Twisted holomorphic chains and vortex equations over non-compact Kähler manifolds

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\textbf{A B S T R A C T}

In this paper, we study twisted holomorphic chains and related gauge equations over non-compact Kähler manifolds. We use the heat flow method to solve the Dirichlet boundary problem for the related gauge equations, and prove a Hitchin–Kobayashi type correspondence for twisted holomorphic chain over some non-compact Kähler manifolds. © 2010 Elsevier Inc. All rights reserved.

1. Introduction

Let $M$ be a compact Kähler manifold and let $E$ be a holomorphic vector bundle over $M$. The Hitchin–Kobayashi correspondence \cite{22,17,9,10,24,26} states that a holomorphic structure is stable if and only if it is simple and admits a Hermitian–Einstein metric. The classical Hitchin–Kobayashi correspondence has several interesting and important generalizations and extensions where some extra structures are added to the holomorphic bundles. For example: Higgs bundles \cite{14,23}; holomorphic pairs \cite{5,6}; holomorphic triples, holomorphic chains \cite{12,7,1}; quiver bundles and twisted quiver bundles \cite{2,3} over Kähler manifolds. There are also other interesting generalizations, see references: \cite{4,8,16,18–21,25}.

In \cite{23}, Simpson also discussed the Higgs bundles over some non-compact Kähler manifolds. Let $M$ be a complex manifold of dimension $m$, with a Kähler metric $\omega$. By \cite{23}, we will make the following assumptions:

\textbf{Assumption 1.} $(M, \omega)$ has finite volume.

\textbf{Assumption 2.} There exists an exhaustion non-negative function $\varphi$ on $(M, \omega)$ with $\Delta \varphi$ bounded.

\textbf{Assumption 3.} There is an increasing function $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ and $\alpha(x) = x$ for $x > 1$, such that if $f$ is a bounded positive function on $(M, \omega)$ with $\Delta f \leq B$ then

\begin{equation}
\frac{\Delta f}{f} \leq B \quad \text{for all $x \in M$.}
\end{equation}
holomorphic chain will be denoted by $H_i$.

Furthermore, if $\Delta f \leq 0$ then $\Delta f = 0$.

From [23, Proposition 2.1 and Proposition 2.2], we know that if $(M, \omega)$ is a compact Kähler manifold, or $(M, \omega)$ is a Zariski open subset of a smooth compact Kähler manifold $M$ and the metric $\omega$ is the restriction of a smooth Kähler metric on $M$, then the above assumptions hold for $(M, \omega)$.

In this paper, we want to discuss twisted holomorphic chains over Kähler manifolds satisfying the above assumptions. A twisted holomorphic chain (as a special case of the twisted quiver bundles [2]) consists of a finite number of holomorphic bundles. For simplicity, we denote: $E = (E_0, E_1, \ldots, E_n)$ is the $(n + 1)$-tuple of holomorphic bundles $E_i$ over $M$, and $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ is an $n$-tuple of bundle holomorphic morphisms $\phi_i \in \text{Hom}(E_i \otimes E, E_{i-1})$, where $\{E_i\}$ is a collection of fixed holomorphic bundles.

A twisted holomorphic chain is the volume form of $\tau_0 \text{Id}_{E_0}$.

\[ \sqrt{-1} \Delta \tau_0 \text{Id}_{E_0}, \]

\[ \sqrt{-1} \Delta \tau \text{Id}_{E_1}, \]

\[ \sqrt{-1} \Delta \tau_n \text{Id}_{E_n}, \]

where $1 \leq i \leq n - 1$. For simplicity, we denote

\[ \theta_i(H; \tau) = \sqrt{-1} \Delta \tau \text{Id}_{E_i}, \]

where we set $\phi_0 = 0$ and $\phi_{n+1} = 0$. We define the $\tau$-degree of chain $C$ with respect to the metric $H$ to be the real number:

\[ \deg_{\tau}(C; H) = \int_M \left( \sum_{i=0}^{n} \text{Tr} \theta_i(H; \tau) \right) dv, \]

where $dv$ is the volume form of $(M, \omega)$. It is well known that Higgs bundles, holomorphic pairs, holomorphic triples can be seen as special cases of the twisted holomorphic chains. In this paper, we want to consider the above chain vortex equations (1.1) on some non-compact Kähler manifolds.

**Main theorem.** Let $(M, \omega)$ satisfy the above Assumptions 1, 2, 3, and suppose $(E, \phi)$ is a twisted holomorphic chain with an $(n + 1)$-tuple of Hermitian metrics $\mathbf{K}$ satisfying $\text{sup}_M (\sum_{i=0}^{n} |\Delta F_i|,|x|) < \infty$, $\text{sup}_M (\sum_{i=0}^{n} |\phi_i|^2) < \infty$, and $\deg_{\tau}(C; K) = 0$. Suppose $C = (E, \phi)$ is analytic $\tau$-stable with respect to the initial metrics $\mathbf{K}$. Then there is an $(n + 1)$-tuple of Hermitian metrics $\mathbf{H}$ with $|\delta E_i (K_i^{-1} H_i)|,|x| < \infty$ for each $i = 0, 1, \ldots, n$, such that it satisfies the chain $\tau$-vortex equations (1.1).

For the proof of the main theorem, we use the heat flow method which is used by Simpson in the Higgs bundles case [23]. Firstly, we consider the evolution equations with respect to chain $\tau$-vortex equations on compact domain. Using exhaustion method, we get the long-time existence result of the heat flow on non-compact Kähler manifold, by the way, we also solved the Dirichlet boundary problem for Eq. (1.1). Then we consider the convergence properties of the heat flow, under the analytic stability condition, we can show that there exists a sequence of Hermitian metrics along the heat flow which converges to a solution of Eq. (1.1). As an application of the main theorem, we obtain a Bogomolov type inequality for Chern numbers at the end of this paper. The paper is organized as follows: in Section 2, we give some estimates and preliminaries which will be used in the proof of the main theorem; in Section 3, we get the long-time existence result of the heat flow; in Section 4, we consider the poisson equation on some non-compact manifolds; in Section 5, we solve the Dirichlet boundary problem of Eq. (1.1); in Section 6, we introduce the definition of analytic $\tau$-stability of twisted holomorphic chains on non-compact Kähler manifolds, and prove that if there exist Hermitian metrics satisfying the chain
\[ \tau \text{-vortex equation (1.1), then the chain must be } \tau \text{-polystable; in Section 7, we give the proof of our main theorem; in the last section, we deduce a Bogomolov type inequality for analytic stable twisted holomorphic chain.} \]

2. Some preliminaries on chain vortex equations

Given a twisted holomorphic chain \( C = (E, \phi) \) on a compact Kähler manifold \((M, \omega)\) (with possible non-empty smooth boundary). Let \( K = (K_0, K_1, \ldots, K_n) \) be the initial \((n+1)\) tuple of Hermitian metrics on chain \( C \). Consider a family of tuples of Hermitian metrics \( H(t) = (H_0(t), H_1(t), \ldots, H_n(t)) \) on \( C \) with initial metric \( H(0) = K \). And denote \( h(t) = (h_0(t), h_1(t), \ldots, h_n(t)) \) to be an \((n+1)\) tuple of endomorphisms \( h_i = K_i^{-1}H_i \). When there is no confusion, we will omit the parameter \( t \) and simply write \( H, h \) for \( H(t), h(t) \). We consider the following heat equations of (1.1)

\[
\begin{align*}
H_0^{-1} \frac{\partial H_0}{\partial t} &= -2\left(\sqrt{-1} \Lambda F_{H_0} + \frac{1}{2} \phi_1 \circ \phi_1^H - \tau_0 \text{Id}_{E_0}\right), \\
H_1^{-1} \frac{\partial H_1}{\partial t} &= -2\left(\sqrt{-1} \Lambda F_{H_1} - \frac{1}{2} \phi_1^H \circ \phi_1 - \phi_1 \circ \phi_1^H + \tau_1 \text{Id}_{E_1}\right), \\
H_n^{-1} \frac{\partial H_n}{\partial t} &= -2\left(\sqrt{-1} \Lambda F_{H_n} - \frac{1}{2} \phi_n^H \circ \phi_n - \tau_n \text{Id}_{E_n}\right),
\end{align*}
\]

where \( 1 \leq i \leq n - 1 \). It is completely equivalent to the following evolution equations

\[
\begin{align*}
\frac{\partial h_0}{\partial t} &= -2\sqrt{-1} \Lambda \tilde{\partial} E_0 \partial_k h_0 + 2\sqrt{-1} \Lambda (\tilde{\partial} E_0 h_0 h_0^{-1} \partial_k h_0) - 2\sqrt{-1} \tau_0 h_0 \partial E_{K_0} + 2\tau_0 h_0 - h_0 \phi_1 h_0^{-1} \phi_1^K h_0, \\
\frac{\partial h_1}{\partial t} &= -2\sqrt{-1} \Lambda \tilde{\partial} E_1 \partial_k h_1 + 2\sqrt{-1} \Lambda (\tilde{\partial} E_1 h_1 h_1^{-1} \partial_k h_1) - 2\sqrt{-1} \tau_1 h_1 \partial E_{K_1} + 2\tau_1 h_1 + \phi_1 h_1 \phi_1^K h_1 - h_0 \phi_1 h_0^{-1} \phi_1^K h_1, \\
\frac{\partial h_n}{\partial t} &= -2\sqrt{-1} \Lambda \tilde{\partial} E_n \partial_k h_n + 2\sqrt{-1} \Lambda (\tilde{\partial} E_n h_n h_n^{-1} \partial_k h_n) - 2\sqrt{-1} \tau_n h_n \partial E_{K_n} + 2\tau_n h_n + \phi_n h_n \phi_n^K h_n - h_0 \phi_1 h_0^{-1} \phi_1^K h_1.
\end{align*}
\]

where we have used the identities

\[
\begin{align*}
F_{H_i} &= F_{K_i} + \tilde{\partial} E_i (h_i^{-1} \partial_k h_i), \\
\phi_i^H &= h_i^{-1} \circ \phi_i^K \circ h_i + \text{Id}_{E_i}, \quad \text{or} \\
\phi_i^H &= h_i^{-1} \circ \phi_i^K \circ h_i^{-1} + \text{Id}_{E_i}.
\end{align*}
\]

We know that the above equations are non-linear parabolic system, as in [9], \( h_i(t) \) are self-adjoint with respect to \( K_i \) for \( t \) > 0 since \( h_i(0) = \text{Id}_{E_i} \). For simplicity, we denote

\[
\Theta^2(H; \tau) = \sum_{i=0}^{n} |\theta_i(H; \tau)|_{H_i}^2, \quad \Phi^2(H) = \sum_{i=1}^{n} |\phi_i|^2_{H_i},
\]

where \( \theta_i \) is defined by (1.2).

**Proposition 2.1.** Let \( H(t) = (H_0(t), H_1(t), \ldots, H_n(t)) \) be a solution of the heat flow (2.1), then

\[
\left( \Delta - \frac{\partial}{\partial t} \right) \Theta^2(H; \tau) \geq 0,
\]

and

\[
\left( \Delta - \frac{\partial}{\partial t} \right) \sum_{i=0}^{n} \text{Tr} \theta_i(H; \tau) = 0.
\]

**Proof.** By calculating directly, we have

\[
\begin{align*}
\frac{\partial \theta_i}{\partial t} &= \sqrt{-1} \Lambda \tilde{\partial} E_i \left( \partial_{H_i} \left( h_i^{-1} \frac{d h_i}{d t} \right) \right) + \frac{1}{2} h_i^{-1} \frac{d h_i}{d t} \phi_i^H \phi_i \\
&\quad - \frac{1}{2} \phi_i^H h_i^{-1} \frac{d h_i}{d t} \phi_i - \frac{1}{2} \phi_i h_i^{-1} \frac{d h_i}{d t} \phi_i^H + \frac{1}{2} \phi_i \phi_i^H h_i^{-1} \frac{d h_i}{d t},
\end{align*}
\]

and

\[
\Delta |\theta_i|^2_{H_i} = 2 \text{Re} \langle -2 \sqrt{-1} \Lambda \tilde{\partial} E_i \theta_i, \theta_i \rangle_{H_i} + \text{Re} \langle 2 \sqrt{-1} \Lambda F_{H_i}, \theta_i \rangle_{H_i} + 2 |\partial H_i \theta_i|^2_{H_i} + 2 |\tilde{\partial} E_i \theta_i|^2_{H_i}.
\]
Using above formulas, we have
\[
\left( \Delta - \frac{\partial}{\partial t} \right) \Theta^2 = 2 \sum_{i=0}^{n} |\nabla \phi_i|^2_H + \sum_{i=0}^{n} \left| |\phi_i \phi_i^*|^2 + |\phi_i \phi_{i+1}^*|^2 + |\phi_i \phi_{i+1}|^2 \right| + |\phi_i \phi_{i+1}|^2 - 2 \langle \phi_{i+1} \phi_i, \phi_i \phi_i \rangle - 2 \langle \phi_{i+1} \phi_{i+1}^*, \phi_{i+1}^* \phi_i \rangle \\
= \sum_{i=0}^{n-1} \left( |\phi_i \phi_i^*|^2 + |\phi_i \phi_{i+1}^*|^2 + |\phi_i \phi_{i+1}|^2 \right) + \sum_{i=0}^{n} \left( |\phi_i \phi_i^*|^2 + |\phi_i \phi_{i+1}^*|^2 + |\phi_i \phi_{i+1}|^2 \right) + \sum_{i=1}^{n-1} \left( |\phi_i \phi_i^*|^2 + |\phi_i \phi_{i+1}^*|^2 + |\phi_i \phi_{i+1}|^2 \right) + \sum_{i=1}^{n} \left( |\phi_i \phi_i^*|^2 + |\phi_i \phi_{i+1}^*|^2 + |\phi_i \phi_{i+1}|^2 \right)
\]
\[
\geq 2 \sum_{i=0}^{n} |\nabla \phi_i|^2_H \geq 0.
\]
(2.8)

The formula (2.6) can be deduced from (2.7) directly. \( \square \)

**Proposition 2.2.** Let \( H(t) = (H_0(t), H_1(t), \ldots, H_n(t)) \) be a solution of the heat flow (2.1), then there exists a positive constant \( C_1 \) depending only on \( \text{rank}(E_1) \) and \( \text{rank}(E_i) \) such that
\[
\left( \Delta - \frac{\partial}{\partial t} \right) \Phi^2 (H) \geq 2 \sum_{i=1}^{n} |\rho_H \Phi_i|^2_H + C_1 \Phi^4 - \max_{1 \leq i \leq n} \{|\tau_i - \tau_{i-1}| \} \Phi^2.
\]
(2.9)

**Proof.** By calculating directly, we have
\[
\left( \Delta - \frac{\partial}{\partial t} \right) |\phi_i|^2_H = 2|\rho_H \phi_i|^2_H + 2|\phi_i \phi_i^*|^2 - \langle \phi_i \phi_i^*, \phi_i \phi_i \rangle + 2(\tau_i - \tau_0)|\phi_i|^2,
\]
\[
\left( \Delta - \frac{\partial}{\partial t} \right) |\phi_i|^2_H = 2|\rho_H \phi_i|^2_H + 2|\phi_i \phi_i^*|^2 - \langle \phi_i \phi_i^*, \phi_i \phi_i \rangle - \langle \phi_i \phi_i^*, \phi_i \phi_i \rangle + 2(\tau_i - \tau_{i-1})|\phi_i|^2,
\]
\[
\left( \Delta - \frac{\partial}{\partial t} \right) |\phi_i|^2_H = 2|\rho_H \phi_i|^2_H + 2|\phi_i \phi_i^*|^2 - \langle \phi_i \phi_i^*, \phi_i \phi_i \rangle - \langle \phi_i \phi_i^*, \phi_i \phi_i \rangle + 2(\tau_i - \tau_{i-1})|\phi_i|^2,
\]
where we have used \( \tilde{\rho}_{E_i} \otimes \tilde{\rho}_{E_{i-1}} \phi_i = 0 \), and Eq. (2.1). On the other hand, one can easily check that
\[
|\phi_i \phi_i^*|^2_{H_{i-1}} = |\phi_i \phi_i^*|^2_{H_{i-1}},
\]
(2.10)
and
\[
|\phi_i \phi_i^*|^2_{H_{i-1}} \geq \frac{1}{\min(\text{rank}(E_1) \times \text{rank}(E_i), \text{rank}(E_{i-1}))} |\phi_i|^4_{H_i}.
\]
(2.11)

From the above inequalities we have
\[
\left( \Delta - \frac{\partial}{\partial t} \right) \Phi^2 \geq 2 \sum_{i=1}^{n} |\rho_H \phi_i|^2 + \left( \frac{1}{\sum_{i=1}^{n} (2^i)} \right) \sum_{i=1}^{n} |\phi_i \phi_i^*|^2 - \max_{1 \leq i \leq n} \{|\tau_i - \tau_{i-1}| \} \Phi^2
\]
\[
\geq 2 \sum_{i=1}^{n} |\rho_H \phi_i|^2 + C_1 \Phi^4 - \max_{1 \leq i \leq n} \{|\tau_i - \tau_{i-1}| \} \Phi^2. \quad \square
\]

Next, we will introduce the Donaldson’s “distance” on the space of Hermitian metrics as follows.

**Definition 2.3.** For any two Hermitian metrics \( H, K \) on vector bundle \( E \) set
\[
\sigma (H, K) = \text{Tr} H^{-1} K + \text{Tr} K^{-1} H - 2 \text{rank } E.
\]
(2.12)

It is obviously that \( \sigma (H, K) \geq 0 \) with equality if and only if \( H = K \). The function \( \sigma \) is not quite a metric but it serves almost equally well in our problem. In particular, a sequence of metrics \( H_i \) converges to \( H \) in the usual \( C^0 \) topology if and only if \( \text{Sup} \sigma (H_i, H) \rightarrow 0 \).
Let $H = (H_0, H_1, \ldots, H_n)$ and $K = (K_0, K_1, \ldots, K_n)$ be two tuples of Hermitian metrics on chain $(E, \phi)$. We define the Donaldson’s distance of two tuples as follows:

$$\sigma(H, K) = \sum_{i=0}^{n} \sigma(H_i, K_i). \quad (2.13)$$

Since the twisted holomorphic chain is a special case of the twisted quiver bundle, so the estimates of quiver vortex equations are valid to chain vortex equation. By [28], we have the following estimates.

**Proposition 2.4.** (See [28, Proposition 3.3].) Let two $(n+1)$-tuples $H(t), K(t)$ be two solutions of the heat flow (2.1), then

$$\Delta - \frac{\partial}{\partial t}\left( \sum_{i=0}^{n} (\text{Tr}(K_i^{-1} H_i) + \text{Tr}(H_i^{-1} K_i)) \right) \geq 0. \quad (2.14)$$

**Corollary 2.5.** Let $H$ and $K$ be two tuples of Hermitian metrics satisfying the chain $\tau$-vortex equation (1.1), then

$$\Delta \sigma(H, K) \geq 0. \quad (2.15)$$

Let two $(n+1)$-tuples $H(t), K(t)$ be two families of Hermitian metrics on the twisted chain $C = (E, \phi)$. From [28], we have

**Proposition 2.6.** (See [28, Proposition 3.5].) Let $H(t), K(t)$ be two families of Hermitian metrics on the twisted chain $C$, then

$$\Delta - \frac{\partial}{\partial t}\left( \sum_{i=0}^{n} (\text{Tr}(K_i^{-1} H_i) + \text{Tr}(H_i^{-1} K_i)) \right) \geq -\left( \sum_{i=0}^{n} |\partial \theta_i(H; \tau)|_K \right). \quad (2.16)$$

where $\theta_i$ is defined in (2.4).

**Corollary 2.7.** Let $(n+1)$-tuples $H(x, t)$ be a solution of the heat flow (2.1) with the initial tuple $K$, then

$$\Delta - \frac{\partial}{\partial t}\left( \sum_{i=0}^{n} (\text{Tr}(K_i^{-1} H_i) + \text{Tr}(H_i^{-1} K_i)) \right) \geq -\left( \sum_{i=0}^{n} |\partial \theta_i(K; \tau)|_K \right). \quad (2.17)$$

**Corollary 2.8.** Let $H$ and $K$ be two $(n+1)$-tuples of Hermitian metrics on the chain $(E, \phi)$, then

$$\Delta \left( \sum_{i=0}^{n} (\text{Tr}(K_i^{-1} H_i) + \text{Tr}(H_i^{-1} K_i)) \right) \geq -\left( \sum_{i=0}^{n} |\partial \theta_i(H)|_H + \sum_{i=0}^{n} |\partial \theta_i(K)|_K \right). \quad (2.18)$$

If the tuple $H$ satisfies the chain $\tau$-vortex equation (1.1), then we have

$$\Delta \left( \sum_{i=0}^{n} (\text{Tr}(K_i^{-1} H_i) + \text{Tr}(H_i^{-1} K_i)) \right) \geq -\sum_{i=0}^{n} |\partial \theta_i(K)|_K. \quad (2.19)$$

**3. The long-time existence of the heat flow**

Let $(M, \omega)$ be a compact Kähler manifold (with possibly non-empty boundary), and $(E, \phi)$ be a twisted holomorphic chain over $M$. We fix Hermitian metrics $K_i$ on the bundles $E_i$ as the initial metrics. If $M$ is closed (i.e. compact without boundary), we consider the following evolution equations

$$H_i^{-1} \frac{\partial H_i}{\partial t} = -2\theta_i(H; \tau), \quad H_i(t)|_{t=0} = K_i, \quad 0 \leq i \leq n. \quad (3.1)$$

Suppose $M$ is a compact manifold with non-empty smooth boundary $\partial M$, and the Kähler metric $\omega$ is smooth and non-degenerate on the boundary. Fix Hermitian metrics $K_i$ on the bundles $E_i$, then we will consider the Neumann boundary problem

$$H_i^{-1} \frac{\partial H_i}{\partial t} = -2\theta_i(H; \tau), \quad H_i(t)|_{t=0} = K_i, \quad \frac{\partial}{\partial \nu} H_i \bigg|_{\partial M} = 0. \quad (3.2)$$
for all $0 \leq i \leq n$, and the Dirichlet boundary problem:

$$
H_i^{-1} \frac{\partial H_i}{\partial t} = -2\theta_i(H; \tau), \quad H_i(t)|_{t=0} = K_i, \quad H_i|_{\partial M} = K_i|_{\partial M}.
$$

(3.3)

for all $0 \leq i \leq n$. Here $\frac{\partial}{\partial t}$ denotes differentiation of sections of bundles in the direction perpendicular to the boundary $\partial M$ using the connections induced by those metric connections $D_{K_i}$.

Since the above heat equations are parabolic, and the Neumann or Dirichlet boundary conditions are good, the solutions exist for short time [13].

**Proposition 3.1.** For sufficiently small $\epsilon > 0$, the heat equations (3.1), (3.2) and (3.3) have smooth solutions defined for $M \times (0, \epsilon)$.

Next we want to prove the long-time existence of the evolution equations. Let tuples $H(t)$ be a solution of the above evolution equations, and $h_i = K_i^{-1}H_i$, for $0 \leq i \leq n$. Then

$$
\left| \frac{\partial}{\partial t} (\log \text{Tr} h_i) \right| = \left| \frac{\text{Tr}(\frac{\partial h_i}{\partial t})}{\text{Tr} h_i} \right| = 2 \left| \frac{\text{Tr}(h_i \theta_i(H; \tau))}{\text{Tr} h_i} \right| 
\leq 2|\theta_i(H; \tau)|_{H_i},
$$

(3.4)

and

$$
\left| \frac{\partial}{\partial t} (\log \text{Tr} h_i^{-1}) \right| \leq 2|\theta_i(H; \tau)|_{H_i}.
$$

(3.5)

**Proposition 3.2.** $\sup_M (\sum_{i=0}^n |\theta_i(H; \tau)|_{H_i}^2)$ is decreasing with time in a solution of the heat equation (3.1), (3.2) or (3.3).

**Proof.** When $H(t)$ is a solution of (3.2), by applying $\frac{\partial}{\partial t}$ to both sides of the heat equation (3.2), $|\theta_i(H; \tau)|_{H_i}^2$ satisfies the corresponding Neumann boundary condition. On the other hand, when $H(t)$ is a solution of (3.3), we have $(\sum_{i=0}^n |\theta_i(H; \tau)|_{H_i}^2)_{|\partial M} = 0$. By Proposition 2.1 and the maximum principle, $\sup_M (\sum_{i=0}^n |\theta_i(H; \tau)|_{H_i}^2)$ is decreasing with time. □

**Theorem 3.3.** Suppose that a smooth solution $H(t)$ to the evolution equation (3.1), (3.2) or (3.3) is defined for $0 \leq t < T$. Then $H(t)$ converges in $C^0$-topology to some $(n+1)$-tuple $H(T)$ of continuous non-degenerate metrics as $t \to T$.

**Proof.** Given $\epsilon > 0$, by continuity at $t = 0$ we can find a $\delta$ such that

$$
\sup_M \sigma(H(t), H(t')) < \epsilon,
$$

for $0 < t, t' < \delta$. Then Proposition 2.4 and the maximum principle imply that

$$
\sup_M \sigma(H(t), H(t')) < \epsilon,
$$

for all $t, t' > T - \delta$. This implies that $H_i(t)$ are uniformly Cauchy sequence and converge to a continuous limiting metric $H_i(T)$, for every $0 \leq i \leq n$. By Proposition 3.2, we know that $\sum_{i=0}^n |\theta_i(H; \tau)|_{H_i}^2$ are bounded uniformly. Using formulas (3.4) and (3.5), one can conclude that $\sigma(H_i(t), K_i)$ are bounded uniformly, therefore $H_i(T)$ is a non-degenerate metric. □

Discussing like that in [9, Lemma 19] or [15, Lemma 4.3.2], one can easily prove the following lemma.

**Lemma 3.4.** (See [23, Lemma 6.4.]) Let $H(t)$, $0 \leq t < T$ (or $\infty$) be any one-parameter family of Hermitian metrics on holomorphic bundle $E$ over compact $\kappa$-H"{a}hler manifold $M$ without boundary (or with non-empty smooth boundary). If $H(t)$ (satisfying Neumann or Dirichlet boundary conditions) converges in $C^0$ topology to some continuous metric $H(T)$ as $t \to T$, and if $\sup_M |\Lambda F_{H_t}|$ is bounded uniformly in $t$, then $H(t)$ is bounded in $C^1$ and also bounded in $L^p_2$ (for any $1 < p < \infty$) uniformly in $t$.

**Theorem 3.5.** Given any initial tuple $K$ of Hermitian metrics, then the evolution equations (3.1), (3.2), (3.3) have a unique solution $H(t)$ which exists for $0 \leq t < \infty$.

**Proof.** Proposition 3.1 guarantees that a solution exists for a short time. Suppose that the solution $H(t)$ exists for $0 \leq t < T$. By Theorem 3.3, $H(t)$ converges in $C^0$-topology to an $(n+1)$-tuple $H(T)$ of non-degenerate continuous limit metrics as $t \to T$. We have known that $\sqrt{-1} \Lambda F_{H_i} - \frac{1}{2}(\phi_i^{H_i} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{H_i}) - \tau_i Id_{E_i}|_{H_i}$ is bounded independently of $t$. Moreover,
Lemma 3.4, Hi(t) is bounded in $C^1$-topology and also bounded in $L^p_2$ (for any $1 < p < \infty$) uniformly in $t$. Since the evolution equations are quadratic in the first derivative of $h_i$, we can apply Hamilton’s method [13] to deduce that $H_i(t) \to H_i(T)$ in $C^\infty$-topology, for every $0 \leq i \leq n$, and the solution can be continued past $T$. Then the evolution equations (3.1), (3.2) and (3.3) have a solution $H(t)$ defined for all time.

On the other hand, suppose that $H'(t)$ is another solution of Eq. (3.1) ((3.2) or (3.3)) with the same initial tuple $K$ of Hermitian metrics. From Proposition 2.4, we have

$$\left(\Delta \frac{\partial}{\partial t}\right) \sigma (\mathbf{H}(t), \mathbf{H}'(t)) \geq 0,$$

and $\sigma (\mathbf{H}, \mathbf{H}')|_{t=0} = 0$. By the maximum principle, we have

$$\sigma (\mathbf{H}(t), \mathbf{H}'(t)) \equiv 0, \quad \text{i.e.} \quad \mathbf{H}(t) \equiv \mathbf{H}'(t).$$

So we have proved the uniqueness of the solution. □

**Proposition 3.6.** Let $M$ satisfy Assumptions 1, 2, 3, and let $(E, \phi)$ be a twisted holomorphic chain over $M$. Suppose the initial $(n+1)$-tuple of Hermitian metrics $K$ satisfies that $\sup_M \left(\sum_{i=0}^{n} |\theta_i(K; \tau)|_K\right) = B_1 < \infty$ for any $0 \leq i \leq n$. Then there is a unique solution $H$ to the heat equations (2.1) with $H_0 = K$, and such that $\sum_{i=0}^{n} \sup_M |H_i|_K < \infty$ on each finite interval of time. For this solution, we have $\sup_M \left(\sum_{i=0}^{n} |\theta_i(H; \tau)|_H\right) \leq \sup_M \left(\sum_{i=0}^{n} |\theta_i(K; \tau)|_K\right)$. Furthermore, if $\Phi^2(K) = \sup_M \left(\sum_{i=1}^{n} |\phi_i|^2\right) = B_2 < \infty$, then the solution $H$ must satisfy $\Phi^2(H) = \sup_M \left(\sum_{i=1}^{n} |\phi_i|^2\right) \leq \max(B_2, B_3)$, where $B_3$ is a positive constant depending only on $\text{rank}(E_i)$, $\text{rank}(\bar{E}_i)$ and $\tau_i$.

**Proof.** For each $a > 0$ such that $\partial M_a$ (where $M_a$ denotes the compact set $\varphi(x) \leq a$) is smooth, let $H_0(\cdot, \tau)$ be the solution with Neumann or Dirichlet boundary conditions given by Theorem 3.5. By Proposition 3.2 we have

$$\sup_{M_0} \left(\sum_{i=0}^{n} |\theta_i(H_0)|_{H_0}\right) \leq \sup_{M} \left(\sum_{i=0}^{n} |\theta_i(K)|_{K}\right).$$

Using formulas (3.4) and (3.5), we can obtain the following estimate

$$\sup_{M_0 \times [0, T]} \sigma (K, H_0) \leq 2 \left(\sum_{i=0}^{n} \text{rank } E_i\right) \left(\exp(2BT) - 1\right).$$

From the above estimate, we know that $\sum_{i=0}^{n} |H_{a,i}|_K$ is bounded on the finite interval $0 \leq t \leq T$, and the bound is independent of $a$. To obtain the $C^0$ convergence as $a \to \infty$, we need the following lemma.

**Lemma 3.7.** (See [23, Lemma 6.7.]) Suppose $u$ is a function on some $M_a \times [0, T]$, satisfying

$$\left(\Delta \frac{\partial}{\partial t}\right) u \geq 0, \quad u|_{t=0} = 0,$$

and suppose there is a bound $\sup_{M_a} u \leq C_0$. Then we have

$$u(x, t) \leq \frac{C_0}{a} \left(\varphi(x) + C_2t\right),$$

where $C_2$ is the bound of $\Delta \varphi$ in Assumption 2.

For a fixed relatively compact set $Z \subset M$, we fix $a_0$ such that $Z \subset M_{a_0}$. Let $u = \sigma (H_0, H_0)$. The $C^0$ bound derived above gives the bound on $u$, and $u$ is a sub-solution for the heat operator with $u(0) = 0$ by Proposition 2.4. So if $1 \leq a_0 < a \leq b$ then

$$\sigma (H_b, H_b) \leq C_0 (1 + C_2T) \frac{a_0}{a}$$

on $Z \times [0, T]$, i.e. $H_0$ is Cauchy for $a \to \infty$.

From (3.6) and (3.7), we know that $\sum_{i=0}^{n} |\Delta F_{H_{a,i}|_K|$ is uniformly bounded in $M_{a_0} \times [0, T]$ for $a > a_0$. Using Lemma 3.4 and discussing like that in [23, Proposition 6.6], we obtain that $H_a(\cdot, t)$ is bounded in $L^p_2(Z)$ for any $p > 0$ as $a \to \infty$. The bound is uniform in $0 \leq t \leq T$. By the evolution equations (2.1), the time derivative of $H_{a,i}$ is bounded in $L^p$ so $H_a$ is bounded in $L^p_2(Z \times [0, T])$. Here $L^p_2(Z \times [0, T])$ denotes the space of metrics with two $L^p$ derivatives in the space direction and one in the time direction. By going to a subsequence we may assume that for each relatively compact set, $H_a \to H$ in $L^p_2/1$. By
the Sobolev embedding $H^s \to H$ in $C^{1,0}$ over compact sets. Therefore the standard parabolic theory shows that, by passing to a subsequence, $H^s$ converges uniformly over any compact subset of $M \times [0, \infty)$ to a smooth $H$ which is a solution of the evolution equations (2.1). And $H$ satisfies the same $C^0$ bound on each finite interval of time as that in (3.7), as well as the bound $\sup_{M} \sum_{i=0}^{n} |f_i(H; \tau)|_H \leq B$. By using Lemma 3.7 again, it is easy to conclude the uniqueness of the solution.

Let $H^{s, \alpha}(t, \tau)$ be the solution with the Dirichlet boundary condition on $M_\alpha$. From Proposition 2.2, we have

$$\left( \frac{\partial}{\partial t} \right) \Phi^2(H_\alpha) \geq C \Phi^4(H_\alpha) - \max_{1 \leq i \leq n} \{ |\tau_i - \tau_{i-1}| \} \Phi^2(H_\alpha).$$

(3.8)

By the maximum principle of the heat flow, we get the following inequality

$$\sup_{M_\alpha} \Phi^2(H_\alpha) \leq \max \left\{ \sup_{M_\alpha} \Phi^2(K), C \max_{1 \leq i \leq n} \{ |\tau_i - \tau_{i-1}| \} \right\}.$$  

(3.9)

So, the last assertion in Proposition 3.6 can be concluded easily by the above inequality. □

4. Poisson equations on non-compact manifolds

Let $M$ be a non-compact Riemannian manifold satisfying Assumptions 1, 2, 3. In this section, we want to solve the following Poisson equation on $M$:

$$\Delta f = g,$$

(4.1)

where $f \equiv g = 0$ and $\sup_M |g| \leq +\infty$. When $M$ is compact, it is known that the Poisson equation (4.1) can be solved. But when $M$ is non-compact, the solvability of (4.1) is not clear. The idea of our discussion comes from Simpson’s paper [23]. The application of this section is to obtain initial Hermitian metric $K$ satisfying $\text{tr} \Lambda F_K = C$, since it can be reduced to solve the poisson equation (4.1) by conformal transformation. In the following part of this paper, the $L^p(M, \omega)$ norm of a function $\psi$ will be denoted by $\|\psi\|_{L^p}$ or $\|\psi\|_p$.

As above, for each $\alpha > 0$, let $M_\alpha$ denote the compact set $\psi(x) \leq \alpha$ with smooth boundary $\partial M_\alpha$. We first consider the following heat equation on $M_\alpha$ with Neumann boundary condition,

$$\begin{cases}
\frac{\partial}{\partial t} f = \Delta f - g, \\
\frac{\partial f}{\partial \nu} \bigg|_{\partial M_\alpha} = 0, \\
f \big|_{t=0} = 0.
\end{cases}$$

(4.2)

Since the above heat equation is strictly parabolic, and the Neumann boundary condition is good, the standard parabolic theory shows that the solution of (4.2) exists for long time. Suppose that $f_\alpha$ is a solution of (4.2), it is easy to check that,

$$\begin{cases}
\left( \frac{\partial}{\partial t} \right) (\Delta f_\alpha - g)^2 \geq 0, \\
\left. \frac{\partial}{\partial \nu} (\Delta f_\alpha - g)^2 \right|_{\partial M_\alpha} = 0.
\end{cases}$$

(4.3)

By the maximum principle, we have

$$\sup_{M_\alpha} |\Delta f_\alpha| \leq 2 \sup_{M_\alpha} |g| \leq 2 \sup_{M} |g|.$$

(4.4)

Similarly as that in Section 3, we want to get $C^0$ convergence of $f_\alpha$ on relatively compact sets $Z$. Let $u_{ab} = (f_\alpha - f_b)^2$. It is easy to prove that

$$\begin{cases}
\left( \frac{\partial}{\partial t} \right) u_{ab} \geq 0, \\
u_{ab} \big|_{t=0} = 0.
\end{cases}$$

By Lemma 3.7, we have

$$u_{ab} \leq C \cdot \frac{a_0}{a} \text{ on } M_\alpha \times [0, T].$$

So, $f_\alpha$ is a Cauchy sequence for $a \to \infty$. On the other hand, since $\sup_{M_\alpha} |\Delta f_\alpha|$ is bounded uniformly, it is easy to conclude that $f_\alpha$ are bounded in $L^2(\Omega)$, where $\Omega$ is a fixed relatively compact $\Omega \subset X$. By passing to a subsequence which we
also denoted by \( \{ f_a \} \), we may assume that for each relatively compact set, \( f_a \to f \) in \( L^p_{2,1} \). The standard parabolic theory shows that the limit \( f \) is a smooth solution of the heat equation
\[
\begin{aligned}
\frac{\partial}{\partial t} f &= \Delta f - g, \\
f|_{t=0} &= 0,
\end{aligned}
\tag{4.5}
\]
on \( M \) and it satisfies the bound \( \sup_M |\Delta f| \leq 2 \sup_M |g| \).

Let \( f_1 \) and \( f_2 \) be two solutions of the heat equation (4.5), and \( \Delta f_1, \Delta f_2 \) be uniformly bounded. It is easy to prove that
\[
\left( \Delta - \frac{\partial}{\partial t} \right) (f_1 - f_2)^2 \geq 0.
\]

Applying Lemma 3.7 to \( u = (f_1 - f_2)^2 \), we have the uniqueness \( f_1 \equiv f_2 \).

So, we have proved the following lemma.

**Lemma 4.1.** Let \( M \) satisfy Assumptions 1, 2, 3 and \( g \) be \( C^\infty \) function defined on \( M \) satisfying \( \sup_M |g| < \infty \). Then there is a unique solution \( f \) to the heat equation (4.5) with \( \sup_M |\Delta f| \leq 2 \sup_M |g| \).

Furthermore, we assume that \( f_M = 0 \). Let \( f_a \) be a solution of (4.2). For any compact set \( Z \subset X \), we have
\[
\int_Z f_a(\cdot,t) = -\int_{M_a \setminus Z} f_a(\cdot,t) + \int_{M_a} f_a(\cdot,t) = -\int_0^t \int_{M_a} g \, dx \, dt - \int_{M_a \setminus Z} f_a(\cdot,t).
\]
So
\[
\left| \int_Z f_a(\cdot,t) \right| \leq 4t \sup_M |g| \Vol(M \setminus Z).
\]

Then
\[
\left| \int_Z f(\cdot,t) \right| \leq 4t \sup_M |g| \Vol(M \setminus Z).
\]

So we have
\[
\int_M f(\cdot,t) = 0. \tag{4.6}
\]

On the other hand,
\[
\Delta e^{f_a} = e^{f_a} \Delta f_a + e^{f_a} \left| \nabla f_a \right|^2.
\]

Using the above equality and integrating over \( M_a \), we have
\[
\int_{M_a} e^{f_a} \left| \nabla f_a \right|^2 = \int_{M_a} -e^{f_a} \Delta f_a \leq C,
\]
where \( C \) is independent of \( a \).

Since \( f_a \to f \) in \( L^p_{2,loc} \), we have
\[
\int_M \left| \nabla f \right|^2(\cdot,t) < +\infty. \tag{4.7}
\]

It is easy to check that
\[
\Delta \ln(e^f + e^{-f}) = \nabla \left( \frac{1}{e^f + e^{-f}} \nabla (e^f + e^{-f}) \right) = \nabla \left( \frac{e^f - e^{-f}}{e^f + e^{-f}} \nabla f \right) \geq -|\Delta f| \tag{4.8}
\]
\[
\ln \left( \frac{1}{2} (e^f + e^{-f}) \right) \leq |f| \leq \ln (e^f + e^{-f}). \tag{4.9}
\]

By Assumption 3 and the bound in Lemma 4.1, we have
\[
|f| \leq \ln (e^f + e^{-f}) \leq C_1 \left( \int_M \ln \left( \frac{1}{2} (e^f + e^{-f}) + \ln 2 \right) \right)
\]
\[
\leq C_3 \int_M |f| + C_4.
\]

where \(C_3\) and \(C_4\) depend only on \(\sup_M |g|\) and \(\text{Vol}(M)\). So we have
\[
\|f\|_\infty \leq C_3 \|f\|_1 + C_4. \tag{4.10}
\]

**Lemma 4.2.**
\[
\frac{d}{dt} \int_M fg + \frac{1}{2} |\nabla f|^2 = - \int_M |\Delta f - g|^2. \tag{4.11}
\]

**Proof.** Many steps in the proof of this lemma require integration by parts, so we have to be careful if \(M\) is not compact. Let \(\varphi\) be the exhaustion non-negative function in Assumption 2, then
\[
\int_{M_s} |d\varphi|^2 = \int_{M_s} (s - \varphi) \Delta \varphi \leq C_5 s,
\]
where \(C_5\) depends only on the bound of \(\Delta \varphi\) and \(\text{Vol}(M)\). On the other hand
\[
\frac{d}{ds} \int_{M_s} |d\varphi|^2 = \int_{\partial M_s} |d\varphi|,
\]
so
\[
\int_0^s \int_{\partial M_s} |d\varphi| \, ds \leq C_5 s.
\]

By (4.7), we have
\[
\int_M |f(\cdot, t_2) \nabla f(\cdot, t_1)|^2 < +\infty,
\]
for any \(0 \leq t_1, t_2 < \infty\). Now
\[
\int_0^s \int_{M_s} \left| \text{div}(f(\cdot, t_2) \nabla f(\cdot, t_1)) \right| \, ds \leq \int_0^s \int_{\partial M_s} \left| f(\cdot, t_2) \nabla f(\cdot, t_1) \right| \, ds
\]
\[
\leq \left( \int_0^s \int_{\partial M_s} \left| f(\cdot, t_2) \nabla f(\cdot, t_1) \right|^2 \, |d\varphi| \right)^{1/2} \left( \int_0^s \int_{\partial M_s} |d\varphi| \right)^{1/2}
\]
\[
\leq C_6 s^{3/2},
\]
where \(C_6\) is a positive constant independent on \(s\). So we can choose a subsequence \(s_i \to +\infty\) such that
\[
\int_{M_{s_i}} \text{div}(f(\cdot, t_2) \nabla f(\cdot, t_1)) \to 0,
\]
as \( i \to \infty \). So

\[
\int_M \text{div}(f(\cdot, t) \Delta f(\cdot, t)) = 0.
\]  

(4.12)

Since \( |\frac{\partial f}{\partial t}| \) is bounded uniformly, we have

\[
|f(\cdot, t) - f(\cdot, t_0)| \leq C_7(t - t_0).
\]  

(4.13)

On the other hand, \( \Delta f(\cdot, t) \) is continuous in \( t \) uniformly on compact sets, and it is bounded. So we have

\[
\lim_{t \to t_0} \int_M |\Delta(f(\cdot, t) - f(\cdot, t_0))| \to 0.
\]  

(4.14)

Then, using (4.12), (4.13), (4.14), we have

\[
\frac{d}{dt} \int_M fg + \frac{1}{2} |\nabla f|^2 \bigg|_{t=t_0} = \lim_{t \to t_0} \frac{1}{t - t_0} \left[ \int_M f(\cdot, t)g + \frac{1}{2} |\nabla f(\cdot, t)|^2 - \int_M f(\cdot, t_0) + \frac{1}{2} |\nabla f(\cdot, t_0)|^2 \right]
\]

\[
= \lim_{t \to t_0} \frac{1}{t - t_0} \left[ \int_M (f(\cdot, t) - f(\cdot, t_0))g - \int_M \frac{1}{2} (f(\cdot, t) \Delta f(\cdot, t) - f(\cdot, t_0) \Delta f(\cdot, t_0)) \right]
\]

\[
= \lim_{t \to t_0} \frac{1}{t - t_0} \int_M [f(\cdot, t) - f(\cdot, t_0)] [g - \Delta f(\cdot, t_0)]
\]

\[
- \frac{1}{2} [f(\cdot, t) - f(\cdot, t_0)] \Delta [f(\cdot, t) - f(\cdot, t_0)]
\]

\[
= - \int_M (\Delta f(\cdot, t_0) - g)^2. \quad \Box
\]  

(4.15)

In the following, we want to prove that \( \int_M |f(\cdot, t)| \) must be bounded uniformly in \( t \). If not, we have \( \limsup_{t \to +\infty} \|f(t)\|_1 = +\infty \). So we can choose a sequence \( t_i \to +\infty \), such that \( \lim_{t \to +\infty} \|f(t_i)\|_1 = +\infty \). Let \( u_i = \|f(t_i)\| < +\infty \), and \( u_i = t_i^{-1} f(t_i) \in C^\infty(M) \). Using (4.10), we have

\[
\|u_i\|_1 = 1, \quad \text{and} \quad \|u_i\|_\infty \leq C_8,
\]  

(4.16)

where \( C_8 \) is a constant depending only on \( \sup_M |g| \) and \( \text{Vol}(M) \).

By (4.15), we have

\[
\int_M u_i u_i g + t_i^2 \frac{1}{2} |\nabla u_i|^2 \leq 0.
\]  

(4.17)

From the above inequality, we have

\[
\frac{1}{2} \int_M |\nabla u_i|^2 \leq - \frac{1}{t_i} \int_M u_i g.
\]

and

\[
\int_M |\nabla u_i|^2 \to 0.
\]  

(4.18)

Then, passing to a subsequence, we have that \( u_i \) converges weakly to \( u_\infty \) in \( L^2_{\text{loc}} \), and \( u_\infty \) is constant almost everywhere. Note that for any relatively compact \( Z \subset X \), \( L^2_{\text{loc}}(Z) \) is compact. So

\[
\int_Z |u_i| \to \int_Z |u_\infty|.
\]

On the other hand, \( \sup_M |u_i| \leq C_8 \) and \( M \) has finite volume, so for any \( \varepsilon > 0 \), we have
where $\tau$ is big enough. Thus $1 \geq \int_Z |u_\infty| \geq 1 - \varepsilon$. So, we have

$$\int_M |u_\infty| = 1, \quad \text{and} \quad u_\infty = C^* \neq 0 \quad \text{a.e.} \quad (4.19)$$

On the other hand, by (4.6) we have

$$\int_M u_i = 0. \quad (4.20)$$

Similarly as the above, we can obtain

$$\int_M u_\infty = 0. \quad (4.21)$$

We get a contradiction, so we have proved that $\int_M |f(\cdot, t)|$ must be bounded uniformly in $t$. From formula (4.10), we have a uniform positive constant such that

$$\sup_M |f(\cdot, t)| \leq C_0. \quad (4.22)$$

By Lemma 4.1 and formula (4.15), we also have uniform bounds on $\Delta f$ and $\int_M |\nabla f|^2$. So we may choose a subsequence $t_i \to \infty$ such that $f(\cdot, t_i) \to f_\infty$ weakly in $L^p_{2,loc}$ and $f_\infty(\Delta f - g)^2(\cdot, t_i) \to 0$. The elliptic regularity implies that $f_\infty$ is a smooth solution of Eq. (4.1) and $\sup_M |f_\infty| < \infty$. In fact, we obtain

**Theorem 4.3.** Let $M$ be a non-compact Riemannian manifold satisfying Assumptions 1, 2, 3, and suppose that $g \in C^\infty(M)$ satisfies $f_M, g = 0$ and $\sup_M |g| \leq +\infty$. Then there is a function $f \in C^\infty(M)$ which satisfies the Poisson equation (4.1) and $\sup_M |f| < \infty$.

### 5. The Dirichlet boundary problem for chain vortex equations

In this section, we will consider the case when $M$ is the interior of compact Kähler manifold $(\bar{M}, \omega)$ with non-empty smooth boundary $\partial M$, the Kähler metric is smooth and non-degenerate on the boundary, and the twisted holomorphic chain $(E, \phi)$ is defined over $\bar{M}$. In [11], Donaldson solved the Dirichlet boundary value problem for Hermitian–Einstein metrics on Kähler manifolds; and in [27], the second author generalized the above Donaldson’s result to general Hermitian case. We will show that the solution of Eq. (3.3) always converges to an $(n + 1)$-tuple of Hermitian metrics which satisfy the chain $\tau$-vortex equations (1.1) and the Dirichlet boundary conditions. In fact, we solve the following Dirichlet boundary problem.

**Theorem 5.1.** Let $E$ be a holomorphic vector bundle over the compact Kähler manifold $\bar{M}$ with non-empty smooth boundary $\partial M$, where $E = (E_0, E_1, \ldots, E_n)$ is the $(n + 1)$-tuple of holomorphic bundles $E_i$ over $M$, and $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ is an $n$-tuple of bundle morphisms $\phi_i \in \text{Hom}(E_i \otimes \bar{E}_i, E_{i-1}) \{1 \leq i \leq n\}$. For any Hermitian metric $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ on the restriction of $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$ to $\partial M$, there is a unique Hermitian metric $H = (H_1, H_2, \ldots, H_n)$ on $E$ such that $H$ satisfies chain $\tau$-vortex equations and the Dirichlet boundary condition:

$$\sqrt{-1} \Delta F_{H_0} + \frac{1}{2} \phi_1 \circ \phi_1^{H} = \tau_0 I_{E_0},$$

$$\sqrt{-1} \Delta F_{H_i} + \frac{1}{2} \left( \phi_i^{H} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{H} \right) = \tau_i I_{E_i},$$

$$\sqrt{-1} \Delta F_{H_n} + \frac{1}{2} \phi_n^{H} \circ \phi_n = \tau_n I_{E_n},$$

$$H_i|_{\partial M} = \varphi_i,$$  \hspace{1cm} (5.1)

where $1 \leq i \leq n - 1$, $\phi_i^{H}$ is the adjoint of $\phi_i$ taken with respect to $H$.

First, we shall need the following lemma.

**Lemma 5.2.** (See [11].) Suppose $g \geq 0$ is a sub-solution of the heat equation on $\bar{M} \times [0, \infty)$, i.e. $\frac{\partial g}{\partial t} - \Delta g \leq 0$. If $g = 0$ on $\partial M$ for all time then $g$ decays exponentially.
where $\mu > 0$ depends only on $M$, and $C$ depends on the initial value of $g$.

Let $H(t)$ be a solution of the evolution equation (3.3) with initial data $K$, we consider the function $\Theta^2 = \sum_{i=0}^{n} |\theta_i(H; \tau)|^2_{H_i}$ on $M \times [0, \infty)$, where $\theta_i$ are defined by (2.4). By Proposition 2.1, we know that $(\Delta - \frac{\partial}{\partial t})\Theta^2 \geq 0$, and the Dirichlet boundary condition satisfied by $H(t)$ implies that, for $t > 0$, $\Theta^2$ vanishes on the boundary of $M$.

Thus Lemma 5.2 tells us that $\Theta^2$ decays exponentially

$$\sup_{M} \Theta^2(t, s) \leq Ce^{-\mu t}, \quad \mu > 0$$

and in particular that

$$\int_{0}^{\infty} \Theta(-t, s) dt < \infty.$$  \hspace{1cm} (5.4)

Let $t_1 \leq t \leq t_2$, and $\tilde{h}_i(x, t) = H_i^{-1}(x, t_1)H_i(x, t)$. It is easy to check that

$$\tilde{h}_i^{-1} \frac{\partial \tilde{h}_i}{\partial t} = -2\theta_i.$$  \hspace{1cm} (5.5)

Then we have

$$|\log \text{Tr} \tilde{h}_i(t_2)| - |\log \text{Tr} \tilde{h}_i(t_1)| = \int_{t_1}^{t_2} \frac{\partial}{\partial s} |\log \text{Tr} \tilde{h}_i(s)| ds \leq \int_{t_1}^{t_2} \left| \frac{\text{Tr} \tilde{h}_i^{-1} \frac{\partial \tilde{h}_i}{\partial t}}{\text{Tr} \tilde{h}_i} \right| ds \leq \int_{t_1}^{t_2} 2\Theta dt$$

and similarly

$$|\log \text{Tr} \tilde{h}_i^{-1}(t_2)| \leq |\log \text{Tr} \tilde{h}_i^{-1}(t_1)| + \int_{0}^{t} 2\Theta dt.$$  \hspace{1cm} (5.6)

By inequalities (5.6) and (5.7), we have

$$\sup_{M} \sigma(H(t_1), H(t_2)) \leq 2 \left( \sum_{i=0}^{n} \text{rank}(E_i) \right) \left( e^{\int_{0}^{t} 2\Theta dt} - 1 \right).$$  \hspace{1cm} (5.8)

From the above inequalities, we know that $\sup_{M} \sigma(H(t_1), H(t_2))$ is uniformly bounded for $t \in [0, \infty)$. Then there exists a subsequence of the $H(t)$ convergence in $C^0$ to some continuous metric $H_\infty$. Using Lemma 3.4 again, we know that $H(t)$ is bounded in $C^1$ and also bounded in $L^p_2$ (for any $1 < p < \infty$) uniformly in $t$. On the other hand, $\Theta$ is bounded uniformly. Then the standard elliptic regularity implies that there exists a subsequence $H_t \rightarrow H_\infty$ in the $C^\infty$ topology. From (5.4), we know that $H_\infty$ is the desired Hermitian metric satisfying the boundary condition. The uniqueness can be easily deduced from Corollary 2.5 and the maximum principle. So we have proved Theorem 5.1.

6. Analytic stability of twisted holomorphic chains

Let $(M, \omega)$ be a Kähler manifold satisfying Assumptions 1, 2, 3; and let $C = (E, \phi)$ be a twisted holomorphic chain over $M$. Set a background metric $K = (K_0, \ldots, K_n)$ on the chain $C = (E, \phi)$, and set the parameters $\tau = (\tau_0, \ldots, \tau_n)$. Make the assumption $\sup_{M} (\sum_{i=0}^{n} |\theta_i(K; \tau)|) \leq B$, then we define the $\tau$-degree of chain $C$ with respect to the metric $K$ to be the real number:

$$\deg_\tau(C; K) = \int_{M} \left( \sum_{i=0}^{n} \text{Tr} \theta_i(K; \tau) \right) d\nu,$$

where $d\nu$ is the volume form of $(M, \omega)$. Then the $\tau$-slope of chain $C$ is defined by
\[ \mu_\tau(C, K) = \frac{\deg_\tau(C, K)}{\sum_{i=0}^n \text{rank } E'_i}. \]  

(6.2)

When \((M, \omega)\) is a closed Kähler manifold, the degree defined in the above is a holomorphic invariant which is the same as that defined in [1].

For further consideration, let us introduce the following definitions.

**Definition 6.1.** Let \( C = (E, \phi) \) be a twisted-holomorphic chain over \( M \).

1. A morphism \( f : C \to C' \) between two twisted holomorphic chains \( C \) and \( C' = (E', \phi') \) with the same twisting bundles \( \tilde{E}_\alpha \) is given by a collection of morphisms \( f_i : E_i \to E'_i \) for each \( 0 \leq i \leq n \), such that \( \phi'_\alpha \circ (f_\alpha \otimes \text{Id}_{E_\alpha}) = f'_\alpha \circ \phi_\alpha \) for each \( 1 \leq \alpha \leq n \).

2. A weakly holomorphic sub-chain of a twisted chain \( C = (E', \phi') \) is a saturated sub-sheaf of \( E_i \) for each \( 0 \leq i \leq n \), and \( \phi' \circ (f_\alpha \otimes \text{Id}_{E_\alpha}) = f'_\alpha \circ \phi_\alpha \) for each \( 1 \leq \alpha \leq n \), where \( f_i : E'_i \to E_i \) are the inclusion morphisms. When each \( E'_i \) is a holomorphic sub-bundle of \( E_i \) for each \( 0 \leq i \leq n \), we call \( C' \) to be a holomorphic sub-chain of \( C \).

3. The weakly holomorphic sub-chain \( C' \hookrightarrow C \) is called proper if \( 0 < \sum_{i=0}^n \text{rank } E'_i < \sum_{i=0}^n \text{rank } E_i \).

4. The twisted holomorphic chain \( C \) is called decomposable if it can be written as a direct sum \( C = C^1 \oplus C^2 \) of two holomorphic sub-chains with \( C^1 \neq C \neq C^2 \). Otherwise, \( C \) is called indecomposable.

5. The twisted holomorphic chain \( C \) is called simple if its only holomorphic endomorphisms are the multiples \( \lambda \text{Id}_C \) of the identity endomorphism.

If \( E'_i \) is a saturated sub-sheaf of \( E_i \) then outside of complex co-dimension 2 it is a sub-bundle of \( E_i \). The metric \( K_i \) restricts to a metric on \( E'_i \) outside complex co-dimension two. Let \( \pi_i : E_i \to E'_i \) denote the projection onto \( E'_i \) using the metric \( K_i \), it is also defined outside complex codimension two. So we can define the degree by integrating outside complex co-dimension two.

**Definition 6.2.** Let \( C = (E, \phi) \) be a twisted holomorphic chain over \( M \). Let \( \tau = (\tau_0, \tau_1, \ldots, \tau_n) \in \mathbb{R}^{n+1} \), and \( K = (K_0, K_1, \ldots, K_n) \) be an \((n+1)\)-tuple of Hermitian metrics on chain \( C \). The \( \tau \)-degree and \( \tau \)-slope of a weakly holomorphic sub-chain \( C' \) with respect to metric \( K \) are defined by

\[ \deg_\tau(C', K) = \sum_{i=0}^n \left[ \frac{1}{2} \sum_{i=0}^n \text{Tr} \left( \pi_i \circ \phi_i^* K \phi_i - \phi_{i+1}^* \phi_i K \right) - \sum_{\alpha=1}^n \left| \phi_\alpha^* K \right| \right] \tau_i \]  

\[ \mu_\tau(C, K) = \frac{\deg_\tau(C', K)}{\sum_{i=0}^n \text{rank } E'_i}, \]

respectively. Here \( \phi_\alpha^* = \pi_{\alpha-1} \circ \phi_\alpha \circ ((\text{Id}_{E_\alpha} - \pi_\alpha) \otimes \text{Id}_{E_\alpha}) \). We say that the twisted holomorphic chain \( C \) is analytic \( \tau \)-stable with respect to metric \( K \) if for all proper weakly holomorphic sub-chain \( C' \hookrightarrow C \),

\[ \mu_\tau(C', K) < (\leq) \mu_\tau(C, K). \]  

(6.3)

A direct sum of \( \tau \)-stable twisted holomorphic chain, all of them with the same \( \tau \)-slope, is called \( \tau \)-polystable.

The degree of sub-chain defined is either a real number or \(-\infty\), and if the degree is not \(-\infty\) then \( \pi_i \in L^2_\tau \) for each \( 0 \leq i \leq n \). On the other hand, a straightforward computation shows that

\[ \frac{1}{2} \sum_{i=0}^n \text{Tr} \left( \pi_i \circ \phi_i^* K \phi_i - \phi_{i+1}^* \phi_i K \right) = - \sum_{\alpha=1}^n \left| \phi_\alpha^* K \right|. \]  

(6.4)

If \((M, \omega)\) is a closed Kähler manifold, by Chern–Weil formula and formula (6.4), the degree \( \deg_\tau(C', K) \) is a holomorphic invariant which is independent of metric \( K \); in fact we have

\[ \deg_\tau(C', K) = \sum_{i=0}^n (\deg(E'_i) - \tau_i \text{rank } E'_i). \]  

(6.5)

Here \( \deg(E'_i) \) is just the degree of the sheaf \( E'_i \).

Suppose that the chain \( C \) has an \((n+1)\)-tuple of Hermitian metrics satisfying the chain \( \tau \)-vortex equations (1.1), then taking traces in (1.1), integrating over \((M, \omega)\), and summing for \( 0 \leq i \leq n \), one sees that the \( \tau \)-parameters are constrained by the relation

\[ \deg_\tau(C, K) = 0. \]  

(6.6)
This means that there are only \( n \) independent parameters among \( \tau_0, \ldots, \tau_n \). In the proof of the Hitchin–Kobayashi correspondence, we shall take \( \tau \) satisfying (6.6) for the equations, for the stability condition, it will be convenient to use \( \alpha = (\alpha_0, \ldots, \alpha_n) \) defined by

\[
\alpha_i = \tau_i - \tau_0, \tag{6.7}
\]

so that \( \alpha_0 = 0 \) and the independent parameters are \( \alpha_1, \ldots, \alpha_n \). By the definition, we have

\[
\mu_\alpha(C, K) = \mu_\tau(C, K) + \text{Vol}(M, \omega)\tau_0, \tag{6.8}
\]

hence the stability condition does not change under global translations on parameter vector. So, \( \tau \)-(semi) stability is equivalent to \( \alpha \)-(semi) stability.

Next, we will show that the \( \tau \)-stability is the necessary condition for the existence of solutions of the chain \( \tau \)-vortex equation (1.1). In fact, we prove the following theorem.

**Theorem 6.3.** Let \( (M, \omega) \) be a Kähler manifold satisfying Assumptions 1, 2, and 3, and \( C = (E, \phi) \) be a twisted holomorphic chain over \( M \). Suppose that the chain \( C \) admits an \( (n+1) \)-tuple \( \mathbf{H} = (H_0, \ldots, H_n) \) of Hermitian metrics satisfying the chain \( \tau \)-vortex equations (1.1), then chain \( C \) must be \( \tau \)-polystable with respect to metric \( H \).

**Proof.** This result is proved in exactly the same way as in [2, Section 3.2], so here we only sketch the proof. We can assume that \( C \) is indecomposable. Let \( C' = (E_0', E_1', \ldots, E_n') \) be a proper weakly holomorphic sub-chain, and \( \pi_i \) be the section of \( E_i^* \otimes E_i \) associated to the saturated sub-sheaf \( E_i^* \). Using (1.1), we have \( \mu_\tau(C, H) = 0 \) and

\[
\mu_\tau(C', H) = -\frac{1}{\text{rank} E_i'} \sum_{i=0}^n \int_M \sum_{i=0}^n (\partial E_i \otimes E_i) + \sum_{i=1}^n \left| \phi_{ij} \right|_{H}^2 \omega^{[n]}.
\]

The indecomposability of \( C \) implies that either \( \partial E_i \otimes E_i \neq 0 \) or \( \phi_{ij} \neq 0 \) for some \( i, \alpha \); thus \( \mu_\tau(C', H) < 0 \), hence \( C \) is analytic \( \tau \)-stable with respect to metric \( H \). \( \square \)

7. Proof of the main theorem

In this section, we will use the analytic stability to deduce that the solution \( H(t) \) of above evolution equations (2.1) must converge to an \((n+1)\)-tuple \( H(\infty) \) metric which satisfies the chain \( \tau \)-vortex equation (1.1). For further discussion, we shall introduce the following machineries. Let \( M_D(K_i, H_i) \) be the Donaldson Lagrangian [9] of two hermitian metrics \( K_i, H_i \) on bundle \( E_i \), and we define the modified Donaldson Lagrangian \( M_{\phi, \alpha} \) of two \((n+1)\)-tuples of hermitian metrics as follows:

\[
M_{\phi, \alpha}(K, H) = \sum_{i=0}^{n} M_D(K_i, H_i) + \sum_{i=1}^{n} \int_M (|\phi|^2_{K} - |\phi|^2_{K}) - 2 \sum_{i=0}^{n} \int_M \alpha_i \text{Tr} (\log (K_i^{-1} H_i)). \tag{7.1}
\]

Here \( K = (K_0, \ldots, K_n) \), \( H = (H_0, \ldots, H_n) \), \( K_i \) and \( H_i \) are hermitian metrics on bundle \( E_i \), and \( \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \).

For reader’s convenience, we recall some notation. Let \( K \) be a fixed hermitian metric on bundle \( E \), denote

\[
S_K(E) = \{ s \in \Omega^0(M, \text{End}(E)) \mid s^* s = s \}. \tag{7.2}
\]

Given \( \rho \in C^\infty(R, R) \) and \( s \in S_K(E) \). We define \( \rho(s) \) as follows. At each point \( x \) on \( M \), choose \( \{e_i\}_i=1 \) to be a unitary basis with respect to metric \( K \), such that \( s(e_i) = \delta_i e_i \). Set

\[
\rho(s)(e_i) = \rho(\delta_i) e_i. \tag{7.3}
\]

Given \( \Psi \in C^\infty(R \times R, R) \), \( s \in S_K(E) \), \( p \in \Omega^0(M, \text{End}(E)) \). In a similar way, we define \( \Psi[s](p) \) as follows. Let \( \{e_i\}_i=1 \) be the dual basis for \( \{e_i\}_i=1 \), then \( p \in \Omega^0(M, \text{End}(E)) \) can be written as

\[
p = \sum p_{ij} e_i^* \otimes e_j.
\]

Set

\[
\Psi[s](p) = \sum \Psi(\delta_i, \delta_j) p_{ij} e_i^* \otimes e_j. \tag{7.4}
\]

In fact, the Donaldson's Lagrangian is defined as follows:
\[ M_D(K, H) = 2 \int_M \left[ \log(K^{-1}H) \sqrt{-1} \Lambda F_{K}^{1,1} \right]_K \\
+ 2 \int_M \left[ \log(K^{-1}H), \sqrt{-1} \Lambda \psi \left( \log(K^{-1}H) \right) \left( \partial_K \log(K^{-1}H) \right) \right]_K, \]  
(7.5)

where \( \psi(x, y) = \frac{e^{x+y}-(x-y)}{(x-y)^2} \).

**Lemma 7.1.**

1. Let \( H^1, H^2, H^3 \) be three \((n+1)\)-tuples of Hermitian metrics on chain \( \mathbf{C} = (E, J, \phi) \), and \( \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \), then

\[ M_{\phi, \alpha}(H^1, H^2) = M_{\phi, \alpha}(H^1, H^2) + M_{\phi, \alpha}(H^2, H^3). \]  
(7.6)

2. Let \( H(t) \) be a solution of Eq. (2.1) with initial tuple \( K \) which satisfies the same conditions in the main theorem, then

\[ \frac{d}{dt} M_{\phi, \tau}(K, H(t)) = - \int_M n \left[ 2 \sqrt{-1} \Lambda F_{H_i} - (\phi_i^* H_i - \phi_{i+1}^* H_{i+1}) - 2 H_i \right]_{H_i}^2. \]  
(7.7)

**Proof.** (1) In [23, Proposition 5.1], Simpson has shown that

\[ M_D(H^1, H^2) = M_D(H^1, H^2) + M_D(H^2, H^3). \]

So formula (7.6) can be deduced directly by the definition of modified Donaldson Lagrangian (7.1).

(2) By Proposition 3.6 we know that \( \sup_M |h_i(H; \tau)|_{H_i} \) and \( \sup_M |\Lambda F_{H_i}|_{H_i} \) are uniformly bounded. Then, there exists a uniform constant \( C \) such that

\[ |h_i^{-1} h_i - \text{Id}_{E_i} |_{H_i} \leq C |t_1 - t_2| \]

for any \( t_1 \) and \( t_2 \) in a finite interval \( 0 \leq t \leq T \), so we have

\[ \frac{d}{dt} \int_M |\phi_i|^2_{H(t)} = \lim_{t \to t_0} \frac{1}{t - t_0} \int_M |\phi_i|^2_{H(t)} - |\phi_i|^2_{H(t_0)} \]

\[ = \lim_{t \to t_0} \frac{1}{t - t_0} \int_M \text{tr} \left( \phi_i \circ h_i^{-1}(t) \otimes \text{Id}_{E_i} \circ \phi_i^* \circ h_i^{-1}(t_0) - \phi_i \circ h_i^{-1}(t_0) \otimes \text{Id}_{E_i} \circ \phi_i^* \circ h_i^{-1}(t_0) \right) \]

\[ = \lim_{t \to t_0} \frac{1}{t - t_0} \int_M \text{tr} \left( \phi_i \circ h_i^{-1}(t) \otimes \text{Id}_{E_i} \circ \phi_i^* \circ h_i^{-1}(t_0) \right) \]

\[ + \lim_{t \to t_0} \frac{1}{t - t_0} \int_M \text{tr} \left( \phi_i \circ h_i^{-1}(t) \otimes \text{Id}_{E_i} \circ \phi_i^* \circ (h_i^{-1}(t) - h_i^{-1}(t_0)) \right) \]

\[ + \lim_{t \to t_0} \frac{1}{t - t_0} \int_M \text{tr} \left( \phi_i \circ h_i^{-1}(t_0) \otimes \text{Id}_{E_i} \circ \phi_i^* \circ (h_i^{-1}(t_0) - h_i^{-1}(t_0)) \right) \]

\[ = \int_M \text{tr} \left( \phi_i \circ h_i^{-1} \frac{\partial h_i}{\partial t} \circ h_i^{-1} \frac{\partial h_i}{\partial t} + h_i \frac{\partial h_i}{\partial t} \circ \phi_i^* \circ \phi_i \right) \bigg|_{t=t_0}. \]

On the other hand, since \( \sup_M |\Lambda F_{H_i}|_{H_i} \) is uniformly bounded and \( H_i \) is bounded for any finite time interval, Simpson proved that [23, Lemma 7.1]

\[ \frac{d}{dt} M_D(K_i, H_i) = 2 \int_M \text{tr} \left( h_i \frac{\partial h_i}{\partial t} \sqrt{-1} \Lambda F_{H_i} \right). \]

Combining above identities, we obtain (7.7). \( \square \)

Let \( H(t) = (H_0(t), \ldots, H_n(t)) \) be a solution of Eq. (2.1) with initial tuple \( K \), and \( h(t) = (h_0(t), \ldots, h_n(t)) \) where \( h_i = K_i^{-1} H_i = \exp(s_i) \). From Corollary 2.8, we have
\[
\Lambda \lg \left\{ \sum_{i=0}^{n} (\text{Tr}(h_i) + \text{Tr}(h_i^{-1})) \right\} \geq - \left( \sum_{i=0}^{n} 2\sqrt{-1} \Lambda F_{K_i} - (\phi_i^* K \phi_i - \phi_{i+1}^* K) - 2\tau_i \text{Id}_{E_i} \bigg|_{K_i} \right) \\
- \left( \sum_{i=0}^{n} 2\sqrt{-1} \Lambda F_{H_i} - (\phi_i^* H \phi_i - \phi_{i+1}^* H) - 2\tau_i \text{Id}_{E_i} \bigg|_{H_i} \right). \tag{7.8}
\]

From Proposition 2.1, we know that \( \sup_M 2\sqrt{-1} \Lambda F_{K_i} - (\phi_i^* K \phi_i - \phi_{i+1}^* K) - \tau_i \text{Id}_{E_i} \big|_{K_i} \) is bounded independently of \( t \). By Assumption 3, there exist two constants \( B_5 \) and \( B_6 \) such that
\[
\left\| \lg \left\{ \sum_{i=0}^{n} (\text{Tr}(h_i) + \text{Tr}(h_i^{-1})) \right\} \right\|_{\infty} \leq B_5 \left( \sum_{i=0}^{n} (\text{Tr}(h_i) + \text{Tr}(h_i^{-1})) \right) + B_6 \tag{7.9}
\]

On the other hand, one can check that
\[
\sum_{i=0}^{n} \| s_i(t) \|_{\infty} \leq B_7 + B_8 \left( \sum_{i=0}^{n} \| s_i(t) \|_1 \right). \tag{7.11}
\]

In the following, we will get \( C^0 \)-estimate for \( H(t) \) under the analytic stability condition.

**Proposition 7.2.** Let \( H(t) = (H_0(t), \ldots, H_n(t)) \) be a solution of Eq. (2.1) with initial tuple \( K \) which satisfies the same conditions in the main theorem. Suppose that \( \text{tr}(\sum_{i=0}^{n} \theta_i(\tau)) = 0 \) and \( C = (E, \phi) \) is analytic \( \tau \)-stable with respect to the initial metrics \( K \). There exists constant \( B_9 > 0 \) such that, for every \( t \in [0, +\infty) \), we have
\[
\sum_{i=0}^{n} \| s_i(t) \|_{\infty} < B_9. \tag{7.12}
\]

**Proof.** We prove it by contradiction. If not, then we must have
\[
\limsup_{t \to \infty} \left( \sum_{i=0}^{n} \| s_i(t) \|_1 \right) = +\infty.
\]

In particular, we can choose a sequence \( \{t_j\}_{j=1}^{\infty} \) such that: \( t_j \to \infty \) and \( \sum_{i=0}^{n} \| s(t_j) \|_1 \to \infty \). Set \( E = E_0 \oplus E_1 \oplus \cdots \oplus E_n \), then \( H = H_0 \oplus H_1 \oplus \cdots \oplus H_n \) is a hermitian metric on \( E \), denote \( h = h_0 \oplus h_1 \oplus \cdots \oplus h_n \), and \( s = s_0 \oplus s_1 \oplus \cdots \oplus s_n \in \text{End}(E) \). The morphisms \( \phi_i : E_i \oplus \bar{E}_i \to E_i \) induce a section \( \bar{\phi} = \bigotimes_{i=1}^{n} \phi_i \) of the bundle \( \bigotimes_{i=1}^{n} \text{Hom}(E_i \oplus \bar{E}_i, E_i^{-1}) \) (or \( \bigotimes_{i=1}^{n} \text{Hom}(E_i, E_i^{-1}) \otimes \bar{E}_i \)). Then, we can define the endomorphisms \( \bar{\phi}^* H \phi, \bar{\phi}^* H \in \bigotimes_{i=1}^{n} \text{End}(E_i) \) in a natural way as that in the introduction. We denote \( \pi_i : E \to E_i \) to be the projection to sub-bundle \( E_i, K = K_0 \oplus K_1 \oplus \cdots \oplus K_n \) to be the initial hermitian metric on bundle \( E \). The chain \( \tau \)-vortex equations (1.1) can be re-written as follows:
\[
\sqrt{-1} \Lambda F_{H} - \frac{1}{2} (\phi_i^* H \phi - \bar{\phi}^* H) \sum_{i=0}^{n} \tau_i \pi_i = 0. \tag{7.13}
\]

Let \( l_j = \| s(t_j) \|_1 \) and \( u_j = l_j^{-1} s(t_j) \in \text{End}(E) \), from the assumption, we know that \( l_j \to \infty \). Using (7.11), we have
\[
\| u_j \|_1 = 1 \quad \text{and} \quad \| u_j \|_{\infty} \leq B_{12} \tag{7.14}
\]
where \( B_{12} \) is a positive constant. By formula (2.7) and the initial assumption \( \text{tr}(\sum_{i=0}^{n} \theta_i(\tau)) = 0 \), we have
\[
\text{tr} s(t) = 0, \tag{7.15}
\]
for every \( 0 \leq t < \infty \). From
\[
l_j \langle \psi[l_j u_j] (\bar{\partial}_E u_j), \bar{\partial}_E u_j \rangle \geq \langle \psi[u_j] (\bar{\partial}_E u_j), \bar{\partial}_E u_j \rangle \tag{7.16}
\]
and (7.1), (7.7), it follows that
Lemma 7.3. Since $u_j$ is bounded uniformly, so $\Psi \geq C > 0$ on the range of the $u_j$'s. We obtain

$$\|\tilde{\delta} E u_j\|_2 \leq B_{14}. \quad (7.17)$$

Then, passing to a subsequence, $u_j$ converges weakly to $u_{\infty}$ in $L_1^2$. Moreover, we have

$$\int_M \left( u_j, 2\sqrt{-1} A F_K - 2 \sum_{i=0}^n t_i \pi_i \right) + 2 \int_M \left( \Psi [u_j] (\tilde{\delta} E u_j), \tilde{\delta} E u_j \right)$$

$$\leq I_j^{-1} \left\{ \int_M \left( s(t_j), 2\sqrt{-1} A F_K - 2 \sum_{i=0}^n t_i \phi_i \right) + 2 \int_M \left( \Psi [s(t_j)] (\tilde{\delta} E s(t_j)), \tilde{\delta} E s(t_j) \right) \right\}$$

$$\leq I_j^{-1} \left\{ M_{\Phi, \tau} (K, H(t_j)) + \int_M \left( (\tilde{\Phi}_H^2) - |\tilde{\Phi}_K^2| \right) \right\} \leq I_j^{-1} C,$$

for some uniform constant $C$. Then, by the same discussion in [23, Lemma 5.4], we have $u_{\infty}$ is non-trivial, and

$$\int_M \left( u_{\infty}, 2\sqrt{-1} A F_K - 2 \sum_{i=0}^n t_i \pi_i \right) + \int_M \left( \Psi [u_{\infty}] (\tilde{\delta} E u_{\infty}), \tilde{\delta} E u_{\infty} \right) \leq 0. \quad (7.18)$$

In the same manner, if $\zeta \in C^\infty (R \times R, R)$ satisfies $\zeta (x, y) \leq (x - y)^{-1}$, whenever $x > y$, then

$$\int_M \left( u_{\infty}, 2\sqrt{-1} A F_K - 2 \sum_{i=0}^n t_i \pi_i \right) + \int_M \left( \zeta [u_{\infty}] (\tilde{\delta} E u_{\infty}), \tilde{\delta} E u_{\infty} \right) \leq 0. \quad (7.19)$$

For any smooth function $\rho : R \rightarrow R$, a straightforward computation gives

$$\hat{\delta} \mathop{\text{Tr}} \rho(u_{\infty}) = \mathop{\text{Tr}} (\delta \rho [u_{\infty}](\tilde{\delta} E u_{\infty})), \quad (7.20)$$

where we set

$$\delta \rho (\lambda, \mu) = \begin{cases} \frac{\rho(\lambda) - \rho(\mu)}{\lambda - \mu}, & \text{if } \lambda \neq \mu, \\ \rho'(\lambda), & \text{if } \lambda = \mu. \end{cases}$$

For any number $N$, we can find a smooth function $\tilde{\rho} : R \times R \rightarrow R$ such that: $\tilde{\rho}(x, y) = \delta \rho(x, x)$; and $N \tilde{\rho}^2(x, y) \leq (x - y)^{-1}$ whenever $x > y$. Then

$$\left| \hat{\delta} \mathop{\text{Tr}} (\rho(u_{\infty})) \right|^2 = \left| \mathop{\text{Tr}} (\delta \rho [u_{\infty}](\tilde{\delta} E u_{\infty})) \right|^2 \leq B_{15} |\tilde{\rho}[u_{\infty}][\tilde{\delta} E u_{\infty}]|^2$$

$$= B_{15} \frac{N}{N} \left[ \left| \tilde{\rho}^2[u_{\infty}][\tilde{\delta} E u_{\infty}], \tilde{\delta} E u_{\infty} \right| \right].$$

By (7.19), we have

$$\left\| \hat{\delta} \mathop{\text{Tr}} (\rho(u_{\infty})) \right\|_2^2 \leq \frac{B_{15}}{N}. \quad (7.21)$$

Since this holds for all $N > 0$, and $\mathop{\text{Tr}} (\rho(u_{\infty}))$ is real-valued, we get that $\mathop{\text{Tr}} (\rho(u_{\infty}))$ is constant almost everywhere. This implies that the eigenvalues of $u_{\infty}$ are constant almost everywhere, so we have proved the following lemma.

**Lemma 7.3.** The eigenvalues of $u_{\infty}$ are constant almost everywhere.

Let $\lambda_1, \ldots, \lambda_l$ denote the distinct eigenvalues of the $u_{\infty}$, listed in ascending order. On the other hand, by (7.15), we have $Tr u_{\infty} = 0$ almost everywhere. Since $u_{\infty}$ is non-trivial, so we must have $l \geq 2$.

For $\alpha < l$, let us define $p_{\alpha} : R \rightarrow R$ to be a smooth positive function such that

$$p_{\alpha}(x) = \begin{cases} 1, & \text{if } x \leq \lambda_{\alpha}, \\ 0, & \text{if } x \geq \lambda_{\alpha + 1}. \end{cases} \quad (7.22)$$

Define

$$\pi_{\alpha} = p_{\alpha} (u_{\infty}). \quad (7.23)$$
Proposition 7.4. Let $\pi'_{\alpha}$ be as above for $\alpha < l$. Then

1. $\pi'_{\alpha} \in L_1^2(S_K(E));$
2. $\pi'_{\alpha} = \pi'_{\alpha} = \pi'_{\alpha};$
3. $(Id - \pi'_{\alpha})\delta_{E \otimes E}(\pi'_{\alpha}) = 0$ almost everywhere;
4. $\|Id - \pi'_{\alpha}\phi\pi'_{\alpha}\|_2^2 = 0.$

Proof. (1), (2), (3) can be deduced directly by the same argument as in [6, Proposition 3.10.2]. So we only need to prove (4).

It is not hard to check that, for large enough $J$,

$$\int_M |\tilde{\phi}|^2_{H(t)} = \int_M |h(t_j)\tilde{\phi}h^{-1}(t_j)|^2_{K}$$

$$= \int_M |e^{l_ju_j}\tilde{\phi}e^{-l_ju_j}|^2_{K}$$

$$\geq \int_M |l_j\Omega[u_j]\tilde{\phi}|^2_{K}.$$  \hspace{1cm} (7.24)

Here $\Omega(\lambda, \mu) = \omega(\mu - \lambda), \omega : R \rightarrow R$ is a smooth positive function such that $\omega$ is compactly supported and $\omega(\lambda) = 0$ whenever $\lambda \leq \epsilon$ for some $\epsilon > 0$. In (7.24), we have used the fact [6, Proposition 3.9.1]: for large enough $l$

$$\omega(\lambda) \leq \frac{e^l}{l}.$$  \hspace{1cm} (7.25)

For any $\alpha < l$, choose $\epsilon$ such that $0 < \epsilon < \lambda_{\alpha+1} - \lambda_{\alpha}$ and define a smooth positive function $\omega_{\alpha} : R \rightarrow R$ such that

$$\omega_{\alpha}(\lambda) = \begin{cases} 0, & \text{if } \lambda \leq \epsilon, \\ 1, & \text{if } \lambda \geq \lambda_{\alpha+1} - \lambda_{\alpha}. \end{cases}$$  \hspace{1cm} (7.26)

It is easy to check that

$$|(Id - \pi'_{\alpha})\phi|_{K} \leq |\Omega_{\alpha}|_{u_{\infty}}|\tilde{\phi}|_{K},$$  \hspace{1cm} (7.27)

where $\Omega_{\alpha}(\lambda, \mu) = \omega_{\alpha}(\mu - \lambda).$ From Proposition 3.6, we know $|\tilde{\phi}|_{H(t)} = \Phi^2 = \sum_{i=1}^{n} |\phi_i|^2_{H(t)}$ is bounded uniformly. So, (4) can be deduced easily from (7.24), (7.27).

From the above proposition, we know that $\pi'_{\alpha}$’s are $L_1^2$-weakly holomorphic sub-bundles of $E$. Let $\pi'_{\alpha} = \pi_{\alpha} \circ \pi_i$ and $\phi'_{\alpha} = \phi_i \circ \pi_{\alpha}$. By Uhlenbeck and Yau’s result [26], we know that $\pi'_{\alpha}$ represents a saturated sub-sheaf $E'_{\alpha}$ of $E_i$. On the other hand, from (4) of Proposition 7.3, we have

$$\phi_i \circ \pi'_{\alpha} \otimes Id_{E_i} = \pi'_{\alpha(i-1)} \circ \phi_i \circ \pi'_{\alpha} \otimes Id_{E_i}.$$  \hspace{1cm} (7.28)

So, from $(E'_{\alpha}, \phi'_{\alpha})$, we have obtained a sequence of proper weakly holomorphic sub-chain $C_{\alpha}$ of $C = (E, J, \phi)$. We define

$$Q(\tau) := \lambda_{\tau} \deg_{\tau}(C) - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) \deg_{\tau}(C_{\alpha}).$$  \hspace{1cm} (7.29)

Then

$$Q(\tau) = \frac{1}{2\pi} \int_M \sqrt{-\Omega} \left\{ \lambda_{\tau} \text{Id}_E - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) \pi'_{\alpha} \right\} \Lambda F_K + \frac{1}{2\pi} \int_M \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) |\delta_{E \otimes E} \pi'_{\alpha}|^2_{K}$$

$$- \frac{\text{Vol}(M)}{2\pi} \sum_{i=0}^{n} \tau_i \left( \lambda_{\tau} \text{rank } E_i - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) \text{rank } \pi'_{\alpha} \right)$$

$$= \frac{1}{2\pi} \int_M \sqrt{-\Omega} \left\{ \lambda_{\tau} \text{Id}_E - \sum_{i=0}^{n} \tau_i \pi_i \right\} K + \frac{1}{2\pi} \int_M \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) |\delta_{E \otimes E} \pi'_{\alpha}|^2_{K}.$$  \hspace{1cm} (7.30)

Using the result and notation of [6, Lemma 3.12.1], we have
\[
\sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) \left| \tilde{\partial} E \otimes \bar{E} \pi_{\alpha} \right|^2 = \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) \left| \tilde{\partial} E \otimes \bar{E} \pi_{\alpha} \right|
\]
\[
= \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) (\delta p_{\alpha})^2 (\bar{E} \otimes \bar{E} u_{\infty}), \bar{E} \otimes \bar{E} u_{\infty}
\]
\[
= \{ \zeta (u_{\infty}) (\bar{E} \otimes \bar{E} u_{\infty}), \bar{E} \otimes \bar{E} u_{\infty} \}.
\]

(7.31)

Here \( \zeta : R \times R \to R \) is defined by \( \zeta = \sum_{\alpha=0}^{l-1} (\lambda_{\alpha+1} - \lambda_{\alpha}) (\delta p_{\alpha})^2 \), hence it satisfies the conditions that \( \zeta (\lambda, \mu) \leq (\lambda - \mu)^{-1} \) for \( \lambda > \mu \). Then, we make use of (7.19), (7.30), (7.31) to deduce that

\[
Q (\tau) \leq 0.
\]

(7.32)

On the other hand, from the definition, the \( \tau \)-stability of chain \( C \) deduce that \( Q (\tau) > 0 \), these conclude a contradiction. \( \square \)

**Proof of main theorem.** By conformal transformations \( \bar{K}_i = e^f K_i \) for all \( 0 \leq i \leq n \), it is easy to check that

\[
\text{tr} \left( \sum_{i=0}^{n} \theta_i (\bar{K} ; \tau) \right) = \text{tr} \left( \sum_{i=0}^{n} \theta_i (K ; \tau) \right) - \left( \sum_{i=0}^{n} \text{rank} E_i \right) \Delta f.
\]

By solving the poisson equation, from Theorem 4.3, we can pick up function \( f \), such that \( \text{tr} \left( \sum_{i=0}^{n} \theta_i (\bar{K} ; \tau) \right) \equiv 0 \), \( \sup M (\sum_{i=0}^{n} |A F_{\bar{K}_i}|) < \infty \), \( \sup M (\sum_{i=0}^{n} |\phi_i|^2) < \infty \) and \( C = (E, \phi) \) is analytic \( \tau \)-stable with respect to \( K \). Considering the above discussion, we can assume that the initial \( (n+1) \)-tuple of hermitian metrics \( K = (K_0, \ldots, K_n) \) satisfies \( \text{tr} \left( \sum_{i=0}^{n} \theta_i (K ; \tau) \right) \equiv 0 \).

By formula (7.1), we have

\[
M_{\phi, \tau} (E, H) \geq -n \int_M |s_i| 2 \sqrt{-1} A F_{H_i} - 2\tau_i Id |K_i|
\]
\[
+ 2 \sum_{i=0}^{n} \int_M \langle \psi | s_i | (\bar{E}_i s_i), \bar{E}_i s_i |K_i\rangle
\]
\[
+ \sum_{i=1}^{n} \int_M \left( |\phi_i|^2_{H_i} - |\phi_i|^2_{K_i} \right).
\]

(7.33)

where \( \psi (x, y) = \psi (y, x) = e^{x-y-(x+y)-1} \). From Proposition 7.2, we know that \( \sum_{i=0}^{n} \|s_i(t)\|_\infty \) is uniformly bounded, it follows that \( \psi \geq B_{10} > 0 \) on the range of the \( s_i(t) \)'s; so that

\[
\int_M \langle \psi | s_i | (\bar{E}_i s_i), \bar{E}_i s_i |K_i\rangle \geq B_{10} \|\bar{E}_i s_i\|^2_2,
\]

(7.34)

for every \( 0 \leq i \leq n \). On the other hand, from Proposition 3.6, we know that \( \sum_{i=1}^{n} |\phi_i|^2_{H(t)} \) is bounded uniformly. Therefore, there exists \( B_{11} > 0 \) such that, for every \( t \in [0, +\infty) \)

\[
M_{\phi, \tau} (E, H(t)) \geq -B_{11}.
\]

(7.35)

Since \( h_i \) is continuous in \( t \) uniformly on compact sets, and it is uniformly bounded, we have

\[
\lim_{t \to t_0} \int_M |h_i(\cdot, t) - h_i(\cdot, t_0)| \to 0.
\]

By (7.33), (7.34), (7.7), we know that \( \|\bar{E}_i s_i\|_2 \) and also \( \|\bar{E}_i h_i\|_2 \) are uniformly bounded. Thus, there exists a subsequence \( t_j \to +\infty \), such that \( h_i(t_j) \) weakly converges to \( h_i(\infty) \) in \( L^2 \), for every \( 0 \leq i \leq n \). By (7.35) and (7.7), we know that \( \sum_{i=1}^{n} |2 \sqrt{-1} A F_{H_i} - (\phi_i^H \phi_i - \phi_{i+1} \phi_{i+1}^H) - 2\tau_i Id |H_i|_{2}(t_j) \) weakly converges to 0 in \( L^2 \). Then, the standard elliptic regularity implies that \( h_i(\infty) \) is smooth and \( H_i(\infty) = K_i h_i(\infty) \) satisfies the chain \( \tau \)-vortex equations (1.1). So, we have proved the main theorem. \( \square \)
8. A Bogomolov type inequality

Let us define the following Chern numbers of holomorphic vector bundle $E$ with respect to Hermitian metric $H$ by

$$C_1(E, H) = \int_M c_1(E, H) \frac{\omega^{m-1}}{(m-1)!} = \frac{-1}{2\pi} \int_M \Lambda_\omega F_H \frac{\omega^m}{m!},$$

(8.1)

and

$$Ch_2(E, H) = \int_M \frac{1}{2} \left\{ c_1(E, H)^2 - 2c_2(E, H) \right\} \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$= \frac{1}{8\pi^2} \int_M \text{tr}(F_H \wedge F_H) \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$= \frac{1}{8\pi^2} \int_M (\sqrt{-1} \wedge F_H |^2_H - |F_H|^2_H) \frac{\omega^m}{m!}.$$  

(8.2)

Since $M$ may not be compact, the above Chern numbers measured with different metrics need not be equal a priori. For further consideration, we need the following lemma in Simpson’s paper [23, Lemma 5.2].

**Lemma 8.1.** Suppose $(M, \omega)$ has an exhaustion function $\phi$ with $\int_M |\Delta \phi| < \infty$, and suppose $\eta$ is a $(2m-1)$-form with $\int_M |\eta|^2 < \infty$. Then if $d\eta$ is integrable, we have

$$\int_M d\eta = 0.$$  

(8.3)

**Proposition 8.2.** Let $(M, \omega)$ satisfy the above Assumptions 1, 2, and suppose $(E, \phi)$ is a twisted holomorphic chain with an $(n+1)$-tuple of Hermitian metrics $H = (H_0, H_1, \ldots, H_n)$ satisfying $\text{sup}_M (\sum_{i=0}^n |\Lambda F_{H_i}|_{H_i}) < \infty$, $\text{sup}_H (\sum_{i=1}^n |\phi_i|_{H_i}) < \infty$, $\sum_{i=1}^n |\partial H \phi_i|_{H_i}$ and $\sum_{i=1}^n |\sqrt{-1} \Lambda_\omega F_{H_i}|_{H_i}$ both belong to $L^2(M, \omega)$. Suppose the $(n+1)$-tuple of Hermitian metrics $H$ satisfies the chain $\tau$-vortex equations (1.1) and $\sqrt{-1} \Lambda_\omega F_{H_i}$ is positive semi-definite for each $i = 1, \ldots, n$, then we have the following Bogomolov type inequality

$$\sum_{i=0}^n \tau_i C_1(E_i, H_i) \geq 2\pi \sum_{i=0}^n Ch_2(E_i, H_i).$$

(8.4)

**Proof.** Using $\tilde{\partial}_E^* \otimes E_i \otimes \tilde{E}_{i-1} \phi_i = 0$, we have

$$\text{Re} \left( \sqrt{-1} \Lambda_\omega F_H (\phi_i), \phi_i |_{H_i} \right) = |\partial H \phi_i|_{H_i}^2 - \frac{1}{2} \Delta |\phi_i|_{H_i}^2,$$

(8.5)

where

$$F_H (\phi_i) = F_{H_{i-1}} \circ \phi_i - \phi_i \circ (F_{H_{i}} \otimes \text{Id}_{E_i} + \text{Id}_{E_i} \otimes F_{H_{i}}).$$

(8.6)

Let us define the Yang–Mills–Higgs functional $YMH_\tau (H, \phi)$ as follows

$$YMH_\tau (H, \phi) = \sum_{i=0}^n \|\partial H \phi_i\|_{L^2}^2 + \sum_{i=0}^n \left\|F_{H_i} \right\|_{L^2}^2 + \left\| \frac{1}{2} (\phi_{i+1} \circ \phi_{i+1}^H - \phi_i^H \circ \phi_i) - \tau_i \text{Id}_{E_i} \right\|_{L^2}^2.$$  

(8.7)

By direct calculation, we have

$$YMH_\tau (H, \phi) = -\sum_{i=0}^n 8\pi^2 Ch_2(E_i, H_i) + \sum_{i=0}^n \|\partial H \phi_i\|_{L^2}^2$$

$$+ \sum_{i=0}^n \left\{ \left\| \sqrt{-1} \Lambda_\omega F_{H_i} \right\|_{L^2}^2 + \left\| \frac{1}{2} (\phi_{i+1} \circ \phi_{i+1}^H - \phi_i^H \circ \phi_i) - \tau_i \text{Id}_{E_i} \right\|_{L^2}^2 \right\}$$

$$= -\sum_{i=0}^n 8\pi^2 Ch_2(E_i, H_i) + \sum_{i=0}^n \|\partial H \phi_i\|_{L^2}^2 + \sum_{i=0}^n 4\pi \tau_i C_1(E_i, H_i)$$

$$+ \sum_{i=0}^n \left\{ \left\| \sqrt{-1} \Lambda_\omega F_{H_i} \right\|_{L^2}^2 + \left\| \frac{1}{2} (\phi_{i+1} \circ \phi_{i+1}^H - \phi_i^H \circ \phi_i) - \tau_i \text{Id}_{E_i} \right\|_{L^2}^2 \right\}.$$
\[
+ \sum_{i=0}^{n} \left\| \sqrt{-1} A_\omega F_{H_i} + \frac{1}{2} (\phi_{i+1} \circ \phi_{i+1}^* - \phi_i^* \circ \phi_i) - \tau_i Id_{E_i} \right\|_{L^2}^2 \\
- \sum_{i=0}^{n} \int_{M} \text{Re} \left( \sqrt{-1} A_\omega F_{H_i} \cdot \phi_i^* - \phi_i^* \circ \phi_i \right)_{H_i}.
\]

(8.8)

If \( \int_{M} \Delta |\phi_i|^2_{H_i} d\mu_\omega = 0 \) by (8.5) and (8.6), then we have

\[
\int_{M} \sum_{i=1}^{n} |\partial H \phi_i|^2_{H_i} = \sum_{i=1}^{n} \int_{M} \text{Re} \left( \sqrt{-1} A_\omega F_H(\phi_i), \phi_i \right)_{H_i} \\
= \sum_{i=1}^{n} \int_{M} \text{Re} \left( \sqrt{-1} A_\omega \left\{ \text{tr} (F_{H_{i-1}} \circ \phi_i \circ \phi_i^*) \\
- \text{tr} (\phi_i \circ (F_{H_i} \otimes Id_{E_i}) \circ \phi_i^*) - \left\{ \phi_i \circ (Id_{E_i} \otimes F_{H_i}) \right\} \phi_i \right)_{H_i} \\
= \sum_{i=1}^{n} \int_{M} \text{Re} \left( \sqrt{-1} A_\omega F_{H_i}, \phi_i^* \circ \phi_i^* - \phi_i^* \circ \phi_i \right)_{H_i} \\
- \sum_{i=1}^{n} \int_{M} \text{Re} \left( \phi_i \circ (Id_{E_i} \otimes \sqrt{-1} A_\omega F_{H_i}), \phi_i \right)_{H_i}.
\]

(8.9)

Combining (8.8) and (8.9), we have

\[
- \sum_{i=0}^{n} 8\pi^2 C_2 (E_i, H_i) + \sum_{i=0}^{n} 4\pi \tau_i C_1 (E_i, H_i) = \text{YM}_{\tau} (H, \phi) + \sum_{i=1}^{n} \int_{M} \text{Re} \left( \phi_i \circ (Id_{E_i} \otimes \sqrt{-1} A_\omega F_{H_i}), \phi_i \right)_{H_i} \\
\geq 0.
\]

(8.10)

So, we only need to prove the following claim.

Claim. \( \int_{M} \Delta |\phi_i|^2_{H_i} d\mu_\omega = 0 \) for each \( i = 1, \ldots, n \).

Let \( \eta = \sqrt{-1} \bar{\partial} |\phi_i|^2_{H_i} \wedge \omega^{n-1-m} \), then \( d\eta = \frac{1}{2} \Delta |\phi_i|^2_{H_i} \omega^{n-1} \). By the Cauchy–Schwarz inequality, we have

\[
|\bar{\partial} |\phi_i|^2_{H_i} |_{H_i} \leq |\bar{\partial} H \phi_i |_{H_i} |_{H_i}.
\]

(8.11)

Since \( |\bar{\partial} H \phi_i |_{H_i} \in L^2(M, \omega) \), then \( \int_{M} |\eta|^2 \leq \infty \). On the other hand, by the assumptions that \( \sum_{i=0}^{n} |\sqrt{-1} A_\omega F_{H_i} |_{H_i} \), \( \sum_{i=0}^{n} |\phi_j |_{H_i} \) are bounded, and \( |\bar{\partial} H \phi_i |_{H_i} + \sqrt{-1} A_\omega F_{H_i} |_{H_i} \in L^2(M, \omega) \), by (8.5), it is easy to see that \( \Delta |\phi_i|^2_{H_i} \) is integrable, i.e. \( d\eta \) is integrable. Then Lemma 8.1 implies the claim. So, we complete the proof. \( \square \)

When \( (M, \omega) \) is compact, the above inequality was proved by L. Alvarez-Consal, O. Garcia-Prada in [2]. Using the main theorem and the above proposition, we have the following theorem.

**Theorem 8.3.** Let \( (M, \omega) \) satisfy Assumptions 1, 2, 3, and suppose the exhaustion function \( \varphi \) in Assumption 2 satisfies \( 0 \leq \sqrt{-1} \bar{\partial} \bar{\partial} \varphi \leq C \varphi \) for some constant \( C \). Let \( (E, \phi) \) be a twisted holomorphic chain with an \( (n + 1) \)-tuple of Hermitian metrics \( K \) satisfying that \( \sum_{i=0}^{n} |A_{FK} |_{K_i} |_{K_i} < \infty \), \( \text{sup}_{M} |\phi_i^* |_{K_i} |_{K_i} < \infty \), \( \text{deg}_\tau (C; K) = 0 \), \( \sum_{i=0}^{n} |\delta H \phi_i |_{K} \) and \( \sum_{i=0}^{n} |\sqrt{-1} A_\omega F_{H_i} |_{H_i} \) both belong to \( L^2(M, \omega) \), and \( \sqrt{-1} A_\omega F_{H_i} \) is positive semi-definite for each \( i = 1, \ldots, n \). Suppose \( C = (E, \phi) \) is analytic \( \tau \)-stable with respect to the metrics \( K \), then we have the following Bogomolov type inequality

\[
\sum_{i=0}^{n} \tau_i C_1 (E_i, K_i) \geq 2 \pi \sum_{i=0}^{n} Ch_2 (E_i, K_i).
\]

(8.12)

**Proof.** By the main theorem, we know that there exists an \( (n + 1) \)-tuple of Hermitian metrics \( H = (H_0, H_1, \ldots, H_n) \) with \( \text{sup}_{M} |\phi_i^* |_{K_i} |_{K_i} < \infty \), \( H_i \) and \( K_i \) are mutually bounded, \( |\bar{\partial} E (K_i^{-1} H_i) |_{K_i} \in L^2(M, \omega) \) for each \( i = 0, \ldots, n \), and satisfying the the chain \( \tau \)-vortex equations (11). As above, we denote \( h_i = K_i^{-1} H_i \). Since \( h_i = h_i^{K_i} \), then we have

\[
|\bar{\partial} e H_i |_{K_i} = |\partial K_i h_i |_{K_i} = |\bar{\partial} K_i h_i |_{K_i}.
\]

(8.13)
So, \(|\tilde{\partial}_E (K_i^{-1} H_i)|_{K_i} \in L^2 (M, \omega)\) implies \(|\partial_X H_i|_{K_i} \in L^2 (M, \omega)\). Considering the following relation

\[
\tilde{\partial}_H \phi_i - \partial_K \phi_i = (\tilde{\partial}_{H_{i-1}} - \partial_{K_{i-1}}) \circ \phi_i - \phi_i \circ ((\partial_{H_{i-1}} - \partial_{K_{i-1}}) \otimes \text{Id}_{E_i})
\]

\[
= (h_{i-1}^{-1} \partial_{K_{i-1}} h_{i-1}) \circ \phi_i - \phi_i \circ ((h_{i-1}^{-1} \partial_{K_{i-1}} h_{i-1}) \otimes \text{Id}_{E_i}).
\]

(8.14)

it is easy to see that the assumptions \(\sup_M (\sum_{i=1}^{n} |\phi_i|^2_{K_i}) < \infty\), \(H_i\) and \(K_i\) are mutually bounded (i.e. \(\sup_M |h_i|_{K_i} < \infty\)) for each \(i = 0, \ldots, n\) and \(\sum_{i=1}^{n} |\partial_K \phi_i|_{K_i} \in L^2 (M, \omega)\)

\[
\sum_{i=1}^{n} |\partial_K \phi_i|_{K_i} \in L^2 (M, \omega).
\]

(8.15)

Then, by Proposition 8.2, we have

\[
\sum_{i=0}^{n} r_i C_1 (E_i, H_i) \geq 2\pi \sum_{i=0}^{n} C_2 (E_i, H_i).
\]

(8.16)

Since \((\sum_{i=0}^{n} |\Delta F_{K_i}|_{K_i})\) and \((\sum_{i=0}^{n} |\Delta F_{H_i}|_{H_i})\) are both bounded on \(M\), \(|\text{tr}(h_i^{-1} \partial_K h_i)|_{E_i} \in L^2 (M, \omega)\), by Lemma 8.1, we have

\[
\int_M \tilde{\partial} \text{tr} (\sqrt{-1} (h_i^{-1} \partial_K h_i)) \wedge \frac{\omega^{m-1}}{(m-1)!} = 0.
\]

(8.17)

where we have used the relation \(F_{H_i} = F_{K_i} + \tilde{\partial}_E (h_i^{-1} \partial_K h_i)\). Then

\[
2\pi C_1 (E_i, H_i) = \int_M A_\omega \text{tr} (\sqrt{-1} F_{H_i}) \frac{\omega^m}{m!} = \int_M \text{tr} (\sqrt{-1} F_{H_i}) \wedge \frac{\omega^{m-1}}{(m-1)!} = \int_M \{\text{tr} (\sqrt{-1} F_{K_i}) + \text{tr} (\sqrt{-1} \tilde{\partial}_E (h_i^{-1} \partial_K h_i))\} \wedge \frac{\omega^{m-1}}{(m-1)!}
\]

\[
= \int_M A_\omega \text{tr} (\sqrt{-1} F_{K_i}) \frac{\omega^m}{m!} + \int_M \tilde{\partial} \text{tr} (\sqrt{-1} (h_i^{-1} \partial_K h_i)) \wedge \frac{\omega^{m-1}}{(m-1)!} = 2\pi C_1 (E_i, K_i).
\]

(8.18)

On the other hand, since the exhaustion function \(\varphi\) satisfies \(0 \leq \sqrt{-1} \leq C\omega\), by the same discussion as that in the proof of Proposition 3.5 in Simpson’s paper [23], we have

\[
\int_M \text{tr} (F_{H_i} \wedge F_{H_i}) \wedge \omega^{m-2} \leq \int_M \text{tr} (F_{K_i} \wedge F_{K_i}) \wedge \omega^{m-2},
\]

(8.19)

i.e.

\[
C_2 (E_i, H_i) \geq C_2 (E_i, K_i),
\]

(8.20)

for each \(i = 0, \ldots, n\). So, (8.16), (8.18) and (8.20) imply the inequality (8.12). □

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