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Elimination of Variables in Linear Solvable Polynomial Algebras and ∂-Holonomicity

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Let k be a field of characteristic 0. Based on the Gelfand–Kirillov dimension computation of modules over solvable polynomial k-algebras, where solvable polynomial algebras are in the sense of A. Kandri-Rody and V. Weispfenning (1990, *J. Symbolic Comput.* **9**, 1–26), we prove that the elimination lemma, obtained from D. Zeilberger (1990, *J. Comput. Appl. Math.* **32**, 321–368) by using holonomic modules over the Weyl algebra $A_n(k)$ and used in the automatic proving of special function identities, holds for a class of solvable polynomial k-algebras without any "holonomicity" restriction. This opens a way to the solution of the extension/contraction problem stemming from the automatic proving of multivariate identities with respect to the ∂-finiteness in the sense of F. Chyzak and B. Salvy (1998, *J. Symbolic Comput.* **26**, 187–227). It also yields a ∂-holonomicity so that automatic proving of multivariate identities may be dealt with by manipulating polynomial function coefficients instead of rational functions.

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0. INTRODUCTION

Throughout this paper we let k denote a computable subfield of the complex number field $\mathbb{C}$. A field k is said to be *computable* if the addition, subtraction, multiplication, and the inverses of elements in k can be performed on a computer [BW, p. 78]. All rings considered in this paper are associative rings with 1, and modules are unitary *left* modules. If A is a ring and S is a nonempty subset of A, we write $\langle S]$ for the *left* ideal of A generated by S.

Holonomic systems or, equivalently, *holonomic modules* over the Weyl algebras were introduced by Bernstein and Kashiwara in the algebraic study of the solutions of linear differential equations. The earliest grand application of holonomic module theory was given by Bernstein [Ber] for an elementary algebraic proof of a famous conjecture of Gelfand concerning the existence of a meromorphic extension of the distribution valued complex function $\lambda \to P^{\lambda}$, where P is a polynomial in several variables over $\mathbb{R}^n$. Based on Bernstein's celebrated holonomic module theory over the Weyl algebras, a large class of *special function* identities including all terminating hypergeometric (alias binomial coefficient) identities has been identified by Zeilberger in the automatic proving of function identities (see [Zei1]). These special functions are named *holonomic functions*. Using noncommutative Gröbner basis theory, more general automatic proving of multivariate identities has been considered (e.g., [CS]) in the context of iterated Ore extensions.

More precisely, let k be a subfield of $\mathbb{C}$ (in computer algebra k is usually taken to be a finite dimensional extension of $\mathbb{Q}$ in $\mathbb{C}$), and let $A_n(k)$ be the nth Weyl algebra over k which is by definition the k-algebra generated by $2n$ elements $\{x_1, \dots, x_n, \partial_1, \dots, \partial_n\}$ subject to the defining relations

$$[x_i, x_j] = [\partial_i, \partial_j] = 0, \qquad i, j = 1, \dots, n,$$
$$\partial_j x_i = x_i \partial_j + \delta_{ij}, \quad i, j = 1, \dots, n.$$

We have

0.1. Definition [Zei1]. Let f be a nonzero member of a family on which the Weyl algebra $A_n(k)$ acts naturally. Put

$$\mathcal{I}_f = \{D \in A_n(k) \mid Df = 0\}.$$

(It is clear that $\mathcal{I}_f$ is a left ideal of $A_n(k)$.) f is said to be a *holonomic function* if the $A_n(k)$-module $A_n(k)/\mathcal{I}_f$ has Gelfand–Kirillov dimension n, denoted by $\text{GK.dim}(A_n(k)/\mathcal{I}_f) = n$, i.e., if $A_n(k)/\mathcal{I}_f$ is a holonomic $A_n(k)$-module in the sense of Bernstein. (We refer to [Bj] for a general theory of holonomic modules over $A_n(k)$.) Examples of holonomic functions may be found, e.g., in [Stan, Zei1, WZ, Lip1–2, CS]: D-finite functions, rational functions $1/P$ where P is a nonzero polynomial in $n \geq 1$ variable(s), all algebraic functions, and all special functions falling in the Askey's scheme [J. Labelle, Tableau d'Askey, in "Polynômes Orthogonauxet Applications," (C. Brezinski et al., Eds.), LNM, Vol. 1171, Springer-Verlag, Berlin/New York, 1985].

Remark. (i) For some historical comments about "special functions" we refer to the proceedings "*Special Funtions: Group Theoretical Aspects and Applications*" (R. A. Askey, T. H. Koornwinder, and W. Schempp, Eds., Reidel, Dordrecht, 1984), and the proceedings of the Hayashibara Forum "*Special Functions*" (M. Kashiwara and T. Miwa, Eds., Springer-Verlag, Berlin/New York, 1991). For an introduction to the automatic proving of function identities we refer to the book written by M. Petkovšek, H. Wilf, and D. Zeilberger ("$A = B$," Peters, 1996).

(ii) Historically, a unified study of the "D-finite functions" and the "P-recursive sequences" was introduced by Stanley [Stan] in the following sense:

- A D-finite function is a solution of a polynomial-coefficient linear ordinary differential equation.
- A P-recursive sequence is a sequence $(u_n)_{n\in\mathbb{N}}$ satisfying linear recurrences with polynomial coefficients; e.g.,

$$P_0(n)u_n + P_1(n)u_{n+1} + P_2(n)u_{n+2} = 0, \qquad n \in \mathbb{N}.$$

- A sequence $(u_n)_{n\in\mathbb{N}}$ is P-recursive if and only if its corresponding generating function $f(x) = \sum_{n=0}^{\infty} u_n x^n$ is D-finite.

In the case of several variables, a similar fusion of the discrete and the continuous succeeded in [Lip2]. Zeilberger used "holonomic function" in the sense of Bernstein and Kashiwara (as defined above) for both "D-finite function" and "P-recursive sequence," as he said in [Zei1]: *In the interest of thawing the cold war between the discrete and the continuous, I have decided to combine these two names into one*.

Holonomic functions form a k-algebra in the usual sense. From a computational point of view, the most important property of a holonomic function is that a holonomic function can be recovered from a finite amount of information. This property may be described more clearly as follows (see [Zei1]). To write down a holonomic function f in "holonomic notation," we may give any set of generators of its corresponding ideal, with the appropriate (finitely many) initial conditions. However, in general it is not clear how many initial conditions are required to uniquely specify the function

f. It is therefore necessary to introduce the so called "canonical holonomic representation" by finding n "ordinary" operators

$$P_i(\partial_i, x_1, \ldots, x_n), \qquad i = 1, \ldots, n, \tag{1}$$

in the Weyl algebra $A_n(k) = k[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ that annihilate f, where each P_i is of degree α_i (in ∂_i). The "initial conditions" are given by

$$\partial_1^{i_1} \cdots \partial_n^{i_n} f(x_0), \qquad 0 \le i_1 < \alpha_1, \ldots, 0 \le i_n < \alpha_n, \tag{2}$$

where x_0 is any point that is not on the "characteristic set" of the system (1) (the characteristic set of a system (1) is the set of common zeros of the leading coefficients of the operators P_i). The wonderful thing is that the existence of system (1) is guaranteed by the following elimination lemma in the Weyl algebra and is recognized by the noncommutative version of Sylvester's dialytic elimination algorithm given by Zeilberger [Zei1]. Moreover, a good "canonical holonomic representation" may be given in terms of Gröbner bases [Tak1–2].

0.2 LEMMA (elimination lemma for the Weyl algebras [Zei1, Lemma 4.1]; proof is due to Bernstein). *Let L be a left ideal in $A_n(k)$ such that $A_n(k)/L$ is a holonomic $A_n(k)$-module. For every $n+1$ generator selected from the $2n$ generators $\{x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\}$ of $A_n(k)$ there is a nonzero member of L that only depends on these $n+1$ generators. In particular, for every $i = 1, \ldots, n$, L contains a nonzero element of the subalgebra $k[x_1, \ldots, x_n, \partial_i] \subset A_n(k)$.*

Remark. In order to provide "real fast" elimination algorithms for the automatic proving of hypergeometric (ordinary and "q") multisum/integral identities, the above lemma was given another, more "effective" proof and was called the "fundamental lemma" in [WZ].

Let $k(x_1, \ldots, x_n)$ be the field of rational functions in n variables and let $B_n(k) = k(x_1, \ldots, x_n)[\partial_1, \ldots, \partial_n]$ be the k-algebra of linear differential operators with rational function coefficients. It is clear that $A_n(k) \subset B_n(k)$.

The relation between the holonomic $A_n(k)$-modules and the $B_n(k)$-modules which are finite dimensional over $k(x_1, \ldots, x_n)$ is given in the following theorem. The importance of this theorem in the automatic proving of holonomic function identities is that it allows the reduction of rational function manipulations to polynomial functions (e.g., [Tak3]). The proof of the "only if" part is given by Bernstein [Ber] and the proof of the "if" part follows from a result of Kashiwara [Kas] concerning holonomic D-modules (an elementary proof was given by Takayama in [Tak3] but this depends again on the nature of holonomic modules).

0.3. THEOREM (Bernstein–Kashiwara). *With notation as above, let $\mathcal{J}$ be a left ideal of $B_n(k)$. Then $M = A_n(k)/(\mathcal{J} \cap A_n(k))$ is a holonomic $A_n(k)$-module if and only if $B_n(k)/\mathcal{J}$ is a finite dimensional vector space over $k(x_1, \dots, x_n)$.*

In this paper we show that the elimination lemma holds for a class of solvable polynomial algebras without any "holonomicity" restriction (Lemma 3.11), where solvable polynomial algebras are defined in the sense of [K-RW]. This makes it possible to solve the extension/contraction problem raised in the automatic proving of multivariate identities with respect to the ∂-finiteness in the sense of [CS] (Theorem 4.3). Also it yields a "∂-holonomicity" for modules (functions) over the algebras under consideration (Section 5), which includes the holonomicity of modules over Weyl algebras. Again, in connection with the automatic proving of multivariate identities it allows the reduction of rational function manipulations to polynomial function manipulations in a more general context. From the work of [CS] it is clear that these results make a more effective use of Gröbner bases for eliminating variables and for constructing algorithms such as *creative telescoping* in the automatic proving of multivariate identities. (In the Weyl algebra case *creative telescoping* is an algorithm written for computing equations satisfied by definite sums or integrals of holonomic functions, e.g., [Zei2]. In [CS] this algorithm has been extended to a more general context.) In this paper we are not going to explore this algorithmic aspect (because we are not experts in computation). However, our proofs of the main results depend on an algorithmic study of dimension theory over solvable polynomial algebras having its root in commutative algebraic geometry (see the remark given in the end of Section 3).

1. BASIC PROPERTIES OF SOLVABLE POLYNOMIAL ALGEBRAS

For a general theory of Gröbner bases in commutative polynomial algebras, we refer to [CLO′, BW]; for the general theory of Gröbner bases in noncommutative solvable polynomial algebras, we refer to the survey paper [K-RW]. For the reader's convenience we recall some basic properties of a left Gröbner basis in a solvable polynomial algebra.

First we give some basic notions and notation. Let $\mathbb{Z}^n_{\geq 0}$ be the set of n-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

By a *monomial ordering* on $\mathbb{Z}^n_{\geq 0}$ we mean any relation $>$ on $\mathbb{Z}^n_{\geq 0}$ satisfying

(1) $>$ is a total (or linear) ordering on $\mathbb{Z}^n_{\geq 0}$.

(2) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.

(3) $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$.

The *lexicographic* ordering on $\mathbb{Z}^n_{\geq 0}$, denoted $>_{\text{lex}}$, is a monomial ordering. Another monomial ordering very often used in computational algebra is the *graded lexicographic* ordering on $\mathbb{Z}^n_{\geq 0}$, denoted $>_{\text{grlex}}$, which is defined as follows: for $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $\alpha >_{\text{grlex}} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \qquad \text{or } |\alpha| = |\beta| \quad \text{and} \quad \alpha >_{\text{lex}} \beta,$$

where $>_{\text{lex}}$ is the lexicographic ordering on $\mathbb{Z}^n_{\geq 0}$.

Let k be a subfield of $\mathbb{C}$, and let $A = k[a_1, \dots, a_n]$ be an affine k-algebra with generating set $\{a_1, \dots, a_n\}$. We call an element of the form $a_1^{\alpha_1} \cdots a_n^{\alpha_n}$ a *standard monomial* in A, where $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, and write

$$\mathrm{SM}(A) = \left\{ a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \,\middle|\, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0} \right\}$$

for the set of all standard monomials. It is clear that any ordering $>$ on $\mathbb{Z}^n_{\geq 0}$ naturally induces an ordering on $\mathrm{SM}(A)$: $a^\alpha > a^\beta$ if and only if $\alpha > \beta$.

An element of the form $c_\alpha a^\alpha \in A$ with $c_\alpha \in k$ and $a^\alpha \in \mathrm{SM}(A)$ is called a *term* in A. We write

$$\mathrm{T}(A) = \left\{ c_\alpha a^\alpha \mid c_\alpha \in k,\ a^\alpha \in \mathrm{SM}(A) \right\}$$

for the set of all terms in A.

Suppose that

(S1) $\mathrm{SM}(A)$ is a k-basis for A.

Then every $f \in A$ has a unique linear expression $f = \sum c_\alpha a^\alpha$, $c_\alpha \in k$, $a^\alpha \in \mathrm{SM}(A)$. Let $>$ be a monomial ordering on $\mathbb{Z}^n_{\geq 0}$. Then we may define

- the *multidegree* of f: $\mathrm{md}(f) = \max\{\alpha \mid c_\alpha \neq 0\}$,
- the *leading monomial* of f: $\mathrm{LM}(f) = a^\alpha$ with $\alpha = \mathrm{md}(f)$,
- the *leading term* of f: $\mathrm{LT}(f) = c_\alpha a^\alpha$ with $\alpha = \mathrm{md}(f)$.

1.1. Definition [K-RW, LW]. An affine k-algebra $A = k[a_1, \dots, a_n]$ is called a *solvable polynomial algebra* with monomial ordering $>$, if A satisfies (S1) and

(S2) if $a^\alpha, a^\beta \in \mathrm{SM}(A)$, then $a^\alpha a^\beta = c_{\alpha,\beta} a^{\alpha+\beta} + \sum c_\gamma a^\gamma$ with $c_{\alpha,\beta} \in k - \{0\}$, $c_\gamma \in k$, and $\alpha + \beta > \mathrm{md}(\sum c_\gamma a^\gamma)$.

Weyl algebras $A_n(k)$ over k and the enveloping algebras of finite dimensional k-Lie algebras are examples of solvable polynomial algebras. More generally, any iterated Ore extension over k is a solvable polynomial algebra (see [K-RW]).

From now on, we always let $A = k[a_1, \dots, a_n]$ denote a solvable polynomial k-algebra with a fixed monomial ordering $>$ on $\mathbb{Z}^n_{\geq 0}$ (or equivalently, on $\mathrm{SM}(A)$).

1.2. LEMMA. *If* $f, g \in A$, *then* $\mathrm{md}(fg) = \mathrm{md}(f) + \mathrm{md}(g)$, $\mathrm{LT}(fg) = \mathrm{LT}(\mathrm{LT}(f)\mathrm{LT}(g))$.

The above lemma shows that the monomial ordering $>$ is *compatible* with the multiplication of A in the sense that if $a^\alpha > a^\beta$ and $a^\gamma \in \mathrm{SM}(A)$, then $\mathrm{LM}(a^\gamma a^\alpha) > \mathrm{LM}(a^\gamma a^\beta)$. It is this property and the well-ordering property of $>$ that make the division algorithm and a noncommutative version of Buchberger's algorithm in A possible (see [K-RW]).

On $\mathbb{Z}^n_{\geq 0}$ we have the Dickson partial order (see [BW]): $\alpha \leq' \beta$ if and only if there exists some $\gamma \in \mathbb{Z}^n_{\geq 0}$ such that $\alpha + \gamma = \beta$. So, for $a^\alpha, a^\beta \in \mathrm{SM}(A)$, we say that a^β *is divisible by* a^α if $\alpha \leq' \beta$. Thus, for two terms $c_\alpha a^\alpha, c_\beta a^\beta \in \mathrm{T}(A)$, we may also say that $c_\beta a^\beta$ *is divisible by* $c_\alpha a^\alpha$ if $\alpha \leq' \beta$.

Warning: Generally, in a solvable polynomial k-algebra, if a^β is divisible by a^α, then it does not mean that $a^\beta = a^\gamma a^\alpha$, for some $\gamma \in \mathbb{N}^n_{\geq 0}$.

1.3. (LEFT) DIVISION ALGORITHM. Let $\mathscr{F} = \{f_1, \dots, f_s\}$ be an ordered s-tuple of elements in A. Then every $f \in A$ can be written as

$$f = h_1 f_1 + \cdots + h_s f_s + r, \tag{$*$}$$

where $h_i, r \in A$, and either $r = 0$ or r is a k-linear combination of standard monomials, none of which is divisible by any term in $\mathrm{LM}(\mathscr{F}) = \{\mathrm{LM}(f_1), \dots, \mathrm{LM}(f_s)\}$. Moreover, if $h_i f_i \neq 0$, then $\mathrm{md}(f) \geq \mathrm{md}(h_i f_i)$.

The element r appearing in $(*)$ is called the *remainder* of f on division by $\mathscr{F}$, and is denoted by $\overline{f}^{\mathscr{F}}$.

Let L be a left ideal of A, and let $\mathscr{G} = \{g_1, \dots, g_s\}$ be an ordered s-tuple of elements in L. If for every $f \in L$, $\overline{f}^{\mathscr{G}} = 0$, then $\mathscr{G}$ is called a *left Gröbner basis* of L (or just a Gröbner basis if no confusion is possible).

1.4. THEOREM [K-RW]. (i) *A Gröbner basis exists for every nonzero left ideal L in A* (*hence A is Noetherian*). *Furthermore*, *if the ground field is computable*, *then a Gröbner basis containing a given generating set of L may be computed in terms of the S-polynomials by using a noncommutative version of Buchberger's algorithm*.

(ii) *Let* $\mathscr{G} = \{g_1, \dots, g_s\}$ *be a Gröbner basis of L. Then for* $f \in A$, $f \in L$ *if and only if* $\overline{f}^{\mathscr{G}} = 0$.

Let L be a left ideal of A. Concerning the k-space A/L, we have the following fact.

1.5. PROPOSITION [K-RW]. *Let $\mathscr{G} = \{g_1, \dots, g_s\}$ be a Gröbner basis of L, and let*

$$B = \left\{a^\alpha \in \mathrm{SM}(A) \,\middle|\, a^\alpha \text{ is not divisible by any } \mathrm{LM}(g_i)\right\}.$$

Then

(i) *$B' = \{[a^\alpha] \mid a^\alpha \in B\}$ is a k-basis of A/L, where $[a^\alpha]$ denotes the coset of a^α modulo L.*

(ii) *A'/L is finite dimensional if and only if for every $1 \le i \le n$, there exists some $g \in \mathscr{G}$ such that $\mathrm{LM}(g)$ is a power of a_i.*

2. ELIMINATION LEMMA FOR HOMOGENEOUS SOLVABLE POLYNOMIAL ALGEBRAS

In the definition of a solvable polynomial k-algebra (Section 1), if we replace the condition (S2) by

(S2′) if $a^\alpha, a^\beta \in \mathrm{SM}(A)$, then $a^\alpha a^\beta = c_{\alpha,\beta} a^{\alpha+\beta}$ with $c_{\alpha,\beta} \in k - \{0\}$,

then it is clear that A is a solvable polynomial algebra with respect to *any monomial order* on $\mathbb{Z}^n_{\ge 0}$. We call such A a *homogeneous solvable polynomial algebra*.

One of the most obvious examples of a homogeneous solvable polynomial algebra is the skew polynomial ring of the form $k[x][y, \sigma]$, where $k[x]$ is the commutative polynomial ring over k in one variable x, σ is an automorphism of $k[x]$ with $\sigma(x) = \lambda x$, $\lambda \in k - \{0\}$, and $yx = \sigma(x)y = \lambda xy$. Another common example is the *quantum n-space* in the sense of Manin, i.e., the $\mathbb{C}$-algebra $\mathscr{O}_q = \mathbb{C}[X_1, \dots, X_n]$ defined by the relations $X_j X_i = q^{-2} X_i X_j$ whenever $i < j$, where q is a nonzero element of $\mathbb{C}$.

In this section, we prove the elimination lemma for a homogeneous solvable polynomial algebra $A = k[a_1, \dots, a_n]$ by further exploring the dimension theory over A which has been studied in [Li]. To this end, we first need to recall from [Li] some basic results concerning left Gröbner bases in A and the dimension computation of a cyclic A-module.

2.1. LEMMA. (i) *A is a positively graded affine k-algebra, i.e., $A = \bigoplus_{m \ge 0} A_m$ with*

$$A_m = \left\{ \sum_{|\alpha|=m} c_\alpha a^\alpha \,\middle|\, a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n},\ |\alpha| = \alpha_1 + \cdots + \alpha_n \right\};$$

in particular, $A_0 = k$, $A_1 = \sum_{i=1}^n k a_i$.

(ii) *With notation as in Section* 1, *if* $c_\alpha a^\alpha, c_\beta a^\beta \in \mathrm{T}(A)$, *then* $c_\beta a^\beta$ *is divisible by* $c_\alpha a^\alpha$ *if and only if there is some* $c_\gamma a^\gamma \in \mathrm{T}(A)$ *such that* $c_\beta a^\beta = c_\alpha c_\gamma a^\gamma a^\alpha$.

A left ideal L of A is said to be a *monomial left ideal* if it has a generating set consisting of standard monomials in $\mathrm{SM}(A)$; i.e., $L = \sum_{\alpha\in I} Aa^\alpha$, $I \subset \mathbb{Z}^n_{\geq 0}$, $a^\alpha \in \mathrm{SM}(A)$. By Lemma 2.1 the following two properties of a monomial left ideal are the noncommutative analogue of those for a commutative monomial ideal in the commutative polynomial algebra $k[x_1, \dots, x_n]$ (e.g., see [CLO′]).

2.2. LEMMA. *Let* $L = \sum_{\alpha\in I} Aa^\alpha$ *be a monomial left ideal of* A. *Then a standard monomial* $a^\beta \in L$ *if and only if* a^β *is divisible by* a^α *for some* $\alpha \in I$.

2.3. LEMMA. *Let* L *be a monomial left ideal of* A, *and let* $f \in A$. *Then the following are equivalent*:

(i) $f \in L$;

(ii) *every term of* f *lies in* L;

(iii) f *is a* k*-linear combination of the standard monomials in* $L \cap \mathrm{SM}(A)$.

Let L be a left ideal of A, and let $\mathrm{LT}(L)$ the set of leading terms of elements of L. We denote by $\langle \mathrm{LT}(L)]$ the monomial left ideal generated by $\mathrm{LT}(L)$ in A. If, furthermore, $\mathscr{G} = \{g_1, \dots, g_s\}$ is a left Gröbner basis of L, then we denote by $\langle \mathrm{LT}(\mathscr{G})]$ the monomial left ideal generated by $\mathrm{LT}(\mathscr{G}) = \{\mathrm{LT}(g_1), \dots, \mathrm{LT}(g_s)\}$ in A. Generally, in a solvable polynomial algebra we do *not* have $\langle \mathrm{LT}(L)] = \langle \mathrm{LT}(\mathscr{G})]$ because of (S2) (it is well known that this is true in the commutative case). However, for a homogeneous solvable polynomial algebra A, it follows from Lemmas 2.2 and 2.3 that the following proposition holds as in the commutative case.

2.4. PROPOSITION. *Let* L *be a left ideal of* A *and* $\mathscr{G} = \{g_1, \dots, g_s\} \subset L$.

(i) *If* L *is a monomial left ideal, then* $L = \langle \mathrm{LT}(L)]$.

(ii) $\mathscr{G}$ *is a Gröbner basis of* L *if and only if* $\langle \mathrm{LT}(L)] = \langle \mathrm{LT}(\mathscr{G})]$.

(iii) *Let* $\mathscr{G}$ *be a Gröbner basis of* L; *then* A/L *has a* k*-basis* $\{[a^\alpha] \mid a^\alpha \in \mathrm{SM}(A) - \langle \mathrm{LT}(\mathscr{G})]\}$, *where* $[a^\alpha]$ *is the class of* a^α *in* A/L. (*Compare with Proposition* 1.5(i).)

Note that A has a natural filtration (or standard filtration) FA:

$$F_m A = \left\{ \sum_{|\alpha|\leq m} c_\alpha a^\alpha \in A \right\}, \qquad m \geq 0.$$

Obviously, $F_0 A = k$, and each $F_m A$ is a finite dimensional k-space.

As in the commutative case, we define the *Hilbert function* of the A-module A/L, denoted ${}^{a}HF_L$, by putting $F_mL = L \cap F_mA$ and

$$ {}^{a}HF_L(m) = \dim_k\left(\frac{F_mA}{F_mL}\right), \qquad m \geq 0. $$

The *key* idea making the Hilbert function computable by using Gröbner bases comes from the following noncommutative version of an observation due to Macaulay (see [CLO′], Chap. 9, Sec. 3, Proposition 4).

2.5. PROPOSITION. *Let A be equipped with the monomial ordering $>_{\text{grlex}}$ (see Section 1 for the definition), and let $\langle \mathrm{LT}(L)]$ be as in Section 2. Then*

$$ {}^{a}HF_L(m) = {}^{a}HF_{\langle \mathrm{LT}(L)]}(m), \qquad m \geq 0. $$

We refer to [Li] for the following 2.6–2.8.

2.6. THEOREM. *Let $A = k[a_1, \dots, a_n]$ be a homogeneous solvable polynomial algebra with $>_{\text{grlex}}$ and let L be a left ideal of A. Let $\mathcal{G}$ be a Gröbner basis of L, and put*

$$ M = \max\left\{\beta_i \,\middle|\, \mathrm{LT}(g) = c_\beta a_1^{\beta_1} \cdots a_n^{\beta_n},\ g \in \mathcal{G},\ 1 \leq i \leq n\right\}. $$

Then there exists a unique polynomial $h(x) \in \mathbb{Q}[x]$ with positive leading coefficient such that

$$ {}^{a}HF_L(m) = h(m), \qquad \text{for all } m \geq n \cdot M. $$

If the ground field is computable, then $h(x)$ and the number $n \cdot M$ can be computed from any given generating set of L.

As for a commutative polynomial algebra, the polynomial $h(x)$ obtained in the above theorem is called the *Hilbert polynomial* of A/L and is denoted by ${}^{a}HP_L$. From the definition of the Hilbert function ${}^{a}HF_L(m)$ one sees that the degree of the polynomial ${}^{a}HP_L$ is nothing but the Gelfand–Kirillov dimension of A/L (e.g., see [KL]) which is usually denoted by GK.dim(A/L), i.e., deg$({}^{a}HP_L) =$ GK.dim(A/L).

In order to compute deg$({}^{a}HP_L)$ without computing ${}^{a}HP_L$, let $\mathcal{G} = \{g_1, \dots, g_s\}$ be a Gröbner basis of L. Put

$$ \begin{aligned} m_j &= \mathrm{LT}(g_j) = \lambda_j a_1^{\alpha_{j1}} \cdots a_n^{\alpha_{jn}}, \qquad j = 1, \dots, s, \quad \lambda_j \in k, \\ M_j &= \left\{i \in \{1, \dots, n\} \,\middle|\, \alpha_{ji} \neq 0\right\}, \qquad j = 1, \dots, s, \\ \mathcal{M} &= \left\{J \subset \{1, \dots, n\} \,\middle|\, J \cap M_j \neq \emptyset,\ 1 \leq j \leq s\right\}. \end{aligned} $$

2.7. PROPOSITION. *With notation as above,*

$$ \deg({}^{a}HP_L) = n - \min\left\{|J| \,\middle|\, J \in \mathcal{M}\right\}. $$

Furthermore, it has also been shown in [Li] that $\deg({}^aHP_L)$ is closely related to the independence of generators of A (modulo) L which is defined as follows. If $U = \{a_{i_1}, \ldots, a_{i_r}\} \subset \{a_1, \ldots, a_n\}$ with $i_1 < i_2 < \cdots < i_r$, then the subalgebra $k[U] = k[a_{i_1}, \ldots, a_{i_r}]$ of A generated by U over k is also a homogeneous solvable polynomial k-algebra. Put

$$\mathrm{T}(U) = \left\{\lambda a_{i_1}^{\alpha_1} \cdots a_{i_r}^{\alpha_r} \,\middle|\, \lambda \in k,\ (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_{\geq 0}^r\right\},$$

which is the set of of all terms in $k[U]$. We say that U is *independent* (modulo) L, if $\mathrm{T}(U) \cap \langle \mathrm{LT}(L)] = \{0\}$, or equivalently, $\mathrm{T}(U) \cap \mathrm{LT}(\mathscr{G}) = \varnothing$ where $\mathscr{G} = \{g_1, \ldots, g_s\}$ is a Gröbner basis of L and $\mathrm{LT}(\mathscr{G}) = \{\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_s)\}$.

2.8. PROPOSITION. *With notation as above, if we put*

$$d = \max\left\{|U| \,\middle|\, U \subset \{a_1, \ldots, a_n\} \text{ independent (modulo) } L\right\},$$

then $\deg({}^aHP_L) = d$.

We now go on to show that the independence condition for subsets of the generating set of A given above may be replaced by a weaker independence condition which will enable us to obtain the elimination lemma for homogeneous solvable polynomial algebras.

With notation as before, we say that a subset $U = \{a_{i_1}, \ldots, a_{i_r}\} \subset \{a_1, \ldots, a_n\}$ with $i_1 < i_2 < \cdots < i_r$ is *weakly independent* (modulo) L if $k[U] \cap L = \{0\}$. It is clear that if U is independent (modulo) L, then U is weakly independent (modulo) L. Hence, if we put

$$d' = \max\left\{|U| \,\middle|\, U \subset \{a_1, \ldots, a_n\} \text{ weakly independent (modulo) } L\right\},$$

it follows from Proposition 2.8 that

$$d' \geq d = \deg({}^aHP_L). \qquad (**)$$

2.9. PROPOSITION. *With notation as before, we have*

$$d' = d = \deg({}^aHP_L).$$

Proof. In view of $(**)$ we only have to show $d \geq d'$.

If $d' = 0$, then since L is a proper nonzero left ideal we have ${}^aHP_L(m) \geq 1 = \binom{m+0}{0}$. So we suppose $d' > 0$, and without loss of generality we let $U = \{a_1, \ldots, a_{d'}\} \subset \{a_1, \ldots, a_n\}$ be weakly independent (modulo) L. Consider the standard filtration $Fk[U]$ on $k[U]$ as we defined for A. Then $F_mk[U] \subset F_mA$ for all $m \geq 0$. Since all standard monomials in variables $a_1, \ldots, a_{d'}$

form a k-basis for $k[U]$, it follows from the weak independence condition $k[U] \cap L = \{0\}$ that

$$\binom{m+d'}{d'} = \dim_k F_m k[U] = \dim_k \frac{F_m k[U] + L}{L}$$
$$\leq \dim_k \frac{F_m A + L}{L}$$
$$= {}^a HF_L(m).$$

Thus for $m \gg 0$ we obtain

$${}^a HP_L(m) \geq \binom{m+d'}{d'} = f(m),$$

where $f(x)$ denotes the polynomial

$$\binom{x+d'}{d'}.$$

Hence $d = \deg({}^a HP_L) \geq \deg f(x) = d'$, as desired. ∎

2.10. Lemma. (*elimination lemma for homogeneous solvable polynomial algebras*). *Let $A = k[a_1, \dots, a_n]$ be a homogeneous solvable polynomial algebra, and let L be a proper left ideal of A such that the A-module A/L has Gelfand–Kirillov dimension d; i.e.,* $\deg({}^a HP_L) = d$. *Then for every $d+1$ generator out of the n generators $\{a_1, \dots, a_n\}$ of A there is a nonzero member of L that only depends on these $d+1$ generators.*

3. ELIMINATION LEMMA FOR LINEAR SOLVABLE POLYNOMIAL ALGEBRAS

The aim of this section is to obtain the elimination lemma for a class of non-homogeneous solvable polynomial algebras. We maintain the notation used in previous sections.

Let $A = k[a_1, \dots, a_n]$ be a solvable polynomial k-algebra with respect to a monomial ordering $>$ on $\mathbb{Z}^n_{\geq 0}$, where k is a subfield of $\mathbb{C}$. We say that A is *linear* if it satisfies

(L1) for $1 \leq i < j \leq n$, $a_j a_i = c_{ij} a_i a_j + \sum_{h=1}^n c_h a_h + c$ with c_{ij}, c_h, $c \in k$;

(L2) $>$ is the graded lexicographic ordering $>_{\text{grlex}}$.

Typical examples of linear solvable polynomial algebras are Weyl algebras and enveloping algebras of finite dimensional Lie algebras. The iterated Ore extensions of the form

$$k[x_1, \dots, x_n][y_1; \sigma_1, \delta_1] \cdots [y_m; \sigma_m, \delta_m],$$

where $k[x_1, \ldots, x_n]$ is the commutative polynomial k-algebra in n variables, are linear solvable polynomial algebras in case

$$\sigma_j(x_i) = \lambda_{ji}x_i, \ \ \lambda_{ji} \in k - \{0\}, \qquad 1 \le j \le m, \quad 1 \le i \le n,$$

$$\delta_j(x_i) = \sum_{h=1}^{n} c_h x_h + \sum_{t=1}^{j-1} c_t y_t + c_i, \quad c_h, c_t, c_i \in k, \quad 1 \le j \le m, \quad 1 \le i \le n,$$

$$\sigma_j(y_i) = \ell_{ji}y_i, \ \ \ell_{ji} \in k - \{0\}, \qquad 1 \le i < j \le m,$$

$$\delta_j(y_i) = \sum_{h=1}^{n} d_h x_h + \sum_{t=1}^{j-1} d_t y_t + d_i, \qquad d_h, d_t, d_i \in k, \quad 1 \le i < j \le m.$$

Let $A = k[a_1, \ldots, a_n]$ be an affine k-algebra. Consider the standard filtration FA on A:

$$F_mA = \left\{ \sum_{|\alpha| \le m} c_\alpha a^\alpha \,\middle|\, c_\alpha \in k, \ a^\alpha \in \mathrm{SM}(A) \right\}, \qquad m \ge 0.$$

Then, $F_0A = k$, $F_1A = k + \sum_{i=1}^{n} ka_i$, and for $m \ge 1$ we have $F_mA = (F_1A)^m$. Let $G(A)$ denote the associated graded ring of A with respect to FA; i.e., $G(A) = \bigoplus_{p \ge 0} G(A)_m$ with $G(A)_m = F_mA/F_{m-1}A$. If we consider the *order function* v: $A \to \mathbb{Z}$ as usual,

$$v(a) = \begin{cases} -\infty, & \text{if } a = 0, \\ m, & \text{if } a \in F_mA - F_{m-1}A, \end{cases}$$

and for $a \in A$ with $v(a) = m$, write $\sigma(a)$ for the class of a in $G(A)_m = F_mA/F_{m-1}A$ (which is usually called the *principal symbol* of a), then it is easy to see that $G(A)$ is an affine k-algebra generated by $\{\sigma(a_1), \ldots, \sigma(a_n)\}$. Moreover, we have the following easy but very useful fact.

3.1. LEMMA. *Let a, b be nonzero elements of A. Then $\sigma(a)\sigma(b) \neq 0$ if and only if $v(a) + v(b) = v(ab)$ if and only if $\sigma(a)\sigma(b) = \sigma(ab)$.*

Another important and easily verified fact concerning the standard filtration FA on a solvable polynomial algebra A with the monomial ordering $>_{\mathrm{grlex}}$ is that $>_{\mathrm{grlex}}$ is *compatible* with FA in the following sense.

3.2. LEMMA. *Let $f \in A$ be a nonzero element. Then $f \in F_mA$ if and only if $|\mathrm{md}(f)| \le m$.*

The results mentioned in 3.3–3.6 below have been obtained in [LW, Li].

3.3 PROPOSITION [LW]. *Let A be a solvable polynomial algebra with respect to $>_{\mathrm{grlex}}$ and let FA be the standard filtration on A. Then $G(A)$ is a domain and hence A is a domain.*

3.4 THEOREM [LW]. *Let $A = k[a_1, \ldots, a_n]$ be an affine k-algebra with the standard filtration FA. Then A is a linear solvable polynomial algebra with respect to $>_{\text{grlex}}$ if and only if the associated graded k-algebra $G(A) = k[\sigma(a_1), \ldots, \sigma(a_n)]$ is a (homogeneous) solvable polynomial algebra with respect to $>_{\text{grlex}}$.*

Let $A = k[a_1, \ldots, a_n]$ be an affine algebra with the standard filtration FA as before, and let L be a nonzero left ideal of A. Consider the filtration FL on L induced by FA, i.e., $F_mL = F_mA \cap L$, $m \geq 0$; then it is clear that the associated graded $G(A)$-module $G(L) = \bigoplus_{m\geq 0} G(L)_m$ with $G(L)_m = F_mL/F_{m-1}L$ may be identified with the graded left ideal

$$\bigoplus_{m\geq 0} \frac{F_mA \cap L + F_{m-1}A}{F_{m-1}A} \subset G(A).$$

Since FA also induces a filtration $F(A/L)$ on A/L with $F_m(A/L) = (F_mA + L)/L$, $m \geq 0$, it is easy to see that, with respect to the induced filtration on L and A/L , $G(A/L) \cong G(A)/G(L)$.

3.5 THEOREM [LW]. *Let A be a solvable polynomial algebra with respect to $>_{\text{grlex}}$, and let FA be the standard filtration on A. With notation as above, for a left ideal L of A with the filtration FL induced by FA, a finite subset $\mathcal{G} = \{g_1, \ldots, g_s\} \subset L$ is a Gröbner basis of L if and only if $\sigma(\mathcal{G}) = \{\sigma(g_1), \ldots, \sigma(g_n)\}$ is a Gröbner basis of $G(L)$ in $G(A)$.*

3.6. THEOREM [Li]. *Let A be a linear solvable polynomial k-algebra, and let FA be the standard filtration on A as before. If L is a nonzero left ideal of A, then there exists a unique polynomial $h(x) \in \mathbb{Q}[x]$ with positive leading coefficient such that for $m \gg 0$,*

$$\dim_k\left(\frac{F_mA}{F_mL}\right) = h(m),$$

where $F_mL = F_mA \cap L$. Moreover, if the ground field is computable, then the polynomial $h(x)$ can be computed from any given generating set of L.

The polynomial $h(x)$ obtained in the above theorem is called the *Hilbert polynomial* of the A-module A/L. It is clear that the degree of the Hilbert polynomial of the A-module A/L is nothing but the Gelfand–Kirillov dimension of A/L; i.e., $\deg h(x) = \text{GK.dim}(A/L)$. Moreover, since A is a linear solvable polynomial algebra, it follows from Theorem 3.4 that $G(A)$ is a homogeneous solvable polynomial algebra in the sense of Section 2; and if we consider the filtrations on L and A/L induced by FA, [Li, proof of Theorem 3.6] yields that

$$\begin{aligned}\deg h(x) = \text{GK.dim}(A/L) &= \text{GK.dim}(G(A/L))\\ &= \text{GK.dim}(G(A)/G(L))\\ &= \deg\left({}^aHP_{G(L)}\right). \qquad (\nabla)\end{aligned}$$

We now proceed to prove the elimination lemma for a linear solvable polynomial algebra. Let $A = k[a_1, \ldots, a_n]$ be a linear solvable polynomial algebra with the standard filtration FA. Note that the condition (L1) on A entails that, for a nonempty subset $U = \{a_{i_1}, \ldots, a_{i_r}\}$ of $\{a_1, \ldots, a_n\}$, the subalgebra $k[U]$ of A generated by U over k may contain elements which are not the linear sum of the standard monomials in variables in U. Hence the method we used for the homogeneous case in Section 2 cannot be carried over to the linear solvable case. However, since $G(A)$ is a homogeneous solvable polynomial algebra, by using the filtered-graded transfer trick again we may still arrive at the elimination lemma for linear solvable polynomial algebras.

First note an easy but important consequence of Lemma 3.2 and Proposition 3.3.

3.7. LEMMA. *For any $f \in A$ we have*

$$\sigma(\mathrm{LT}(f)) = \mathrm{LT}(\sigma(f))$$

with respect to the standard filtration FA on A.

Let L be a left ideal of A with the filtration FL induced by FA and let $\mathscr{G} = \{g_1, \ldots, g_s\}$ be a Gröbner basis of L. It follows from Theorem 3.5 that $\sigma(\mathscr{G}) = \{\sigma(g_1), \ldots, \sigma(g_s)\}$ is a Gröbner basis of $G(L)$ in $G(A)$. Writing

$$m_j = \mathrm{LT}(g_j) = \lambda_j a_1^{\alpha_{j1}} \cdots a_n^{jn}, \qquad j = 1, \ldots, s, \quad \lambda_j \in k, \tag{$*$}$$

Proposition 3.3 and Lemma 3.7 yield

$$\begin{aligned} \mathrm{LT}\big(\sigma(g_j)\big) &= \sigma\big(\mathrm{LT}(g_j)\big) \\ &= \lambda_j \sigma(a_1)^{\alpha_{j1}} \cdots \sigma(a_n)^{\alpha_{jn}}. \end{aligned} \tag{Δ}$$

Using notation as in Section 2,

$$M_j = \big\{ i \in \{1, \ldots, n\} \,\big|\, \alpha_{ji} \neq 0 \big\}, \qquad j = 1, \ldots, s,$$
$$\mathscr{M} = \big\{ J \subset \{1, \ldots, n\} \,\big|\, J \cap M_j \neq \emptyset,\ 1 \le j \le s \big\},$$

we may also compute $\mathrm{GK.dim}(A/L)$ as follows.

3.8. PROPOSITION. $\mathrm{GK.dim}(A/L) = n - \min\big\{ |J| \,\big|\, J \in \mathscr{M} \big\}.$

Proof. This follows immediately from (∇), (Δ), and Proposition 2.7. ∎

Now, for a nonempty subset $U = \{a_{i_1}, \ldots, a_{i_r}\} \subset \{a_1, \ldots, a_n\}$ with $i_1 < i_2 < \cdots < i_r$, we write

$$\mathrm{T}(U) = \big\{ \lambda a_{i_1}^{\alpha_1} \cdots a_{i_r}^{\alpha_r} \,\big|\, \lambda \in k,\ (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_{\ge 0}^r \big\},$$

and we say that U is *independent* (modulo) L if $\mathrm{T}(U) \cap \mathrm{LT}(\mathscr{G}) = \emptyset$, where $\mathrm{LT}(\mathscr{G}) = \{m_1, \ldots, m_s\}$ is given by ($*$), above.

3.9. PROPOSITION. *With notation as above, and putting*

$$d' = \max\left\{|U| \,\middle|\, U \subset \{a_1, \ldots, a_n\} \text{ independent (modulo) } L\right\},$$

then $\mathrm{GK.dim}(A/L) = d'$.

Proof. Let $J \in \mathcal{M}$ with $J = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$. Then $J \cap M_j \neq \emptyset$, $j = 1, \ldots, s$. Write $U = \{a_{j_1}, \ldots, a_{j_{n-r}}\}$ with $\{j_1, \ldots, j_{n-r}\} = \{1, \ldots, n\} - J$. It is easy to see that $\mathrm{T}(U) \cap \mathrm{LT}(\mathcal{G}) = \emptyset$; i.e., $n - r = |U| \leq d'$. Since J is arbitrary in $\mathcal{M}$ it follows from Proposition 3.8 that $\mathrm{GK.dim}(A/L) \leq d'$.

To obtain the opposite inequality, let $U = \{a_{i_1}, \ldots, a_{i_{d'}}\} \subset \{a_1, \ldots, a_n\}$ be independent (modulo) L, and put $J = \{1, \ldots, n\} - \{i_1, \ldots, i_{d'}\}$. Then we see that $J \in \mathcal{M}$. Thus, $d' = |U| = n - |J| \leq \mathrm{GK.dim}(A/L)$ by Proposition 3.8. This proves the proposition. ∎

Furthermore, let $U = \{a_{i_1}, \ldots, a_{i_r}\} \subset \{a_1, \ldots, a_n\}$ with $i_1 < i_2 < \cdots < i_r$, and let

$$\mathrm{SM}(U) = \left\{a_{i_1}^{\alpha_1} \cdots a_{i_r}^{\alpha_r} \,\middle|\, (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r_{\geq 0}\right\}.$$

Write $\mathbf{V}(U)$ for the *k-vector space* spanned by $\mathrm{SM}(U)$ in A. We say that U is *weakly independent* (modulo) L, if $\mathbf{V}(U) \cap L = \{0\}$.

3.10. THEOREM. *With notation as above, if we put*

$$d'' = \max\left\{|U| \,\middle|\, U \subset \{a_1, \ldots, a_n\} \text{ weakly independent (modulo) } L\right\},$$

then $\mathrm{GK.dim}(A/L) = d''$.

Proof. We first prove that if $U = \{a_{i_1}, \ldots, a_{i_r}\} \subset \{a_1, \ldots, a_n\}$ is independent (modulo) L with respect to some fixed Gröbner basis $\mathcal{G}$ of L, then U is weakly independent (modulo) L, and hence $\mathrm{GK.dim}(A/L) \leq d''$ by Proposition 3.9. To see this, let $\mathcal{G} = \{g_1, \ldots, g_s\}$, and suppose that $f \in \mathbf{V}(U) \cap L$. If $f \neq 0$, then f has a Gröbner presentation by $\mathcal{G}$:

$$f = \sum_{i=1}^{s} h_i g_i, \qquad \mathrm{md}(f) \geq \mathrm{md}(h_i g_i) \text{ whenever } h_i g_i \neq 0.$$

Hence, $\mathrm{LM}(f)$ appears as one of the $\mathrm{LM}(\mathrm{LT}(h_i)\mathrm{LT}(g_i))$ (Lemma 1.2). But $\mathrm{LM}(f) \in \mathrm{SM}(U)$; this implies that some $\mathrm{LT}(g_i)$ must be contained in $T(U) \cap \mathrm{LT}(\mathcal{G})$, contradicting U being independent (modulo) L.

To prove $d'' \leq \mathrm{GK.dim}(A/L)$, let $h(x)$ be the Hilbert polynomial of the A-module A/L as given in Theorem 3.6. If $d'' = 0$, then since L is a proper left ideal we have $h(m) \geq 1 = \binom{m+0}{0}$. So we may suppose $d'' > 0$, and without loss of generality we let $U = \{a_1, \ldots, a_{d''}\} \subset \{a_1, \ldots, a_n\}$ be weakly independent (modulo) L. Consider the filtration $F\mathbf{V}(U)$ on the k-vector space $\mathbf{V}(U)$ induced by FA: $F_m\mathbf{V}(U) = F_mA \cap \mathbf{V}(U)$, $m \geq 0$, and put

$$\mathrm{SM}(U)_{\leq m} = \left\{a_1^{\alpha_1} \cdots a_{d''}^{\alpha_{d''}} \;\middle|\; \alpha_1 + \cdots + \alpha_{d''} \leq m\right\}, \qquad m \geq 0.$$

Then $\mathrm{SM}(U)_{\leq m} \subset F_m\mathbf{V}(U)$, and it follows from the weak independence condition $\mathbf{V}(U) \cap L = \{0\}$ that for $m \gg 0$

$$\begin{aligned}\binom{m+d''}{d''} = |\mathrm{SM}(U)_{\leq m}| &= \dim_k \frac{F_m\mathbf{V}(U)+L}{L} \\ &\leq \dim_k \frac{F_mA+L}{L} \\ &= h(m).\end{aligned}$$

Thus we obtain

$$h(m) \geq \binom{m+d''}{d''} = f(m), \qquad m \gg 0,$$

where $f(x)$ denotes the polynomial

$$\binom{x+d''}{d''}.$$

Hence $\mathrm{GK.dim}(A/L) = \deg h(x) \geq \deg f(x) = d''$, as desired. ∎

We arrive at the following lemma.

3.11. Lemma (elimination lemma for linear solvable polynomial algebras). *Let $A = k[a_1, \ldots, a_n]$ be a linear solvable polynomial algebra, and let L be a proper left ideal of A such that the A-module A/L has Gelfand–Kirillov dimension d; i.e., the Hilbert polynomial of the A-module A/L has degree d. Then for every subset $U = \{a_{i_1}, \ldots, a_{i_{d+1}}\} \subset \{a_1, \ldots, a_n\}$, $\mathbf{V}(U) \cap L \neq \{0\}$; i.e., there is a nonzero member of L that only depends on the generators in U.*

Remark. In the commutative case, the notion of (weak) independence modulo a polynomial ideal $I \subset k[x_1, \ldots, x_n]$ was introduced by Gröbner in [Grö]. Its usefulness in algebraic geometry was realized after combination with the algorithmic techniques of Gröbner bases in order to compute the dimension $\dim \mathbf{V}(I)$ of the affine algebraic set $\mathbf{V}(I)$ determined by the ideal I, or the degree of the Hilbert polynomial of the $k[x_1, \ldots, x_n]$-module $k[x_1, \ldots, x_n]/I$. The notion of (strong) independence modulo a polynomial ideal I was introduced in [KW] as a *key* link between the (weak) independence modulo I and a Gröbner basis of I.

4. AN APPLICATION TO THE EXTENSION/CONTRACTION PROBLEM W.R.T. ∂-FINITENESS

Let k be a subfield of $\mathbb{C}$. Given the sets $\{x_1, \ldots, x_n\}$, $\{\partial_1, \ldots, \partial_m\}$ of *symbols* with $n, m \geq 1$, in this section we consider the k-algebra $A = k[x_1, \ldots, x_n, \partial_1, \ldots, \partial_m]$ which is defined by the defining relations

$$\begin{aligned}
&x_i x_j = x_j x_i, \qquad 1 \leq i, j \leq n,\\
&\partial_\ell x_i = c_{\ell i} x_i \partial_\ell + \mathscr{E}_{i\ell} + c,\ \ c_{\ell i},\\
&\qquad\qquad c \in k, c_{\ell i} \neq 0, \quad 1 \leq i \leq n, \quad 1 \leq \ell \leq m,\\
&\partial_h \partial_k = c_{hk} \partial_k \partial_h + \mathscr{E}_{hk} + d,\\
&\qquad\qquad c_{hk}, d \in k, c_{hk} \neq 0, \quad 1 \leq h, \quad k \leq m, \quad h > k,
\end{aligned} \tag{$\Diamond$}$$

where

- $\mathscr{E}_{i\ell}$ are elements of the linear k-space spanned by $\{x_1, \ldots, x_n, \partial_\ell\}$,
- $\mathscr{E}_{hk}$ are elements of the linear k-space spanned by $\{x_1, \ldots, x_n, \partial_1, \ldots, \partial_m\}$.

Obviously, A contains the commutative k-algebra $k[x_1, \ldots, x_n]$ as a subalgebra and every element $D \in A$ may be written as

$$\begin{aligned}
D &= \sum c_{(\alpha,\beta)} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_m^{\beta_m}\\
&= \sum P_\beta(x_1, \ldots, x_n) \partial_1^{\beta_1} \cdots \partial_m^{\beta_m},
\end{aligned}$$

where $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \in \mathbb{Z}_{\geq 0}^{n+m}$ and $P_\beta(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$. Hence, we also write A as

$$A = k[x_1, \ldots, x_n][\partial_1, \ldots, \partial_m].$$

We assume that A satisfies the following two conditions:

(P1) $S = k[x_1, \ldots, x_n] - \{0\}$ forms a left and right Ore set in A; i.e., for any given $s \in S$, $f \in A$, there are $s', s'' \in S$ and $f', f'' \in A$ such that $s'f = f's$ and $fs'' = sf''$.

(P2) The set of standard monomials

$$\left\{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_m^{\beta_m} \;\middle|\; (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \in \mathbb{Z}_{\geq 0}^{n+m} \right\}$$

forms a k-basis of A.

As a matter of fact, condition (P2) is algorithmically checkable; namely, the algebra A satisfies (P2) if and only if the defining relations of A form a Gröbner basis in the free k-algebra $k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_m \rangle$ with respect to the graded lexicographic ordering $>_{\text{grlex}}$ (in the sense of [Mor]; see [LWZ]). So except for the Weyl algebras, one easily finds other examples satisfying (P1), (P2); in particular, the iterated Ore extensions subject to the commutation rules listed in [CS, Table 2] are such algebras, except for those including Mahlerian operators.

In what follows we let A be a k-algebra defined by the defining relation $(\Diamond)$ and satisfying (P1), (P2).

From (P1) we know that the localization of A at the Ore set $S = k[x_1, \dots, x_n] - \{0\}$, denoted $S^{-1}A$, exists, and from (P2) it follows that A is a linear solvable polynomial algebra (in the sense of Section 3) with respect to $\partial_m >_{\text{grlex}} \cdots >_{\text{grlex}} \partial_1 >_{\text{grlex}} x_n >_{\text{grlex}} \cdots >_{\text{grlex}} x_1$. Hence A is a Noetherian domain (Proposition 3.3). Thus, we may view A as a subring of $S^{-1}A$ and write $A \subset S^{-1}A$. Keeping this in mind, every element $D \in S^{-1}A$ may be written as

$$D = \sum Q_\beta \partial_1^{\beta_1} \cdots \partial_m^{\beta_m}, \qquad Q_\beta \in k(x_1, \dots, x_n),$$

where $k(x_1, \dots, x_n)$ is the rational function field in variables $x_1, \dots, x_n$; i.e., $S^{-1}A$ is indeed a $K(x_1, \dots, x_n)$-algebra with generating set $\{\partial_1, \dots, \partial_m\}$. So we may write

$$S^{-1}A = k(x_1, \dots, x_n)[\partial_1, \dots, \partial_m].$$

4.1. DEFINITION. A left ideal $\mathcal{J}$ of $S^{-1}A$ is said to be *∂-finite* if $S^{-1}A/\mathcal{J}$ is a finite dimensional vector space over $k(x_1, \dots, x_n)$.

Note that the "∂" in the above definition is only a symbol and has no relaion with the generators $\partial_1, \dots, \partial_m$ of $S^{-1}A$. The name of a ∂-finite ideal comes from [CS] where a class of holonomic-like functions (sequences) was defined by using the ∂-finite ideals in an iterated Ore extension as annihilator ideals to generalize the work of [Zei1-2, Lip1-2, Tak1-2]. Unfortunately, since there is no a general version of the Bernstein–Kashiwara theorem for general iterated Ore extensions, the generalization in [CS] could only be done at the level of manipulating rational function coefficients, i.e., in an algebra of type $S^{-1}A$ instead of in A. This, in turn, leads to the extension/contraction problem (see [CS, Sect. 4]). However, even if one only deals with rational function coefficients, the nice properties of ∂-finite functions (sequences) still yield the relative simplicity of the corresponding algorithm in the automatic proving of (∂-finite) function identities (see [CS, Sects. 1–3]).

Using the results of the foregoing sections, we now proceed to deal with the extension/contraction problem posed in [CS, Sect. 4] with respect to ∂-finiteness defined above.

4.2. Proposition. *Let $\mathcal{J}$ be a left ideal of $S^{-1}A$. $\mathcal{J}$ is ∂-finite if and only if for each $i = 1, \ldots, m$,*

$$\mathcal{J} \cap k[x_1, \ldots, x_n][\partial_i] \neq \{0\}.$$

Proof. If $\mathcal{J}$ is ∂-finite, then for each i, $\{1, \partial_i, \partial_i^2, \ldots\}$ spans a finite dimensional vector space over $k(x_1, \ldots, x_n)$ in $S^{-1}A/\mathcal{J}$. It follows that there is a nonzero element $P_i(\partial_i) \in k[x_1, \ldots, x_n][\partial_i] \cap \mathcal{J}$. Conversely, suppose that for each $i = 1, \ldots, m$ there is a nonzero $P_i(\partial_i) \in k[x_1, \ldots, x_n][\partial_i] \cap \mathcal{J}$ with degree k_i. Note that since S is an Ore set in A, if $f/g \in k(x_1, \ldots, x_n)$ and $\alpha_i \geq 1$, then there exist $s \in S$, $a \in A$ such that

$$s\partial_i^{\alpha_i} = ag \qquad \text{in } A, \tag{1}$$

$$\frac{\partial_i^{\alpha_i}}{1} \cdot \frac{f}{g} = \frac{af}{s} \qquad \text{in } S^{-1}A. \tag{2}$$

The defining relations of A and (1) entail that a is a polynomial in ∂_i with coefficients in $k[x_1, \ldots, x_n]$, and the degree of a with respect to ∂_i is equal to α_i. Thus, the defining relations of A and (2) entail that $\frac{af}{s}$ is a polynomial in ∂_i with coefficients in $k(x_1, \ldots, x_n)$ and of degree α_i. Paying attention to the multiplication in $S^{-1}A$, a formal division by $\{P_1(\partial_1), \ldots, P_m(\partial_m)\}$ in $S^{-1}A$ yields that $S^{-1}A/\mathcal{J}$ is spanned by $\{\partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}\}_{0 \leq \alpha_i < k_i}$ as a $k(x_1, \ldots, x_n)$-space. ∎

The extension/contraction problem that appeared in the automatic proving of function identities may be described as follows. Let A be an algebra of the type we are considering, and let $S^{-1}A$ be the corresponding localization of A at $S = k[x_1, \ldots, x_n] - \{0\}$. If $\mathcal{J}$ is a left ideal of $S^{-1}A$, then the left ideal $\mathcal{J}^c = \mathcal{J} \cap A$ of A is called the *contraction* of $\mathcal{J}$ in A; if L is a left ideal of A, then the left ideal $L^e = S^{-1}AL$ of $S^{-1}A$ is called the *extension* of L in $S^{-1}A$. Note that $\mathcal{J}^{ce} = \mathcal{J}$ and L^{ec} is usually *larger* than L. Since the ∂-finiteness is defined at the level of $S^{-1}A$, Proposition 4.2 suggests the study of the elimination information contained in the contraction ideal $\mathcal{J}^c$ at the level of A. Since A is a linear solvable polynomial algebra, all results obtained in Section 3 may be applied to A and the next theorem may be thought of as a solution to the extension/contraction problem.

For a subset $U = \{x_{i_1}, \ldots, x_{i_r}, \partial_{j_1}, \ldots, \partial_{j_t}\} \subset \{x_1, \ldots, x_n, \partial_1, \ldots, \partial_m\}$ with $i_1 < i_2 < \cdots < i_r < j_1 < j_2 < \cdots < j_t$, as in Section 3 we write $\mathbf{V}(U)$ for the k-vector space spanned by

$$\left\{x_{i_1}^{\alpha_1} \cdots x_{i_r}^{\alpha_r} \partial_{j_1}^{\beta_1} \cdots \partial_{j_t}^{\beta_t} \,\middle|\, (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_t) \in \mathbb{Z}_{\geq 0}^{r+t}\right\}.$$

4.3. Theorem. *With notation as before, let L be a left ideal of A, and let $\mathcal{J}$ be a proper left ideal of $S^{-1}A$.*

(i) *Let* $U = \{x_{i_1}, \dots, x_{i_r}, \partial_{j_1}, \dots, \partial_{j_t}\} \subset \{x_1, \dots, x_n, \partial_1, \dots, \partial_m\}$ *be as above. If* $L \cap \mathbf{V}(U) = \{0\}$, *then* $\text{GK.dim}(A/L) \geq r + t$.

(ii) *If* $\text{GK.dim}(A/L) = n$, *then for each* $i = 1, \dots, m$, $U = \{x_1, \dots, x_n, \partial_i\}$,

$$L \cap \mathbf{V}(U) \neq \{0\}.$$

Hence, the extension left ideal L^e *of* L *in* $S^{-1}A$ *is* ∂*-finite.*

(iii) $\text{GK.dim}(A/\mathcal{J}^c) \geq n$.

(iv) *If* $\text{GK.dim}(A/L) = n$ *and* $L \cap k[x_1, \dots, x_n] = \{0\}$, *then* $\text{GK.dim}(A/L^{ec}) = n$.

Consequently, if $\text{GK.dim}(A/\mathcal{J}^c) = n$ *then* $\mathcal{J}$ *is* ∂*-finite in* $S^{-1}A$.

Proof. (i) and (ii) follow from Theorem 3.10 and Lemma 3.11. Since $\mathcal{J}$ is a proper left ideal of $S^{-1}A$ and the localization (or the extension) of the contraction left ideal $\mathcal{J}^c$ is equal to $\mathcal{J}$, it is clear that $\mathcal{J}^c \cap k[x_1, \dots, x_n] = \{0\}$. Hence (iii) follows from (i). Finally, (iv) follows from (ii), (iii), and the natural A-module epimorphism $A/L \to A/L^{ec}$ because L^e is now a proper left ideal of $S^{-1}A$ by the assumption on L. ∎

5. THE ∂-HOLONOMICITY

In this section, A denotes a k-algebra satisfying (P1), (P2), as defined in Section 4, and we maintain notation as before.

Let $S^{-1}A$ be the localization of A at $S = k[x_1, \dots, x_n] - \{0\}$, and let $\mathcal{F}$ be a family of "functions" including $k[x_1, \dots, x_n]$. Suppose that $S^{-1}A$ (and hence A) acts on $\mathcal{F}$ naturally, or that $\mathcal{F}$ forms a left $S^{-1}A$-module (hence an A-module). For a nonzero member $f \in \mathcal{F}$, we consider all elements of $S^{-1}A$, respectively of A, that annihilate f:

$$\mathcal{J}_f = \{D \in S^{-1}A \mid Df = 0\},$$
$$\mathcal{I}_f = \{D \in A \mid Df = 0\}.$$

Clearly, $\mathcal{J}_f$, respectively $\mathcal{I}_f$, is a left ideal of $S^{-1}A$, respectively of A. It is also clear that $\mathcal{J}_f$ is a proper left ideal of $S^{-1}A$, $\mathcal{J}_f \cap k[x_1, \dots, x_n] = \{0\}$, and $\mathcal{I}_f \cap k[x_1, \dots, x_n] = \{0\}$. In particular, by Theorem 4.3 we have

$$\text{GK.dim}(A/\mathcal{J}_f^c) \geq n, \qquad \text{GK.dim}(A/\mathcal{I}_f) \geq n.$$

In view of Theorem 4.3 and the above remark, it is natural to define ∂-finite functions and ∂-holonomic functions as follows.

5.1. DEFINITION. With notation as above, let f be a nonzero member of $\mathcal{F}$.

(i) f is said to be a ∂-finite function if $\mathcal{J}_f$ is a ∂-finite left ideal of $S^{-1}A$ in the sense of Definition 4.1; i.e., $S^{-1}A/\mathcal{J}_f$ is a finite dimensional $k(x_1, \ldots, x_n)$-space.

(ii) f is said to be a ∂-holonomic function if $\mathrm{GK.dim}(A/\mathcal{I}_f) = n$.

Since for a left ideal L of a linear solvable polynomial algebra A the Gelfand–Kirillov dimension of the A-module A/L is always computable in terms of the Gröbner basis (Section 3), we see that ∂-holonomicity defined above is algorithmically recognizable.

Next we show that the automatic proving of ∂-finite function identities, respectively of ∂-holonomic function identities, is feasible in the following sense: if f and g are ∂-finite functions, respectively ∂-holonomic functions, such that $f - g$ is the same type of function, then, to prove f and g are the same function, it is sufficient to show that $f - g$ is identically equal to the zero function by using the ∂-finiteness, respectively the ∂-holonomicity, as one did in the context of Weyl algebras. In particular, the automatic proving of ∂-holonomic function identities can be done at the level of manipulating polynomial function coefficients.

5.2. PROPOSITION. *With notation as above, let f, $g \in \mathcal{F}$ be ∂-finite, resp. ∂-holonomic, functions in the sense of Definition* 5.1. *Then $f + g$ is a ∂-finite, resp. a ∂-holonomic, function.*

Proof. If $f, g \in \mathcal{F}$ are ∂-finite, resp. ∂-holonomic functions, then we have the $S^{-1}A$-module $S^{-1}Af \oplus S^{-1}Ag$, resp. the A-module $Af \oplus Ag$. Considering the element (f, g) in both modules and the $S^{-1}A$-submodule $S^{-1}A(f, g)$, resp. the A-submodule $A(f, g)$, we obtain the exact sequences of modules

$$0 \longrightarrow \frac{S^{-1}A}{\mathcal{J}_{(f,g)}} \longrightarrow \frac{S^{-1}A}{\mathcal{J}_f} \bigoplus \frac{S^{-1}A}{\mathcal{J}_g}$$

$$\frac{S^{-1}A}{\mathcal{J}_{(f,g)}} \longrightarrow \frac{S^{-1}A}{\mathcal{J}_{f+g}} \longrightarrow 0$$

$$0 \longrightarrow \frac{A}{\mathcal{I}_{(f,g)}} \longrightarrow \frac{A}{\mathcal{I}_f} \bigoplus \frac{A}{\mathcal{I}_g}$$

$$\frac{A}{\mathcal{I}_{(f,g)}} \longrightarrow \frac{A}{\mathcal{I}_{f+g}} \longrightarrow 0,$$

where $\mathcal{J}_{(f,g)}$, resp. $\mathcal{I}_{(f,g)}$, is the (left) annihilator ideal of (f, g) in $S^{-1}A$, resp. in A. Since

$$\dim_{k(x_1,\ldots,x_n)}\left(\frac{S^{-1}A}{\mathcal{J}_f} \bigoplus \frac{S^{-1}A}{\mathcal{J}_g}\right) < \infty, \qquad \mathrm{GK.dim}\left(\frac{A}{\mathcal{I}_f} \bigoplus \frac{A}{\mathcal{I}_g}\right) = n,$$

it follows from Theorem 4.3(iii) that

$$\dim_{k(x_1,\ldots,x_n)} \frac{S^{-1}A}{\mathcal{J}_{f+g}} < \infty, \qquad \text{GK.dim}\frac{A}{\mathcal{J}_{f+g}} = n,$$

as desired. ∎

Finally, we discuss the possibility of extending the Bernstein–Kashiwara theorem to algebras of type A.

From the definition and Theorem 4.3 we see that if f is a ∂-holonomic function then f is a ∂-finite function. Conversely, if f is a ∂-finite function such that $\text{GK.dim}(A/\mathcal{J}_f^c) = n$, then f is also ∂-holonomic. It follows that that if ∂-finiteness always implied ∂-holonomicity, then we might replace ∂-finiteness by ∂-holonomicity, and consequently the automatic proving of ∂-finite function identities could be reduced from manipulating rational function coefficients to manipulating polynomial function coefficients. More precisely, we might expect that the equality in Theorem 4.3(iii) holds, and hence we would have an analogue of the Bernstein–Kashiwara theorem (Theorem 0.3) for algebras of type A. By Theorem 1.4 and Proposition 3.8 the most obvious case where the equality $\text{GK.dim}(A/\mathcal{J}^c) = n$ holds is when $\mathcal{J}^c$ contains elements of the form $\partial_j^{\beta_j} + Q_j$ of A, where the Q_j are elements of A with $\beta_j = \text{md}(\partial_j^{\beta_j}) >_{\text{grlex}} \text{md}(Q_j)$, $j = 1, \ldots, m$. However, it seems to us that the equality of Theorem 4.3(iii) does not hold for an arbitrary ∂-finite left ideal in $S^{-1}A$. Nevertheless, we prove that for certain algebras of type A (different from the Weyl algebras) the equality in Proposition 4.3(iii) may hold for arbitrary ∂-finite left ideals of $S^{-1}A$.

5.3. Proposition. *If A is generated by $n+1$ elements $x_1, \ldots, x_n, \partial$, i.e., $A = k[x_1, \ldots, x_n, \partial]$, and $\mathcal{J}$ is any proper left ideal of $S^{-1}A$, then* $\text{GK.dim}(A/\mathcal{J}^c) = n$. *Moreover, every proper left ideal in $S^{-1}A$ if ∂-finite.*

Proof. (that $\text{GK.dim}(A/\mathcal{J}^c) = n$ follows from Theorems 4.3 and 3.10). The ∂-finiteness of $\mathcal{J}$ follows from a formal division as in the proof of Proposition 4.2 by considering the polynomial $P(\partial)$ in ∂ with coefficients in $k[x_1, \ldots, x_n]$ which is contained in $\mathcal{J}$ and has the smallest degree. ∎

Another example is obtained by considering the algebras $A = k[x_1, \ldots, x_n, \partial_1, \ldots, \partial_m]$ with $m > 1$ but we assume that in $S^{-1}A$,

$$\partial_j \cdot \frac{1}{f} = \frac{f\partial_j - \delta(f)}{f^2}, \qquad j = 1, \ldots, m,$$

$$\text{where } f, \delta(f) \in k[x_1, \ldots, x_n] \text{ with } \deg \delta(f) \le \deg f. \qquad (\bullet)$$

One easily finds examples of such algebras (including Weyl algebras as a special case).

5.4. PROPOSITION. *Let A be a k-algebra as above and let $S^{-1}A$ be the localization of A at $S = k[x_1, \dots, x_n] - \{0\}$. If $\mathcal{J}$ is a ∂-finite proper left ideal of $S^{-1}A$, then* $\mathrm{GK.dim}(A/\mathcal{J}^c) = n$.

Proof. By Theorem 4.3(iii) we only have to show that GK.dim $(A/\mathcal{J}^c) \le n$.

Let $\bar{s}_1, \dots, \bar{s}_t$ be a basis of the $k(x_1, \dots, x_n)$-space $S^{-1}A/\mathcal{J}$. We may assume that $\bar{s}_j$ are classes of the monomials $s_j = \partial_1^{\beta_{j1}} \cdots \partial_m^{\beta_{jm}}$ in $S^{-1}A/\mathcal{J}$ and that $s_1 = 1$. Since $\{\bar{s}_1 = 1, \bar{s}_2, \dots, \bar{s}_t\}$ is a basis, there exists a $p \in k[x_1, \dots, x_n]$ and $q_{vj}^u \in k[x_1, \dots, x_n]$, $1 \le u \le m$, $1 \le v, j \le t$, such that

$$p\overline{\partial_u s_v} = \sum_{j=1}^{t} q_{vj}^u \bar{s}_j. \tag{$\bullet\bullet$}$$

Let p be the polynomial we fixed in ($\bullet\bullet$), and consider the k-subspace M of $S^{-1}A/\mathcal{J}$ which is defined as follows:

$$M = \sum_{j=1}^{t} k[x_1, \dots, x_n, p^{-1}]\bar{s}_j.$$

Putting

$$\begin{aligned} E &= \max_{1\le u\le m,\ 1\le v,j\le t} \left\{\deg p + 1, \deg q_{vj}^u\right\}, \\ T &= 2E, \end{aligned}$$

M has a filtration consisting of k-subspaces,

$$F_w M = \left\{ p^{-w} \sum_{j=1}^{t} g_j s_j \,\middle|\, g_j \in k[x_1, \dots, x_n], \deg g_j \le wT \right\}, \qquad w \ge 0,$$

and moreover $\dim_k F_w M = t \cdot \binom{wT+n}{n}$ which is a polynomial in w of degree n. If we consider the filtration on $(A + \mathcal{J})/\mathcal{J} \cong A/\mathcal{J}^c$ induced by FA, it follows from Theorem 3.6 and the formula (∇) before Lemma 3.7 that we can finish the proof by showing that

$$F_w(A/\mathcal{J}^c) \subset F_{2^w} M, \qquad w \ge 0.$$

Indeed, let $D = \overline{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_m^{\beta_m}}$ be a monomial in $A/\mathcal{J}^c$ such that $|\alpha| + |\beta| \le w$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $|\beta| = \beta_1 + \cdots + \beta_m$. If we start with $\overline{\partial_m}$ in the foregoing ($\bullet\bullet$),

$$p\overline{\partial_m} = p\overline{\partial_m s_1} = \sum_{j=1}^{t} q_{1j}^m \bar{s}_j \qquad \text{implying } \overline{\partial_m} = \frac{1}{p} \sum_{j=1}^{t} q_{1j}^m \bar{s}_j,$$

then by the assumption (•) we obtain

$$\begin{aligned}
\overline{\partial_m^2} = \partial_m \overline{\partial_m} = \partial_m \cdot \frac{1}{p} \sum_{j=1}^{t} q_{1j}^m \bar{s}_j
&= \frac{1}{p^2} \sum_{j=1}^{t} (p\partial_m - \delta(p)) q_{1j}^m \bar{s}_j \\
&= \frac{1}{p^2} \sum_{j=1}^{t} \Big(p\big(q_{1j}^m \partial_m + \delta(q_{1j}^m)\big)\bar{s}_j + \delta(p)\bar{s}_j \Big) \\
&= \frac{1}{p^2} \sum_{j=1}^{t} \Big(q_{1j}^m p \overline{\partial_m s_j} + p\delta(q_{1j}^m)\bar{s}_j + \delta(p)\bar{s}_j \Big) \\
&= \frac{1}{p^2} \sum_{j=1}^{t} \left(q_{1j}^m \left(\sum_{h=1}^{t} q_{jh}^m \overline{s_h} \right) + p\delta(q_{1j}^m)\bar{s}_j + \delta(p)\bar{s}_j \right) \\
&= \frac{1}{p^2} \sum_{j=1}^{t} q_j \bar{s}_j
\end{aligned}$$

with $q_j \in k[x_1, \ldots, x_n]$ and $\deg q_j \le 2E$. A repetition of this procedure yields

$$\overline{\partial_m^{\beta_m}} = \partial_m \overline{\partial_m^{\beta_m - 1}} = \frac{1}{p^{2^{\beta_m - 1}}} \sum_{j=1}^{t} q_j \bar{s}_j \qquad \text{with}$$

$$q_j \in k[x_1, \ldots, x_n], \quad \deg q_j \le 2^{\beta_m - 1} \cdot E,$$

$$\vdots$$

$$\overline{\partial_1^{\beta_1} \cdots \partial_m^{\beta_m}} = \frac{1}{p^{2^{|\beta| - 1}}} \sum_{j=1}^{t} q'_j \bar{s}_j \qquad \text{with}$$

$$q'_j \in k[x_1, \ldots, x_n], \quad \deg q'_j \le 2^{|\beta| - 1} \cdot E,$$

and hence

$$\begin{aligned}
D = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \overline{\partial_1^{\beta_1} \cdots \partial_m^{\beta_m}} &= \frac{1}{p^{2^{|\beta| - 1}}} \sum_{j=1}^{t} x_1^{\alpha_1} \cdots x_n^{\alpha_n} q'_j \bar{s}_j \\
&= \frac{1}{p^{2^{|\alpha| + |\beta|}}} \sum_{j=1}^{t} p^{2^{|\alpha| + |\beta|} - 2^{|\beta| - 1}} x_1^{\alpha_1} \cdots x_n^{\alpha_n} q'_j \bar{s}_j.
\end{aligned}$$

Here

$$\begin{aligned}\deg\left(p^{2^{|\alpha|+|\beta|}-2^{|\beta|-1}}x_1^{\alpha_1}\cdots x_n^{\alpha_n}q_j'\right) &\le (2^{|\alpha|+|\beta|}-2^{|\beta|-1})\deg p+|\alpha|+2^{|\beta|-1}\cdot E\\ &\le (2^{|\alpha|+|\beta|}-2^{|\beta|-1})\deg p+2^{|\alpha|}+2^{|\beta|-1}\cdot E\\ &\le 2^{|\alpha|+|\beta|}\cdot E+\left(2^{|\alpha|}+2^{|\beta|}\right)\cdot E\\ &\le 2\cdot 2^{|\alpha|+|\beta|}\cdot E\\ &= 2^{|\alpha|+|\beta|}\cdot T\\ &\le 2^{w}\cdot T.\end{aligned}$$

This shows that $D \in F_{|\alpha|+|\beta|}M \subset F_{2^w}M$, as desired. ∎

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