Theorem 3.3 we immediately derive a formula for the inverse Fourier transform of $|\zeta|^v J_{-\nu}(|\zeta|)$, namely,

$$\mathcal{F}^{-1}[|\zeta|^v J_{-\nu}(|\zeta|)](x) = \begin{cases} \frac{(1 - |x|^2)^{-v-(n/2)}}{2^{-v} \pi^{n/2} \Gamma\left(1 - v - \frac{n}{2}\right)}, & 0 < |x| < 1, \\ 0, & 1 < |x|. \end{cases} \tag{3.23}$$

Since

$$\Gamma\left(1 - \nu - \frac{n}{2}\right) \Gamma\left(\nu + \frac{n}{2}\right) = \frac{\pi}{\sin\left[\left(\nu + \frac{n}{2}\right)\pi\right]},$$

we may rewrite (3.23) as

$$\begin{aligned} & \mathcal{F}^{-1}[|\xi|^\nu J_{-\nu}(|\xi|)](x) \\ &= \begin{cases} \frac{2^\nu \sin\left[\left(\nu + \frac{n}{2}\right)\pi\right] \Gamma\left(\nu + \frac{n}{2}\right)}{\pi^{(n/2)+1}} (1 - |x|^2)^{-\nu - (n/2)}, & 0 < |x| < 1, \\ 0, & 1 < |x|. \end{cases} \end{aligned} \quad (3.24)$$

3. The Inverse Fourier Transform of $|\xi|^\nu N_\nu(|\xi|)$

Theorems 3.2 and 3.3 imply the following result

THEOREM 3.4. *The inverse Fourier transform of $|\xi|^\nu N_\nu(|\xi|)$ is the distribution defined by*

$$\begin{aligned} & \mathcal{F}^{-1}[|\xi|^\nu N_\nu(|\xi|)](x) \\ &= \begin{cases} -\cos\left(\frac{n\pi}{2}\right) \frac{2^\nu \Gamma\left(\frac{n}{2} + \nu\right)}{\pi^{(n/2)+1}} (1 - |x|^2)^{-(n/2) - \nu}, & 0 < |x| < 1, \\ -\cos(\nu\pi) \frac{2^\nu \Gamma\left(\frac{n}{2} + \nu\right)}{\pi^{(n/2)+1}} (|x|^2 - 1)^{-(n/2) - \nu}, & 1 < |x|. \end{cases} \end{aligned} \quad (3.25)$$

Proof. Recall that

$$N_\nu(|\xi|) = \frac{J_\nu(|\xi|) \cos(\nu\pi) - J_{-\nu}(|\xi|)}{\sin(\nu\pi)}.$$

Thus, from formulas (3.18) and (3.24) we get

$$\begin{aligned} & \mathcal{F}^{-1}[|\xi|^v N_v(|\xi|)](x) \\ &= \begin{cases} \frac{\cos(v\pi)}{\sin v\pi} \frac{2^v \Gamma\left(\frac{n}{2} + v\right)}{\pi^{(n/2)+1}} (1 - |x|^2)^{-(n/2)-v}, & 0 < |x| < 1, \\ -\cos(v\pi) \frac{2^v \Gamma\left(\frac{n}{2} + v\right)}{\pi^{(n/2)+1}} (|x|^2 - 1)^{-(n/2)-v}, & 1 < |x|, \end{cases} \\ & - \begin{cases} \frac{\sin\left[\left(v + \frac{n}{2}\right)\pi\right]}{\sin(v\pi)} \frac{2^v \Gamma\left(\frac{n}{2} + v\right)}{\pi^{(n/2)+1}} (1 - |x|^2)^{-(n/2)-v}, & 0 < |x| < 1, \\ 0, & 1 < |x|. \end{cases} \end{aligned}$$

Now

$$\sin\left[\left(v + \frac{\pi}{2}\right)\pi\right] = \sin(v\pi) \cos\left(\frac{n\pi}{2}\right) + \cos(v\pi) \sin\left(\frac{n\pi}{2}\right)$$

and so formula (3.25) follows at once. ■

4. A FUNDAMENTAL SOLUTION WITH SUPPORT IN $\overline{D''}$

We return to the problem (2.1) and, in order to obtain a formula for a fundamental solution with support in the hyperbolic region, we have to modify formula (2.13). Consider the function

$$\tilde{F}_-(\xi, y) = \begin{cases} \frac{3^{2/3} \Gamma(4/3)}{2^{1/3}} \left(\frac{t}{|\xi|}\right)^{1/3} J_{1/3}(t|\xi|) & \text{for } y \leq 0 \\ 0, & \text{for } y \geq 0, \end{cases} \tag{4.1}$$

where $t = (2/3)(-y)^{3/2}$. From formula (6.10) in the Appendix I, it follows that the limit of $\tilde{F}_-(\xi, y)$ as $y \rightarrow 0^-$ is equal to zero and so condition (i) in Section 2 holds. Also it follows from the same formula that

$$\partial_y \left\{ \left(\frac{t}{|\xi|}\right)^{1/3} J_{1/3}(t|\xi|) \right\} \Big|_{y=0} = -\frac{2^{1/3}}{3^{2/3} \Gamma(4/3)},$$

hence, the y -derivative of $\tilde{F}_-(\xi, y)$ at $y = 0^-$ is equal to -1 , and so condition (ii) is also satisfied. By calculating the inverse Fourier transform of $\tilde{F}_-(\xi, y)$ we have the following result:

THEOREM 4.1. *The inverse Fourier transform of $\tilde{F}_-(\xi, y)$ is the distribution*

$$F_t(x, y) = \begin{cases} \frac{3^n \Gamma\left(\frac{4}{3}\right)}{2^{2/3} \pi^{n/2} \Gamma\left(\frac{4}{3} - \frac{n}{2}\right)} |9|x|^2 + 4y^3|^{(1/3)-(n/2)} & \text{in } D_-^n \\ 0 & \text{elsewhere,} \end{cases} \quad (4.2)$$

where $D_-^n = \{(x, y) \in \mathbb{R}^{n+1} : 9|x|^2 + 4y^3 < 0\}$. It is a fundamental solution for the operator (1.1) whose support is the closure of the region D_-^n .

Proof. We must evaluate the inverse Fourier transform of $(t/|\xi|)^{1/3} J_{1/3}(t|\xi|)$. Recall that if $G(x)$ is the inverse Fourier transform of $f(\xi)$, then $(1/a^n) G(x/a)$ is the inverse Fourier transform of $f(a\xi)$. By applying formula (3.22) for $\nu = 1/3$, we obtain

$$\begin{aligned} & \mathcal{F}^{-1}[(t/|\xi|)^{1/3} J_{1/3}(t|\xi|)](x) \\ &= \begin{cases} \frac{t^{(2/3)-n}}{2^{1/3} \pi^{n/2} \Gamma\left(\frac{4}{3} - \frac{n}{2}\right)} \left(1 - \frac{|x|^2}{t^2}\right)^{(1/3)-(n/2)}, & 0 < |x| < t, \\ 0, & t < |x|. \end{cases} \\ &= \begin{cases} \frac{1}{2^{1/3} \pi^{n/2} \Gamma\left(\frac{4}{3} - \frac{n}{2}\right)} (t^2 - |x|^2)^{(1/3)-(n/2)}, & 0 < |x| < t, \\ 0, & t < |x|. \end{cases} \\ &= \begin{cases} \frac{3^{n-(2/3)}}{2^{1/3} \pi^{n/2} \Gamma\left(\frac{4}{3} - \frac{n}{2}\right)} |9|x|^2 + 4y^3|^{(1/3)-(n/2)} & \text{in } D_-^n \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

Multiplication by the constant $3^{2/3} \Gamma(4/3)/2^{1/3}$ in formula (4.1) yields (4.2) which proves the theorem. \blacksquare

Remarks. **1.** If $n = 1$, then $F_-(x, y)$ coincides with the distribution defined by formula (1.4).

2. For all values of n the support of $F_-(x, y)$ is the closure of the region D_-^n . This follows from results in [4] about the generalized function P_+^λ where $P(x)$ is the quadratic polynomial

$$P(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2,$$

$p, q \geq 1$. See also the example in [8] about the distribution Pf. s^λ , where s denotes the hyperbolic distance in \mathbb{R}^N , $N > 1$.

5. A FUNDAMENTAL SOLUTION WITH SUPPORT IN $\overline{D_+^n}$

Consider the function $\tilde{F}_+(\xi, y)$ defined by

$$\tilde{F}_+(\xi, y) = \begin{cases} \gamma \cdot \left(\frac{s}{|\xi|}\right)^{1/3} K_{1/3}(x|\xi|) & \text{if } y \geq 0 \\ \delta \cdot \left(\frac{t}{|\xi|}\right)^{1/3} N_{-1/3}(t|\xi|) & \text{if } y \leq 0, \end{cases} \tag{5.1}$$

where $s = (2/3) y^{3/2}$, $t = (2/3)(-y)^{3/2}$, and the constants γ and δ are respectively given by

$$\gamma = -\frac{2^{2/3}}{3^{4/3}\Gamma(2/3)} \quad \text{and} \quad \delta = \frac{2\pi}{2^{1/3}3^{4/3}\Gamma(2/3)}, \tag{5.2}$$

and where $N_{-1/3}$ is the Neumann function defined by (6.4). By using the formulas in the Appendix I, it is a matter of verification that the conditions (i) and (ii) in Section 2 are satisfied. Thus its inverse Fourier transform, denoted by $F^\#(x, y)$, defines a fundamental solution for the operator (1.1). The following theorem gives an explicit expression for $F^\#(x, y)$.

THEOREM 5.1. *The inverse Fourier transform of $\tilde{F}_+(\xi, y)$ is the distribution defined by*

$$F^\#(x, y) = \begin{cases} -\frac{3^{n-2}}{2^{2/3}\pi^{n/2}} \frac{\Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{\Gamma(2/3)} (9|x|^2 + 4y^3)^{(1/3)-(n/2)} & \text{in } D_+^n \\ -\cos\left(\frac{n\pi}{2}\right) \frac{2^{1/3}3^{n-2}}{\pi^{n/2}} \frac{\Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{\Gamma(2/3)} |9|x|^2 + 4y^3|^{(1/3)-(n/2)} & \text{in } D_-^n. \end{cases} \tag{5.3}$$

Proof. 1. We start by evaluating the inverse Fourier transform of $(s/|\xi|)^{1/3} K_{1/3}(s|\xi|)$, $s = 2y^{3/2}/3$, $y \geq 0$. Since $K_\nu(z) = K_{-\nu}(z)$, it follows from formula (3.5) that

$$\mathcal{F}^{-1}[(s/|\xi|)^{1/3} K_{1/3}(s|\xi|)] = \frac{2^{-4/3} \Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{\pi^{n/2}} (s^2 + |x|^2)^{(1/3) - (n/2)}.$$

After multiplying by the constant γ in (5.2) and reintroducing the variables x and y , we can see that the right-hand side of the last expression is equal to

$$-\frac{3^{n-2} \Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{2^{2/3} \pi^{n/2} \Gamma(2/3)} (9|x|^2 + 4y^3)^{(1/3) - (n/2)}, \quad y \geq 0, \quad (5.4)$$

that if $F^\#(x, y)$ in $D_+^n \cap \{y \geq 0\}$.

2. Next we evaluate the inverse Fourier transform of $(t/|\xi|)^{1/3} N_{-1/3}(t|\xi|)$, $t = 2(-y)^{3/2}/3$, $y \leq 0$. From formula (3.25) we obtain

$$\begin{aligned} & \mathcal{F}^{-1}[(t|\xi|)^{-1/3} N_{1/3}(t|\xi|)] \\ &= \begin{cases} -\cos\left(\frac{n\pi}{2}\right) \frac{2^{-1/3} \Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{\pi^{(n/2)+1}} (t^2 - |x|^2)^{(1/3) - (n/2)}, & 0 < |x| < t, \\ -\left(\frac{1}{2}\right) \frac{2^{-1/3} \Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{\pi^{(n/2)+1}} (|x|^2 - t^2)^{(1/3) - (n/2)}, & t < |x|. \end{cases} \end{aligned}$$

After multiplying both sides by the constant δ in (5.2) and reverting to the variables x and y , we can see that the right-hand side of the last expression can be written as

$$\begin{cases} -\frac{3^{n-2} \Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{2^{2/3} \pi^{n/2} \Gamma(2/3)} (9|x|^2 + 4y^3)^{(1/3) - (n/2)}, & \text{in } D_+^n \cap \{y \leq 0\} \\ -\cos\left(\frac{n\pi}{2}\right) \frac{2^{1/3} 3^{n-2} \Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{\pi^{n/2} \Gamma(2/3)} |9|x|^2 + 4y^3|^{(1/3) - (n/2)}, & \text{in } D_-^n, \end{cases} \quad (5.5)$$

and so (5.5) coincides with $F^\#(x, y)$ in $D_+^n \cap \{y \leq 0\} \cup D_-^n$. Therefore, from (5.4) and (5.5) we obtain (5.3) and the theorem is proved. ■

Remarks. 1. In view of the results in [4] relative to the generalized function P_+^λ (see Remark 2 after the proof of Theorem 4.1) the supports of the distributions

$$(9|x|^2 + 4y^3)^{(1/3)-(n/2)} \quad \text{and} \quad |9|x|^2 + 4y^3|^{(1/3)-(n/2)}$$

that appear in formula (5.3) are the closures of D_+^n and D_-^n , respectively. In other words, for no value of n can these supports be just the boundaries of these regions.

2. If n is *odd*, then $\cos(n\pi/2) = 0$, and we rewrite formula (5.3), using the notation $F_+(x, y)$ instead of $F^\#(x, y)$, as follows:

$$F_+(x, y) = \begin{cases} -\frac{3^{n-2}}{2^{2/3}\pi^{n/2}} \frac{\Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{\Gamma(2/3)} (9|x|^2 + 4y^3)^{(1/3)-(n/2)} & \text{in } D_+^n \\ 0 & \text{elsewhere.} \end{cases} \tag{5.6}$$

This is a fundamental solution whose support is $\overline{D_+^n}$. In particular, if $n = 1$ we obtain formula (1.3).

3. If n is *even*, then $F^\#(x, y)$ is not necessarily identically zero in D_-^n and its support may be the whole of \mathbb{R}^{n+1} . Suppose that $n = 2k, k > 0$, and let us compare the constant in formula (5.3), relative to the region D_-^n , to the constant in the expression (4.2) of $F_-(x, y)$. Let

$$A = \frac{(-1)^{k+1} 2^{1/3} 3^{2(k-1)}}{\pi^k} \frac{\Gamma(k - \frac{1}{3})}{\Gamma(2/3)}$$

be the constant in (5.3) and let

$$B = \frac{3^{2k}}{2^{2/3}\pi^k} \frac{\Gamma(4/3)}{\Gamma(\frac{4}{3} - k)} = \frac{(-1)^{k+1} 3^{2k-1}}{2^{2/3}\pi^k} \frac{\Gamma(k - \frac{1}{3})}{\Gamma(2/3)}$$

be the constant in (4.2). Since $3A - 2B = 0$ it follows that the distribution

$$F_+(x, y) = 3F^\#(x, y) - 2F_-(x, y) \tag{5.7}$$

is now a fundamental solution for the operator (1.1) supported in $\overline{D_+^n}$ and we have for this $F_+(x, y)$ the same expression as that of (5.6). In conclusion, for all values on n we always get two fundamental solutions: one whose support is the closure of D_+^n and another whose support is the closure of D_-^n .

APPENDIX

I. *Bessel Functions*

The function $J_\nu(z)$ of a complex variable z defined by

$$J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{\nu+2r}}{2^{\nu+2r} r! \Gamma(\nu+r+1)}, \quad |z| < \infty, \quad |\arg z| < \pi, \quad (\text{A.1})$$

is called the *Bessel function of the first kind of order ν* .

We also need the Bessel functions $I_\nu(z)$ and $K_\nu(z)$ defined by

$$I_\nu(z) = \sum_{r=0}^{\infty} \frac{z^{\nu+2r}}{2^{\nu+2r} r! \Gamma(\nu+r+1)}, \quad |z| < \infty, \quad |\arg z| < \pi, \quad (\text{A.2})$$

and

$$K_\nu(z) = \frac{\pi \csc(\nu\pi)}{2} \{I_{-\nu}(z) - I_\nu(z)\}, \quad \nu \neq 0, \pm 1, \pm 2, \dots \quad (\text{A.3})$$

as well as the *Neumann function*

$$N_\nu(z) = \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}. \quad (\text{A.4})$$

Note that throughout this work, we only deal with Bessel functions of order $\pm 1/3$. Recall that $Ai(z)$ was defined in formula (2.6) by

$$Ai(z) = \frac{z^{1/2}}{3} \left[I_{-1/3} \left(\frac{2}{3} z^{3/2} \right) - I_{1/3} \left(\frac{2}{3} z^{3/2} \right) \right] = \frac{1}{\pi} \left(\frac{z}{3} \right)^{1/2} K_{1/3} \left(\frac{2}{3} z^{3/2} \right).$$

If we set $s = \frac{2}{3} z^{3/2}$, then we may rewrite $Ai(z)$ as

$$Ai(z) = \frac{1}{3^{2/3} 2^{1/3}} s^{1/3} [I_{-1/3}(s) - I_{1/3}(s)]. \quad (\text{A.5})$$

From the series expansion of $I_\nu(z)$ it follows that

$$s^{1/3} I_{-1/3}(s) = \frac{1}{2^{-1/3} \Gamma(2/3)} + \frac{s^2}{2^{5/3} \Gamma(5/3)} + \dots \quad (\text{A.6})$$

and

$$s^{1/3} I_{1/3}(s) = \frac{s^{2/3}}{2^{1/3} \Gamma(4/3)} + \frac{s^{8/3}}{2^{7/3}} + \dots \quad (\text{A.7})$$

Consequently from (A.5), (A.6), and (A.7) we obtain

$$Ai(0) = \frac{3^{-2/3}}{\Gamma(2/3)},$$

the first expression in formula (2.8). Similarly, by differentiating $Ai(z)$ and setting $z = 0$, we get the second expression in (2.8)

$$Ai'(0) = -\frac{3^{-4/3}}{\Gamma(4/3)}.$$

In an analogous way, recall that $Bi(z)$ in (2.6) was defined by

$$Bi(z) = \left(\frac{z}{3}\right)^{1/2} \left[I_{-1/3} \left(\frac{2}{3} z^{3/2}\right) + I_{1/3} \left(\frac{2}{3} z^{3/2}\right) \right].$$

If we set $s = \frac{2}{3}z^{3/2}$, then the last expression becomes

$$Bi(z) = \frac{1}{2^{1/3}3^{1/6}} s^{1/3} [I_{-1/3}(s) + I_{1/3}(s)], \tag{A.8}$$

and again from (A.6) and (A.7) we obtain

$$Bi(0) = \frac{3^{-1/6}}{\Gamma(2/3)}, \quad Bi'(0) = \frac{3^{-5/6}}{\Gamma(4/3)},$$

which are the two expressions in (2.9).

Finally, from these results we obtain the value of the Wronskian of $Ai(z)$ and $Bi(z)$ at $z = 0$:

$$W(Ai(z), Bi(z))|_{z=0} = \frac{2 \cdot 3^{-3/2}}{\Gamma(2/3) \Gamma(4/3)} = 1/\pi,$$

because $\Gamma(2/3) \Gamma(4/3) = 2\pi \cdot 3^{-3/2}$.

For future reference, we also need the following expansions derived from the series that defines $J_\nu(z)$:

$$t^{1/3}J_{-1/3}(t) = \frac{1}{2^{-1/3}\Gamma(2/3)} - \frac{t^2}{2^{5/3}\Gamma(5/3)} + \dots \tag{A.9}$$

and

$$t^{1/3}J_{1/3}(t) = \frac{t^{2/3}}{2^{1/3}\Gamma(4/3)} - \frac{t^{8/3}}{2^{7/3}\Gamma(7/3)} + \dots \tag{A.10}$$

II. Hypergeometric Series and Functions

Let a , b , and c be arbitrary complex numbers and let z be a complex variable. The power series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \quad (\text{A.11})$$

where

$$(a, 0) = 1, \quad (\text{A.12})$$

$$(a, n) = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1) \cdots (a+n-1), \quad n = 1, 2, \dots$$

and we assume that $c \neq 0, -1, -2, \dots$, is called a *hypergeometric series*. It is known [12] that the series (A.11) is a solution, valid near $z=0$, of the *hypergeometric equation*

$$z(1-z) \frac{d^2u}{dz^2} + \{c - (a+b+1)z\} \frac{du}{dz} - abu = 0, \quad (\text{A.13})$$

for which every point is an ordinary point, except 0, 1, and ∞ , that are regular singular points.

If either a or b is a negative integer, then the series (A.11) terminates; if c is a negative integer, the series is meaningless because all terms after the $(1-c)$ th have a zero denominator. As it is known [3], it is possible to redefine the series (A.11) so that it still is a solution of the hypergeometric equation. We exclude this possibility from our considerations because, in the cases that interest us, c is never a negative integer.

The hypergeometric series is absolutely convergent for $|z| < 1$ and so defines, in the open disk, an analytic function of z which is regular at $z=0$. The point $z=1$ is however a branch point and if a cut is made from 1 to $+\infty$ along the x -axis, it can be shown [12] that series can be continued analytically and defines an analytic function throughout the cut plane that we still denote by $F(x, b; c; z)$. If $\text{Re}(c) > \text{Re}(b) > 0$, this analytic extension can be represented by Euler's formula

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} (1-tz)^{-a} dt, \quad (\text{A.14})$$

for $|\arg(1-z)| < \pi$.

In general, the hypergeometric series (A.11) diverges for $|z| = 1$. However, if $\operatorname{Re}(c - a - b) > 0$, we have absolute convergence for $|z| = 1$. Moreover,

$$F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}. \quad (\text{A.15})$$

Hypergeometric functions satisfy among themselves quite a number of important relations of which we just list the following two that we used in Section 3:

$$F(a, b; c; z) = (1 - z)^{-a} F\left(a, c - b; c; \frac{z}{z - 1}\right) \quad (\text{A.16})$$

and

$$F(a, b; c; z) = (1 - z)^{c - a - b} F(c - a, c - b; c; z). \quad (\text{A.17})$$

For a complete list of such relations, the reader should consult Erdély [3].

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