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Some boundary value problems of fractional differential equations and inclusions

Bashir Ahmad [a,*], Juan J. Nieto [b], Johnatan Pimentel [b]

[a] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[b] Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782, Santiago de Compostela, Spain

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ABSTRACT

In this paper, we study the existence of solutions for nonlinear fractional differential equations and inclusions of order $q \in (1, 2]$ with families of mixed and closed boundary conditions. In case of inclusion problems, the existence results are established for convex as well as nonconvex multivalued maps. Our results are based on Leray–Schauder degree theory, nonlinear alternative of Leray–Schauder type, and some fixed point theorems for multivalued maps. Some interesting special cases are also discussed.

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1. Introduction

Differential equations and inclusions of fractional order have recently been addressed by several researchers for a variety of problems. The fractional calculus has found its applications in various disciplines of science and engineering such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [1–4]. For some recent work on fractional differential equations and inclusions, see [5–15] and the references therein.

In this paper, we consider for $T > 0$ and $1 < q \leq 2$ the following fractional differential equation

$$^cD^q x(t) = f(t, x(t)), \quad t \in [0, T], \tag{1.1}$$

where $^cD^q$ denotes the Caputo fractional derivative of order q and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$. We study (1.1) subject to two families of boundary conditions:

(i) Mixed boundary conditions

$$Tx'(0) = -ax(0) - bx(T) \qquad Tx'(T) = bx(0) + \mathrm{d}x(T). \tag{1.2}$$

(ii) Closed boundary conditions

$$x(T) = \alpha x(0) + \beta Tx'(0), \qquad Tx'(T) = \gamma x(0) + \delta Tx'(0), \tag{1.3}$$

where $a, b, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ are given constants.

Here we remark that the boundary conditions (1.2) interpolate between Neumann ($a = b = d = 0$) and Dirichlet ($a = b = d = \infty$) boundary conditions while (1.3) include quasi-periodic boundary conditions ($\beta = \gamma = 0$) and interpolate between periodic ($\alpha = \delta = 1, \beta = \gamma = 0$) and antiperiodic ($\alpha = \delta = -1, \beta = \gamma = 0$) boundary conditions. Notice that Zaremba boundary conditions $x(0) = 0, x'(T) = 0$ can be considered either as mixed boundary conditions with

* Corresponding author. Tel.: +966 509303423.
E-mail addresses: bashir_qau@yahoo.com (B. Ahmad), juanjose.nieto.roig@usc.es (J.J. Nieto), jspimentel75@gmail.com (J. Pimentel).

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$a = \infty, b = d = 0$ or as quasi-periodic boundary conditions with $\alpha = \infty, \gamma = \delta = 0$. For more details on Zaremba boundary conditions, see [16–18].

Let us recall some definitions of fractional calculus [1–3].

Definition 1.1. For a function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)\mathrm{d}s, \quad n-1 < q < n,\ n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 1.2. The Riemann–Liouville fractional integral of order q is defined as

$$I^{q}g(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}\mathrm{d}s, \quad q > 0,$$

provided the integral exists.

Definition 1.3. The Riemann–Liouville fractional derivative of order q for a function $g(t)$ is defined by

$$D^{q}g(t) = \frac{1}{\Gamma(n-q)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}}\mathrm{d}s, \quad n = [q]+1,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

To solve the boundary value problems (1.1) and (1.2), and (1.1) and (1.3), we need the following known result.

Lemma 1.1 ([3]). *For $q > 0$, the general solution of the fractional differential equation $^{c}D^{q}x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1$ $(n = [q]+1)$.

In view of Lemma 1.1, it follows that

$$I^{q\,c}D^{q}x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, \tag{1.4}$$

for some $c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1$ $(n = [q]+1)$.

Lemma 1.2. *For $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, a unique solution of the boundary value problem (1.1) and (1.2) is given by*

$$x(t) = \int_0^T G_1(t,s)f(s,x(s))\mathrm{d}s,$$

where $G_1(t, s)$ is the Green function given by

$$G_1(t,s) = \begin{cases} \dfrac{(t-s)^{q-1}}{\Gamma(q)} - \dfrac{1}{\Delta_1}\left(\dfrac{[T(b+d)+(b^2-ad)t](T-s)^{q-1}}{T\Gamma(q)}\right. \\ \quad \left. + \dfrac{[(a+b)t-(1+b)T](T-s)^{q-2}}{\Gamma(q-1)}\right), \quad 0 \le s \le t \le T, \\ -\dfrac{1}{\Delta_1}\left(\dfrac{[T(b+d)+(b^2-ad)t](T-s)^{q-1}}{T\Gamma(q)} + \dfrac{[(a+b)t-(1+b)T](T-s)^{q-2}}{\Gamma(q-1)}\right), \\ \quad 0 \le t \le s \le T, \end{cases} \tag{1.5}$$

with

$$\Delta_1 = (1+b)(b+d) - (a+b)(d-1) \neq 0. \tag{1.6}$$

Proof. Using (1.4), for some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = I^{q}f(t, x(t)) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s, x(s))\mathrm{d}s - c_0 - c_1 t. \tag{1.7}$$

In view of the relations $^{c}D^{q}I^{q}x(t) = x(t)$ and $I^{q}I^{p}x(t) = I^{q+p}x(t)$ for $q, p > 0, x \in L(0, T)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}f(s, x(s))\mathrm{d}s - c_1.$$

Using the boundary conditions (1.2) in (1.7), we find that

$$c_0 = \frac{1}{\Delta_1}\left\{(b+d)\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds - T(1+b)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds\right\},$$

$$c_1 = \frac{1}{\Delta_1}\left\{\frac{(b^2-ad)}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds + (a+b)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds\right\},$$

where Δ_1 is given by (1.6). Substituting the values of c_0 and c_1 in (1.7), we obtain

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\right. \\ &\quad \left. + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds\right\} \\ &= \int_0^T G_1(t,s)f(s,x(s))ds, \end{aligned}$$

where $G_1(t,s)$ is given by (1.5). This completes the proof. □

Lemma 1.3. *For $f \in C([0,T]\times\mathbb{R},\mathbb{R})$, the unique solution of the boundary value problem (1.1) and (1.3) is given by*

$$x(t) = \int_0^T G_2(t,s)f(s,x(s))ds,$$

where $G_2(t,s)$ is the Green function given by

$$G_2(t,s) = \begin{cases} \dfrac{(t-s)^{q-1}}{\Gamma(q)} - \dfrac{1}{\Delta_2}\left(\dfrac{[T(1-\delta)+\gamma t](T-s)^{q-1}}{T\Gamma(q)}\right. \\ \quad \left. + \dfrac{[(1-\alpha)t-(1-\beta)T](T-s)^{q-2}}{\Gamma(q-1)}\right), \quad 0\le s\le t\le T, \\ -\dfrac{1}{\Delta_2}\left(\dfrac{[T(1-\delta)+\gamma t](T-s)^{q-1}}{T\Gamma(q)} + \dfrac{[(1-\alpha)t-(1-\beta)T](T-s)^{q-2}}{\Gamma(q-1)}\right), \\ \quad 0\le t\le s\le T, \end{cases} \tag{1.8}$$

with

$$\Delta_2 = \gamma(1-\beta)+(1-\alpha)(1-\delta) \neq 0. \tag{1.9}$$

Proof. We do not provide the proof as it is similar to that of Lemma 1.2. □

2. Existence of solutions

In relation to the problems (1.1) and (1.2), and (1.1) and (1.3), we define

$$\mu_1 = \frac{1}{\Gamma(q+1)}\left\{1+\frac{|b+d+b^2-ad|+q|a-1|}{|\Delta_1|}\right\}, \tag{2.1}$$

$$\mu_2 = \frac{1}{\Gamma(q+1)}\left\{1+\frac{|1-\delta+\gamma|+q|\alpha-\beta|}{|\Delta_2|}\right\}, \tag{2.2}$$

where Δ_1 and Δ_2 are given by (1.6) and (1.9) respectively.

Theorem 2.1. *Assume that there exist constants $0 \le \kappa < \frac{1}{\mu_1}$ and $M > 0$ such that $|f(t,x)| \le \frac{\kappa}{T^q}|x| + M$ for all $t \in [0,T]$, $x \in C[0,T]$. Then the boundary value problem (1.1) and (1.2) has at least one solution.*

Proof. Using Lemma 1.2, the problem (1.1) and (1.2) can be transformed into a fixed point problem as

$$x = \mathcal{F}x, \tag{2.3}$$

where $\mathcal{F} : C[0, T] \to C[0, T]$ is given by

$$(\mathcal{F}x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d) + (b^2 - ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s))\mathrm{d}s + [(a+b)t - (1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s, x(s))\mathrm{d}s\right\}, \quad t \in [0, T].$$

Thus we just need to prove the existence of at least one solution $x \in C[0, T]$ satisfying (2.3). Define a suitable ball $B_R \subset C[0, T]$ with radius $R > 0$ as

$$B_R = \{x \in C[0, T] : \max_{t\in[0,T]} |x(t)| < R\},$$

where R will be fixed later. Then, it is sufficient to show that $\mathcal{F} : \overline{B}_R \to C[0, T]$ satisfies

$$x \neq \lambda\mathcal{F}x, \quad \forall x \in \partial B_R \text{ and } \forall\lambda \in [0, 1]. \tag{2.4}$$

Let us set

$$H(\lambda, x) = \lambda\mathcal{F}x, \quad x \in C(\mathbb{R})\ \lambda \in [0, 1].$$

Then, by the Arzela–Ascoli theorem, $h_\lambda(x) = x - H(\lambda, x) = x - \lambda\mathcal{F}x$ is completely continuous. If (2.4) is true, then the following Leray–Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\lambda, B_R, 0) &= \deg(I - \lambda\mathcal{F}, B_R, 0) = \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_r,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray–Schauder degree, $h_1(t) = x - \lambda\mathcal{F}x = 0$ for at least one $x \in B_R$. In order to prove (2.4), we assume that $x = \lambda\mathcal{F}x$ for some $\lambda \in [0, 1]$ and for all $t \in [0, T]$ so that

$$\begin{aligned}|x(t)| &= |\lambda\mathcal{F}x(t)|\\
&\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s, x(s))|\mathrm{d}s + \frac{|T(b+d) + (b^2 - ad)t|}{T|\Delta_1|}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}|f(s, x(s))|\mathrm{d}s\\
&\quad + \frac{|(a+b)t - (1+b)T|}{|\Delta_1|}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s, x(s))|\mathrm{d}s\\
&\le \left(\frac{\kappa}{T^q}|x| + M\right)\left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\mathrm{d}s + \frac{|T(b+d) + (b^2 - ad)t|}{T|\Delta_1|}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\mathrm{d}s\right.\\
&\quad \left. + \frac{|(a+b)t - (1+b)T|}{|\Delta_1|}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\mathrm{d}s\right]\\
&\le \left(\frac{\kappa}{T^q}|x| + M\right)\frac{T^q}{\Gamma(q+1)}\left[1 + \frac{|b + d + b^2 - ad| + q|a-1|}{|\Delta_1|}\right],\end{aligned}$$

which, on taking norm ($\sup_{t\in[0,T]} |x(t)| = \|x\|$) and using (2.1) yields

$$\|x\| \le \frac{MT^q\mu_1}{(1 - \kappa\mu_1)}.$$

Letting $R = \frac{MT^q\mu_1}{(1-\kappa\mu_1)} + 1$, (2.4) holds. This completes the proof. □

Example 2.1. Consider the following boundary value problem

$$\begin{cases} {}^cD^q x(t) = \dfrac{1}{(4\pi)}\sin\left(\dfrac{2\pi x}{T^q}\right) + \dfrac{|x|}{1+|x|}, & t \in [0, T],\ 1 < q \le 2,\\ Tx'(0) = -2x(0) - x(T) & Tx'(T) = x(0) + x(T). \end{cases} \tag{2.5}$$

Clearly

$$|f(t, x)| = \left|\frac{1}{(4\pi)}\sin\left(\frac{2\pi x}{T^q}\right) + \frac{|x|}{1+|x|}\right| \le \frac{1}{2T^q}\|x\| + 1,$$

with $\kappa = \frac{1}{2} < \frac{4\Gamma(q+1)}{5+q}$ for $1 < q \le 2$ and $M = 1$. Thus, the conclusion of Theorem 2.1 applies to the problem (2.5).

Theorem 2.2. *Assume that there exist constants* $0 \leq \overline{\kappa} < \frac{1}{\mu_2}$ *and* $\overline{M} > 0$ *such that* $|f(t,x)| \leq \frac{\overline{\kappa}}{T^q}|x| + \overline{M}$ *for all* $t \in [0,T], x \in C[0,T]$. *Then the boundary value problem (1.1) and (1.3) has at least one solution.*

Proof. Using Lemma 1.3 together with the arguments employed in the proof of Theorem 2.1, the proof can easily be constructed. So we omit the details. □

Remark 2.1. For positive constants N_1, N_2, we can modify the assumption on the nonlinear function $f(t,x)$ in Theorems 2.1 and 2.2 respectively as

$$|f(t,x)| \leq \frac{N_1}{T^q \mu_1}, \quad \forall t \in [0,T],\ x \in [-N_1, N_1],$$

$$|f(t,x)| \leq \frac{N_2}{T^q \mu_2}, \quad \forall t \in [0,T],\ x \in [-N_2, N_2],$$

where μ_1, μ_2 are respectively given by (2.1) and (2.2).

3. Fractional differential inclusions

In this section, we consider the fractional differential inclusions

$$^cD^q x(t) \in F(t, x(t)), \quad t \in [0,T],\ T > 0,\ 1 < q \leq 2, \tag{3.1}$$

where $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact-valued map, and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. We will study the existence of solutions for (3.1) subject to two families of boundary conditions (1.2) and (1.3).

Let $C([0,T])$ denote a Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Let $L^1([0,T],\mathbb{R})$ be the Banach space of measurable functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)| \mathrm{d}t$.

Now let us recall some basic concepts of multivalued maps [19,20] and set terminology.

For a normed space $(X, \|.\|)$, let $P(X) = \{Y \subseteq X : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\mathrm{Fix}\, G$. A multivalued map $G : [0,1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 3.1. A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \longmapsto F(t,x)$ is upper semi-continuous for almost all $t \in [0,T]$;
(iii) for each $q > 0$, there exists $\varphi_q \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_q(t) \quad \text{for all } \|x\|_\infty \leq q \text{ and for a.e. } t \in [0,T].$$

For each $y \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0,T]\}.$$

Let E be the Banach space, X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $x, y \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,T] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.2. Let Y be a separable metric space and let $\Phi : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say Φ has a property $(\mathcal{CD})$ if Φ is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0, T]. \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0, T] \times \mathbb{R}) \to \mathcal{P}(L^1([0, T], \mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0, T], \mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0, T]\},$$

which is called the Nymetzki operator associated with F.

Definition 3.3. Let $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nymetzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let (X, d) be a metric space. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

where $d(a, B) = \inf_{b\in B} d(a, b)$. H_d is the (generalized) Pompeiu–Hausdorff functional. It is known that $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [21]).

Definition 3.4. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) $\overline{\gamma}$-Lipschitz if and only if there exists $\overline{\gamma} > 0$ such that

$$H_d(N(x), N(y)) \le \overline{\gamma} d(x, y) \quad \text{for each } x, y \in X;$$

(b) a contraction if and only if it is $\overline{\gamma}$-Lipschitz with $\overline{\gamma} < 1$.

The following lemmas will be used in what follows.

Lemma 3.1 (*[22]*). *Let X be a Banach space. Let $F : [0, T] \times \mathbb{R} \to P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0, T], X)$ to $C([0, T], X)$. Then the operator*

$$\Theta \circ S_F : C([0, T], X) \to P_{cp,c}(C([0, T], X)), x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0, T], X) \times C([0, T], X)$.

Lemma 3.2 (*[23]*). *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, T], \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0, T], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 3.3 (*[24]*). *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Theorem 3.1. *Assume that*

(H_1) *$F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has compact and convex values;*
(H_2) *there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in L^1([0, T], \mathbb{R}^+)$ such that*

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \le p(t)\psi(\|x\|_\infty) \quad \text{for each } (t, x) \in [0, T] \times \mathbb{R};$$

(H_3) *there exists a number $M_1 > 0$ such that*

$$\frac{\Gamma(q)M_1}{T^{q-1}\left(1 + \frac{|b+d+b^2-ad|+(q-1)|a-1|}{|\Delta_1|}\right)\psi(M_1)\|p\|_{L^1}} > 1.$$

Then the boundary value problem (3.1) and (1.2) has at least one solution on $[0, T]$.

Proof. Define an operator $\Omega : C([0, T], \mathbb{R}) \to \mathcal{P}(C([0, T], \mathbb{R}))$ as

$$\Omega(x) = \Bigg\{h \in C([0, T], \mathbb{R}) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \frac{1}{\Delta_1}\Bigg\{\frac{[T(b+d) + (b^2 - ad)t]}{T} \times \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s + [(a+b)t - (1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s\Bigg\}, f \in S_{F,x}\Bigg\}.$$

We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. The proof consists of several steps. As a first step, we show that $\Omega(x)$ is convex for each $x \in C([0, T], \mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0, T]$, we have

$$h_i(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_i(s)\mathrm{d}s \right.$$
$$\left. + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_i(s)\mathrm{d}s\right\}, \quad i = 1, 2.$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0, T]$, we have

$$[\omega h_1 + (1-\omega)h_2](t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega)f_2(s)]\mathrm{d}s - \frac{1}{\Delta_1}$$
$$\times \left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega)f_2(s)]\mathrm{d}s \right.$$
$$\left. + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}[\omega f_1(s) + (1-\omega)f_2(s)]\mathrm{d}s\right\}.$$

Since $S_{F,x}$ is convex (F has convex values), it follows that $\omega h_1 + (1-\omega)h_2 \in \Omega(x)$.

Next, we show that $\Omega(x)$ maps bounded sets into bounded sets in $C([0, T], \mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0, T], \mathbb{R}) : \|x\|_\infty \le r\}$ be a bounded set in $C([0, T], \mathbb{R})$. Then, for each $h \in \Omega(x),\ x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s \right.$$
$$\left. + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s\right\}$$

and

$$|h(t)| \le \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |f(s)|\mathrm{d}s + \frac{1}{|\Delta_1|}\left\{\frac{|T(b+d)+(b^2-ad)t|}{T}\int_0^T \frac{|T-s|^{q-1}}{\Gamma(q)} |f(s)|\mathrm{d}s \right.$$
$$\left. + |(a+b)t-(1+b)T|\int_0^T \frac{|T-s|^{q-2}}{\Gamma(q-1)} |f(s)|\mathrm{d}s\right\}$$
$$\le \frac{T^{q-1}}{\Gamma(q)}\left(1 + \frac{|b+d+b^2-ad| + (q-1)|a-1|}{|\Delta_1|}\right)\int_0^T \varphi_r(s)\mathrm{d}s.$$

Thus,

$$\|h\|_\infty \le \frac{T^{q-1}}{\Gamma(q)}\left(1 + \frac{|b+d+b^2-ad| + (q-1)|a-1|}{|\Delta_1|}\right)\int_0^T \varphi_r(s)\mathrm{d}s.$$

Now we show that Ω maps bounded sets into equicontinuous sets of $C([0, T], \mathbb{R})$. Let $t', t'' \in [0, T]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0, T], \mathbb{R})$. For each $h \in \Omega(x)$, we obtain

$$|h(t'') - h(t')| = \left|\int_0^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \int_0^{t'} \frac{(t'-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[(b^2-ad)(t''-t')]}{T}\right.\right.$$
$$\left.\left. \times \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s + (a+b)(t''-t')\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s\right\}\right|$$
$$\le \left|\int_0^{t'} \frac{[(t''-s)^{q-1} - (t'-s)^{q-1}]}{\Gamma(q)} f(s)\mathrm{d}s\right| + \left|\int_{t'}^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s\right|$$
$$+ \left|\frac{(t''-t')}{\Delta_1}\left\{\frac{(b^2-ad)}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s + (a+b)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s\right\}\right|.$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t''-t' \to 0$. As Ω satisfies the above three assumptions, it follows by the Ascoli–Arzela theorem that $\Omega : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω has a closed graph. Let $x_n \to x_*$, $h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$h_n(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_n(s)\mathrm{d}s + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_n(s)\mathrm{d}s\right\}.$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$h_*(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)\mathrm{d}s + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)\mathrm{d}s\right\}.$$

Let us consider the continuous linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$f \mapsto \Theta(f)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s\right\}.$$

Observe that

$$\begin{aligned}\|h_n(t)-h_*(t)\| = \Bigg\| &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))\mathrm{d}s \\ &- \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))\mathrm{d}s\right. \\ &\left.+ [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}(f_n(s)-f_*(s))\mathrm{d}s\right\}\Bigg\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus, it follows by Lemma 3.1 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we have

$$h_*(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)\mathrm{d}s + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)\mathrm{d}s\right\}$$

for some $f_* \in S_{F,x_*}$.

Finally, we discuss a priori bounds on solutions. Let x be a solution of (1.1). Then there exists $f \in L^1([0,T],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,T]$, we have

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\mathrm{d}s + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\mathrm{d}s\right\}.$$

In view of (H_2), for each $t \in [0,T]$, we obtain

$$|x(t)| \le \frac{T^{q-1}}{\Gamma(q)}\left(1+\frac{|b+d+b^2-ad|+(q-1)|a-1|}{|\Delta_1|}\right)\psi(\|x\|_\infty)\int_0^T p(s)\mathrm{d}s.$$

Consequently, we have

$$\frac{\Gamma(q)\|x\|_\infty}{T^{q-1}\left(1+\frac{|b+d+b^2-ad|+(q-1)|a-1|}{|\Delta_1|}\right)\psi(\|x\|_\infty)\|p\|_{L^1}} \le 1,$$

In view of (H_3), there exists M_1 such that $\|x\|_\infty \ne M_1$. Let us set

$$U = \{x \in C([0,T],\mathbb{R}) : \|x\|_\infty < M_1 + 1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,T],\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type [25], we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (3.1) and (1.2). This completes the proof. □

Theorem 3.2. *Assume that* (H_1), (H_2) *and the following condition hold:*

($\overline{H}_3$) *there exists a number* $\overline{M}_1 > 0$ *such that*

$$\frac{\Gamma(q)\overline{M}_1}{T^{q-1}\left(1+\frac{|1-\delta+\gamma|+(q-1)|\alpha-\beta|}{|\Delta_2|}\right)\psi(\overline{M}_1)\|p\|_{L^1}} > 1.$$

Then the boundary value problem (3.1) *and* (1.3) *has at least one solution on* $[0,T]$.

Proof. We omit the proof as it employs the arguments used in the proof of Theorem 3.1. □

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with these problems is based on the nonlinear alternative of Leray–Schauder type together with the selection theorem of Bressan and Colombo [23] for lower semi-continuous maps with decomposable values.

Theorem 3.3. *Assume that* (H_2)–(H_3) *and the following conditions hold:*

(H_4) $F : [0,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
- (a) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L}\otimes\mathcal{B}$ *measurable,*
- (b) $x \longmapsto F(t,x)$ *is lower semi-continuous for each* $t \in [0,T]$;

(H_5) *for each* $\sigma > 0$, *there exists* $\varphi_\sigma \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le \varphi_\sigma(t) \quad \text{for all } \|x\|_\infty \le \sigma \text{ and for a.e. } t \in [0,T].$$

Then the boundary value problem (3.1) *and* (1.2) *has at least one solution on* $[0,T]$.

Proof. It follows from (H_4) and (H_5) that F is of l.s.c. type. Then from Lemma 3.2, there exists a continuous function $f : C([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,T],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), & t\in[0,T],\ T>0,\ 1<q\le 2,\\ Tx'(0) = -ax(0) - bx(T) & Tx'(T) = bx(0) + \mathrm{d}x(T). \end{cases} \tag{3.2}$$

Observe that if $x \in C^2([0,T])$ is a solution of (3.2), then x is a solution to the problem (3.1) and (1.2). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\overline{\Omega}x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(x(s))\mathrm{d}s \right.$$
$$\left. + [(a+b)t - (1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(x(s))\mathrm{d}s\right\}.$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. □

Remark 3.1. The analogue form of Theorem 3.3 for the problem (3.1) and (1.3) replaces the assumption (H_3) by ($\overline{H}_3$).

Next we prove the existence of solutions for the problem (3.1) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [24].

Theorem 3.4. *Assume that the following conditions hold:*

(H_6) $F : [0, T] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(., x) : [0, T] \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$.
(H_7) $H_d(F(t, x), F(t, \bar{x})) \le m(t)|x-\bar{x}|$ *for almost all* $t \in [0, T]$ *and* $x, \bar{x} \in \mathbb{R}$ *with* $m \in L^1([0, T], \mathbb{R}^+)$ *and* $d(0, F(t, 0)) \le m(t)$ *for almost all* $t \in [0, T]$.

Then the problem (3.1) *and* (1.2) *has at least one solution on* $[0, T]$ *if*

$$\frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)}\left(1+\frac{|b+d+b^2-ad|+(q-1)|a-1|}{|\Delta_1|}\right) < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0, T], \mathbb{R})$ by the assumption (H_6), so F has a measurable selection (see Theorem III.6 [26]). Now we show that the operator Ω satisfies the assumptions of Lemma 3.2. To show that $\Omega(x) \in P_{cl}((C[0, T], \mathbb{R}))$ for each $x \in C([0, T], \mathbb{R})$, let $\{u_n\}_{n\ge 0} \in \Omega(x)$ be such that $u_n \to u(n \to \infty)$ in $C([0, T], \mathbb{R})$. Then $u \in C([0, T], \mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0, T]$,

$$\begin{aligned} u_n(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_n(s)\mathrm{d}s\right. \\ &\left. + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_n(s)\mathrm{d}s\right\}. \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0, T], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0, T]$,

$$\begin{aligned} u_n(t) \to u(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v(s)\mathrm{d}s\right. \\ &\left. + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v(s)\mathrm{d}s\right\}. \end{aligned}$$

Hence $u \in \Omega(x)$.

Next we show that there exists $\gamma_1 < 1$ such that

$$H_d(\Omega(x), \Omega(\bar{x})) \le \gamma_1\|x-\bar{x}\|_\infty \quad \text{for each } x, \bar{x} \in C([0, T], \mathbb{R}).$$

Let $x, \bar{x} \in C([0, T], \mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0, T]$,

$$\begin{aligned} h_1(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_1(s)\mathrm{d}s\right. \\ &\left. + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_1(s)\mathrm{d}s\right\}. \end{aligned}$$

By (H_4), we have

$$H_d(F(t, x), F(t, \bar{x})) \le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t)-w| \le m(t)|x(t)-\bar{x}(t)|, \quad t \in [0, T].$$

Define $U : [0, T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t)-w| \le m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [26]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0, T]$, we have $|v_1(t)-v_2(t)| \le m(t)|x(t)-\bar{x}(t)|$.

For each $t \in [0, T]$, let us define

$$h_2(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)\mathrm{d}s - \frac{1}{\Delta_1}\left\{\frac{[T(b+d)+(b^2-ad)t]}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_2(s)\mathrm{d}s + [(a+b)t-(1+b)T]\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_2(s)\mathrm{d}s\right\}.$$

Thus

$$\begin{aligned}|h_1(t)-h_2(t)| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|v_1(s)-v_2(s)|\mathrm{d}s + \frac{1}{|\Delta_1|}\left\{\frac{|T(b+d)+(b^2-ad)t|}{T}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\right.\\ &\quad \left.\times |v_1(s)-v_2(s)|\mathrm{d}s + |(a+b)t-(1+b)T|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|v_1(s)-v_2(s)|\mathrm{d}s\right\}\\ &\le \frac{T^{q-1}}{\Gamma(q)}\left(1+\frac{|b+d+b^2-ad|+(q-1)|a-1|}{|\Delta_1|}\right)\int_0^T m(s)\mathrm{d}s\|x-\bar{x}\|.\end{aligned}$$

Hence

$$\|h_1(t)-h_2(t)\|_\infty \le \frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)}\left(1+\frac{|b+d+b^2-ad|+(q-1)|a-1|}{|\Delta_1|}\right)\|x-\bar{x}\|_\infty.$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$\begin{aligned}H_d(\Omega(x),\Omega(\bar{x})) &\le \gamma_1\|x-\bar{x}\|_\infty\\ &\le \frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)}\left(1+\frac{|b+d+b^2-ad|+(q-1)|a-1|}{|\Delta_1|}\right)\|x-\bar{x}\|_\infty.\end{aligned}$$

Since Ω is a contraction, it follows by Lemma 2.2 that Ω has a fixed point x which is a solution of (1.1)–(1.2). This completes the proof. □

Theorem 3.5. *Assume that* (H_6) *and* (H_7) *hold. Then the problem* (3.1)–(1.3) *has at least one solution on* $[0, T]$ *if*

$$\frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)}\left(1+\frac{|1-\delta+\gamma|+(q-1)|\alpha-\beta|}{|\Delta_2|}\right) < 1.$$

Proof. We do not provide the proof as it can easily be traced on the pattern of the proof of Theorem 3.4. □

Example 3.1. Consider the following inclusion boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t, x(t)), \quad t \in [0,1],\\ x'(0) = -x(0) - \frac{1}{3}x(1) \qquad x'(1) = \frac{1}{3}x(0) + \frac{2}{3}x(1), \end{cases} \tag{3.3}$$

where $q = 3/2$, $T = 1$, $a = 1$, $b = 1/3$, $d = 2/3$ and $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \left[\frac{x^3}{x^3+3} + t^3 + 3, \frac{x}{x+1} + t + 1\right].$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{x^3}{x^3+3} + t^3 + 3, \frac{x}{x+1} + t + 1\right) \le 5, \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le 5 = p(t)\psi(\|x\|_\infty), \quad x \in \mathbb{R},$$

with $p(t) = 1$, $\psi(\|x\|_\infty) = 5$. Further, using the condition

$$\frac{\Gamma(q)M_1}{T^{q-1}\left(1+\frac{|b+d+b^2-ad|+(q-1)|a-1|}{|\Delta_1|}\right)\psi(M_1)\|p\|_{L^1}} > 1,$$

we find that $M_1 > \frac{25}{2\sqrt{\pi}}$. Clearly, all the conditions of Theorem 3.1 are satisfied. So there exists at least one solution of the problem (3.3) on [0, 1].

4. Discussion

This paper studies the existence of solutions for some nonlinear boundary value problems of fractional differential equations of order $q \in (1, 2]$. Our results give rise to various interesting situations. Some of them are listed below:

(i) The results for a nonlinear boundary value problem of fractional order $q \in (1, 2]$ with quasi-periodic (quasi-antiperiodic) boundary conditions follow as a special case of the result (Theorem 2.2) of this paper by taking $\beta = \gamma = 0$.

(ii) The results for an antiperiodic boundary value problem of fractional differential equations of order $q \in (1, 2]$ can be obtained by taking $\alpha = -1 = \delta$, $\beta = \gamma = 0$.

(iii) For $q = 2$, we obtain new results for second order boundary value problems with mixed and closed boundary conditions. In this case, the Green functions $G_1(t, s)$ and $G_2(t, s)$ take the form

$$G_1(t,s) = \begin{cases} (t-s) - \dfrac{1}{\Delta_1}\left(\dfrac{[T(b+d)+(b^2-ad)t](T-s)}{T} + [(a+b)t-(1+b)T]\right), & 0 \le s \le t \le T, \\ -\dfrac{1}{\Delta_1}\left(\dfrac{[T(b+d)+(b^2-ad)t](T-s)}{T} + [(a+b)t-(1+b)T]\right), & 0 \le t \le s \le T, \end{cases}$$

$$G_2(t,s) = \begin{cases} (t-s) - \dfrac{1}{\Delta_2}\left(\dfrac{[T(1-\delta)+\gamma t](T-s)}{T} + [(1-\alpha)t-(1-\beta)T]\right), & 0 \le s \le t \le T, \\ -\dfrac{1}{\Delta_2}\left(\dfrac{[T(1-\delta)+\gamma t](T-s)}{T} + [(1-\alpha)t-(1-\beta)T]\right), & 0 \le t \le s \le T. \end{cases}$$

The Green functions $G_2(t, s)$ for the second order antiperiodic boundary value problem ($\alpha = -1 = \delta$, $\beta = \gamma = 0$) is

$$G_2(t,s) = \begin{cases} \dfrac{1}{4}(-T-2t+2s), & 0 \le t < s \le T, \\ \dfrac{1}{4}(-T+2t-2s), & 0 \le s \le t \le T. \end{cases}$$

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