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Short and long time behavior of the Fokker–Planck equation in a confining potential and applications

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Abstract

We consider the linear Fokker–Planck equation in a confining potential in space dimension $d \geqslant 3$. Using spectral methods, we prove bounds on the derivatives of the solution for short and long time, and give some applications.
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Résumé

On considère l'équation de Fokker–Planck avec un potentiel confinant en dimension $d \geqslant 3$. Avec des méthodes spectrales on donne des bornes sur les dérivées de la solution en temps petit et grand, et quelques applications.
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Keywords: Hypocoercivity; Fokker–Planck; Semigroup; Poisson–Emden; Vlasov–Poisson–Fokker–Planck; Exponential decay

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1. Introduction and results

In this article, we consider the linear Fokker–Planck equation in $\mathbb{R}^{2d}_{x,v}$ for $d \geqslant 3$ which reads after scaling

$$\begin{cases} \partial_t f + v.\partial_x f - \partial_x V.\partial_v f - \gamma \partial_v.(\partial_v + v) f = 0, \\ f|_{t=0} = f_0, \end{cases} \tag{1}$$

where V is a given external confining potential, γ is a positive physical constant, and f is the distribution function of the particles. This equation is a linear model for plasmas or stellar systems, and γ has to be understood as a friction–diffusion coefficient. The aim of this article is to study the short and long time behavior of the solution of this equation, without the help of the explicit Green function, which is known only in very special cases (i.e. V quadratic), and give an application to a mollified Vlasov–Poisson–Fokker–Planck equation.

Let us now precise our notations and hypothesis. For the potential V, we suppose the following:

$$\text{(H1)} \quad e^{-V} \in \mathcal{S}\big(\mathbb{R}^d_x\big), \quad \text{with } V \geqslant 0 \text{ and } V'' \in W^{\infty,\infty}.$$

Note that the assumption $0 \leqslant V$ can be relaxed by assuming that V is bounded from below and adding to it a sufficiently large constant. Let us also note that these assumptions easily imply that $V \in \mathcal{C}^\infty(\mathbb{R}^d_x)$ and $\lim_{x\to\infty} V(x) = +\infty$. We introduce now the so-called Maxwellian, which is the unique L^1-normalized steady solution of Eq. (1):

$$\mathcal{M}(x,v) = \frac{e^{-(v^2/2+V(x))}}{\int e^{-(v^2/2+V(x))}\,dx\,dv}. \tag{2}$$

To this function we associate a weighted space here called B^2 built from the standard L^2 space after conjugation with a half power of the Maxwellian:

$$B^2 \overset{\text{def}}{=} \mathcal{M}^{1/2} L^2 = \big\{ f \in \mathcal{D}' \text{ s.t. } f/\mathcal{M} \in L^2(\mathcal{M}\,dx\,dv)\big\} \tag{3}$$

with the natural norm defined by

$$\|f\|^2_{B^2} = \int (f/\mathcal{M})^2 \mathcal{M}\,dx\,dv.$$

This space is standard (e.g. [13,14]) for the study of the Fokker–Planck operator

$$K = v.\partial_x - \partial_x V.\partial_v - \gamma\partial_v.(\partial_v + v) \tag{4}$$

for which $\mathcal{M}$ is the unique fundamental state. It is shown in [13] that K is maximal accretive with $\mathcal{M}^{1/2}\mathcal{S}$ as a core. It is also shown in [14] that the associated semigroup has smoothing properties under slightly weaker hypotheses. Anyway a number of results from there are still true (see Section 2 here and in particular Remark 2.3). The first theorem concerns accurate estimates about the short time behavior of the associated semigroup.

Theorem 1.1. *There exists a constant C such that for all $t > 0$, we have the following*:

(i) $(-\partial_v + v)e^{-tK}$ *is bounded by* $C(1 + t^{-1/2})$ *and*
(ii) $(-\partial_x + \partial_x V)e^{-tK}$ *is bounded by* $C(1 + t^{-3/2})$,

as bounded operators on B^2. Here C depends only on $\|V''\|_{L^\infty}$ (and γ).

In order to study the long time behavior of the system (1) we introduce an additional hypothesis on V. We first define an intermediate operator called the Witten Laplacian (on 0-forms) naturally associated to the linear Fokker–Planck operator K

$$\Lambda^2 = -\gamma\partial_v(\partial_v + v) - \gamma\partial_x(\partial_x + \partial_x V). \tag{5}$$

The closure of this operator defined in B^2 has also 0 as single eigenvalue for the eigenfunction $\mathcal{M}$. We shall assume the following:

$$\text{(H2)} \qquad \begin{cases} \text{operator } \Lambda^2 \text{ has a spectral gap in } B^2 \\ \text{with first non-zero eigenvalue denoted } \alpha. \end{cases}$$

This hypothesis may seem complicated, but in the particular case when $\partial_x V \to \infty$ it is immediate since then Λ^2 is with compact resolvent. Under this hypothesis, we have the following result about the (short and) long time behavior of the solution of the Fokker–Planck equation.

Theorem 1.2. *Suppose conditions* (H1) *and* (H2) *are fulfilled. Then there exist constants C and A depending only on $\|V''\|_{L^\infty}$ (and γ) such that if $f(t)$ is the solution of* (1) *for an L^1-normalized initial datum $f_0 \in B^2$, we have*:

(i) $\|f(t) - \mathcal{M}\|_{B^2} \leqslant 3e^{-\alpha t/A}\|f_0 - \mathcal{M}\|_{B^2}$,
(ii) $\|(-\partial_v + v)f(t)\|_{B^2} \leqslant C(1 + t^{-1/2})e^{-\alpha t/A}\|f_0\|_{B^2}$,
(iii) $\|(-\partial_x + \partial_x V)f(t)\|_{B^2} \leqslant C(1 + t^{-3/2})e^{-\alpha t/A}\|f_0\|_{B^2}$.

We give now an application to a nonlinear problem. We want to study the following mollified Vlasov–Poisson–Fokker–Planck equation:

$$\begin{cases} \partial_t f + v.\partial_x f - (E + \partial_x V).\partial_v f - \gamma\partial_v.(\partial_v + v)f = 0, \\ E(t,x) \stackrel{\text{def}}{=} \partial_x V_{\text{nl}}(t,x) = -\zeta * \dfrac{\kappa}{|S^{d-1}|}\dfrac{x}{|x|^d} *_x \rho(t,x), \quad \text{where } \rho(t,x) = \displaystyle\int f(t,x,v)\,dv, \\ f|_{t=0} = f_0, \end{cases} \tag{6}$$

where $\zeta \in \mathcal{S}$ (depending only on x). Here $\kappa \in \mathbb{R}$ has to be understood as the total charge of the system. In the usual VPFP equation there is no convolution with ζ, but we were not able to reach similar result in this case. The unique steady state of this equation is given by

$$\mathcal{M}_\infty(x,v) = \frac{e^{-(v^2/2+V(x)+V_\infty(x))}}{\int e^{-(v^2/2+V(x)+V_\infty(x))}\,dx\,dv},$$

where V_∞ is a solution of the Poisson–Emden type equation

$$-\Delta V_\infty = \kappa\zeta *_x \frac{e^{-(V+V_\infty)}}{\int e^{-(V(x)+V_\infty(x))}dx}. \tag{7}$$

It is easy to see that under hypothesis (H1) this equation has a unique (Green) solution $V_\infty \in W^{\infty,\infty}$ thanks to the ellipticity properties of the Laplacian. We immediately check that the associated total potential $V + V_\infty$ satisfies hypothesis (H1), and that $\mathcal{M}_\infty \in \mathcal{S}$. We define also the associated spaces $\mathcal{B}^2_\infty = \mathcal{M}_\infty^{1/2} L^2$ and we impose in addition that $V + V_\infty$ satisfies an hypothesis of type (H2): As in (5) we define the corresponding Witten Laplacian

$$\Lambda^2_\infty = -\gamma\partial_v(\partial_v + v) - \gamma\partial_x(\partial_x + \partial_x V + \partial_x V_\infty)$$

which closure in $\mathcal{B}^2_\infty$ has 0 as single eigenvalue associated with the eigenfunction $\mathcal{M}_\infty$; we shall assume the following:

$$\text{(H2bis)} \quad \begin{cases} \text{operator } \Lambda^2_\infty \text{ has a spectral gap in } \mathcal{B}^2_\infty \\ \text{with first non-zero eigenvalue denoted } \alpha_\infty. \end{cases}$$

Now we state a result about the existence, the uniqueness and convergence to equilibrium or the solution of (6). We call here solution on $[0,T[$ a function $f \in \mathcal{C}([0,T[,\mathcal{B}^2_\infty)$ such that $\|E\|_{L^\infty([0,T[\times\mathbb{R}^d)} < \infty$ and

$$f(t) = e^{-tK} f_0 + \int_0^t e^{-(t-s)K} E(s)\partial_v f(s)\,ds.$$

We call it a global solution if it is a solution for all $T > 0$.

Theorem 1.3. *Suppose that condition* (H1) *is satisfied. Then Eq.* (6) *has a unique global solution for a given L^1-normalized initial datum $f_0 \in \mathcal{B}^2_\infty$.*

Besides if in addition (H2bis) *is fulfilled, then there exist constants A_∞ and C_∞ only depending on second order derivatives of $V + V_\infty$ and γ* (*and uniform in κ varying in a fixed compact set*) *such that for any $\kappa \leqslant \alpha_\infty / C_\infty$*

$$\|f(t,\cdot) - \mathcal{M}_\infty\|_{\mathcal{B}^2_\infty} \leqslant 6\|f_0 - \mathcal{M}_\infty\|_{\mathcal{B}^2_\infty} e^{-\frac{\alpha_\infty}{2A_\infty}t}.$$

As a corollary of Theorem 1.3 we also get the following result concerning the decay of the relative entropy.

Corollary 1.4. *Consider the solution given by Theorem* 1.3. *Then* (*with the notations of Theorem* 1.3 *and in particular for* $\kappa \leqslant \alpha_\infty / C_\infty$) *we have*

$$0 \leqslant H\big(f(t), \mathcal{M}_\infty\big) \stackrel{\text{def}}{=} \iint f(t) \ln\left(\frac{f(t)}{\mathcal{M}_\infty}\right) dx\, dv \leqslant C'_\infty \|f_0\|_{\mathcal{B}^2_\infty} \|f_0 - \mathcal{M}_\infty\|_{\mathcal{B}^2_\infty} e^{-\frac{\alpha_\infty}{2A_\infty} t},$$

where C'_∞ *only depends on second order derivatives of* $V + V_\infty$ *and* γ (*and is uniform in* κ *varying in a fixed compact set*).

Considering the short time linear diffusion estimates for hypoelliptic operators, we mention the cases $V = 0$ known since [16] (see also the computations in the case $V = x^2$ in [17]) where the Green function is explicit. Numerous nonlinear result already quoted use this fact. For generic hypoelliptic operators, this was studied by many authors in the selfadjoint case, in the spirit of the study of sum of squares of vector fields theorem with underlying Lie group structure. We refer to the book [5] and references therein for this subject and point out that it is linked with the subelliptic estimates for semigroups of operators. The author was unable to find any general result concerning the non-selfadjoint case (type II operators), and the estimates given in Theorem 1.1 in this article seem to be new. Concerning the general study of the semigroup of globally hypoelliptic operators we also mention the recent works [1,10,13–15].

Concerning the long time behavior of Fokker–Planck type operators, we mention [24] for the exponential decay with probabilistic method, and [14] for the explicit exponential decay using hypoelliptic tools developed first in [9,11]. Recent results about more general kinetic equations on the torus can be found in works by Guo (e.g. [12]). We quote [7] for the convergence in t^{-N} for all N with the use of entropy-dissipating methods, and [15] for explicit exponential decay using hypoelliptic tools close to the ones in this paper. Let us also mention the work [17] where invariant manifolds methods are used in the case without external potential.

The systematic use of crossed derivatives in order to get short-time, long-time and derivative estimates follows from [14] and was also developed in [18–25]. These studies, concerning type II operators follow common ideas and features sometimes called hypocoercivity.

For the Vlasov–Poisson–Fokker–Planck equation there is a huge literature on the subject (e.g. [2,6,19,20,22]). Essentially when $d = 3$ these results use the explicit Green function and L^p estimates available in this case. The case of a general confining potential was not studied and in fact L^p diffusion estimates on the semigroup seem to be hard to get in this case. This is the reason why we only deal with a mollified equation in the last part of this paper. For the trend to the equilibrium, we quote [3,4,8,23].

The plan of the article is the following. In Section 2, we give some results of functional analysis to be used later, essentially taken from [13,14]. In Section 3 we prove Theorem 1.1 about the short-time diffusion estimate for a general Fokker–Planck operator K. There is a similar gain as in the explicit case when the Green function is known (see e.g. [2]) and obtained through hypoelliptic techniques. It will play a crucial role in the study of the mollified VPFP equation and in particular close to the equilibrium, where the potential in $V + V_\infty$ is *not known*. In Section 4 we give a new proof of the exponential time decay toward the equilibrium, based on an abstract Hilbert lemma given in the first subsection there. In the last section we apply the linear results first in the case when an additional external field is added. Then we deal with the mollified VPFP equation and prove Theorem 1.3 and Corollary 1.4.

2. Functional analysis

We work here with a potential V independent of time and satisfying condition (H1). We denote by B^2 the space defined in (3), and recall that it is an Hilbert with respect to the scalar product

$$\langle f, g\rangle = \iint fg\mathcal{M}^{-1}\,dx\,dv = \iint \frac{f}{\mathcal{M}}\frac{g}{\mathcal{M}}\mathcal{M}\,dx\,dv, \tag{8}$$

for adequate f and g. Recall that the spaces $\mathcal{C}_0^\infty$ and $\mathcal{M}^{1/2}\mathcal{S}$ are dense in the B^2.

We now state some results about the linear Fokker–Planck operator, say in L^1. First the Hamiltonian vector field of $v^2/2 + V(x)$ is denoted by

$$X_0 = v\partial_x - \partial_x V(x)\partial_v,$$

and it is easy to check that it is formally skew-adjoint with respect to the scalar product (8) since X_0 commutes with the multiplication with $\mathcal{M}$. We also introduce the differential ($(d,1)$-matricial) operators

$$a = \gamma^{1/2}\big(\partial_x + \partial_x V(x)\big), \qquad b = \gamma^{1/2}(\partial_v + v). \tag{9}$$

For the scalar product defined in (8), their formal adjoint are the following $(1,d)$-matricial operators:

$$a^* = -\gamma^{1/2}\partial_x, \qquad b^* = -\gamma^{1/2}\partial_v. \tag{10}$$

With these notations the Fokker–Planck operator and its adjoint with respect to the scalar product (8) read

$$K = X_0 + b^*b, \qquad K^* = -X_0 + b^*b. \tag{11}$$

Recall also the definition of Witten Laplacian (on 0-forms) in velocity and spatial coordinates

$$\Lambda^2 = a^*a + b^*b,$$

which is the naturally associated formally self-adjoint operator. All these operators are linked thanks to the following remarkable algebraic properties:

$$a = [b, X_0], \qquad b = -\operatorname{Hess} V[a, X_0].$$

We want to study the linear Cauchy problem,

$$\partial_t f + Kf = 0, \qquad f|_{t=0} = f_0$$

in B^2. We first quote some results from [13,14].

Proposition 2.1. [13,14] *Operators K and K^* defined as the closure of* (11) *with domain $\mathcal{C}_0^\infty$ are maximal accretive. They define semigroups of contraction and positivity preserving denoted e^{-tK} (respectively e^{-tK^*}).*

We shall also need the following chain of Sobolev spaces based on B^2. In the spirit of [14] we denote

$$\Lambda_a^2 = 1 + a^* a, \qquad \Lambda_b^2 = 1 + b^* b,$$

where a and b were defined in (9), (10). Operators Λ^2, Λ_a^2 and Λ_b^2 are maximal accretive with $\mathcal{M}^{1/2}\mathcal{S}$ as a core and we denote by the same letter their closure in B^2 (see the reference already quoted for instance). In this sense b^*b is an harmonic oscillator and a^*a is the Witten Laplacian associated to V. We introduce the natural chain of Sobolev space for $k, l \in \mathbb{R}$

$$\mathbb{H}^{l,k} = \{ f \in \mathcal{M}^{1/2}\mathcal{S}' \text{ s.t. } \Lambda_a^k \Lambda_b^l f \in B^2 \}, \tag{12}$$

for which $l \leqslant l'$ and $k \leqslant k'$ imply $\mathbb{H}^{l,k} \hookrightarrow \mathbb{H}^{l',k'}$ and $\mathbb{H}^{0,0} = B^2$. We first write a result which proof is essentially contained in [13,14] about the parabolic (smoothing) properties of operator K in B^2.

Proposition 2.2. [13,14] *For all $t > 0$ and $k \in \mathbb{R}$, e^{-tK} maps $\mathcal{M}^{1/2}\mathbb{H}^{-k,-k}$ to $\mathcal{M}^{1/2}\mathbb{H}^{k,k}$. Besides for a given $k \geqslant 0$ there is constants $C_{k,k}$ and N_k such that for any initial data $f_0 \in \mathbb{H}^{k,k}$ we have*

$$\| f(t) \|_{\mathbb{H}^{k,k}} \leqslant C_{k,k} (t^{N_k} + t^{-N_k}) \| f_0 \|_{\mathbb{H}^{-k,-k}}. \tag{13}$$

Besides the same result holds for K^*.

Remark 2.3. The proof of this result is included in [14]. Let us just notice some differences. Here there is no growing assumptions for $\partial_x V$, anyway the definition of the commutators and the pseudodifferential calculus are still valid with based metric $g_0 = dx^2 + d\xi^2 + d\eta^2 + dv^2$ where (ξ, η) are the dual variables of (x, v). On the contrary some assertions about compactness (of the resolvent of K, $\Lambda^2, \dots$) are not true anymore. It corresponds in [14, Appendix A] to the case $n = 1/2$. In particular $\bigcap_{k\in\mathbb{R}} \mathbb{H}^{k,k} \neq \mathcal{S}$ and e^{-tK} does not anymore send $\mathcal{M}^{1/2}\mathcal{S}'$ to $\mathcal{M}^{1/2}\mathcal{S}$. Anyway the proofs of the other result there remain true under the hypothesis (H1) here.

3. Short time behavior

The purpose of the following section is to prove Theorem 1.1 about the short time behavior of the semigroup associated to K. In particular we want to ameliorate the estimate for small t in (13), at least in the case $k = 1$, and with explicit bounds. It is based on the construction of a particular Lyapounov functional $A(t)$ taking into account the hypoelliptic properties of K. In fact Theorem 1.1 is included in the following proposition.

Proposition 3.1. *There exists a constant C_2 such that for all $t > 0$, we have the following*:

(i) *$e^{-tK} b^*$ is bounded by $C_2(1 + t^{-1/2})$ and*
(ii) *$e^{-tK} a^*$ is bounded by $C_2(1 + t^{-3/2})$,*

as bounded operators on B^2. Here C_2 depends only on $\|V''\|_{L^\infty}$ (and γ). Besides we have the same bounds as in (i) *for the operators $b^\natural e^{-tK^\sharp}$ and $e^{-tK^\sharp} b^\natural$ and as in* (ii) *for operators $a^\natural e^{-tK^\sharp}$ and $e^{-tK^\sharp} a^\natural$, where $\sharp$ and $\natural$ are either nothing or $*$.*

Proof. We shall in a moment prove the results for ae^{-tK} and be^{-tK}. Taking these bounds for given we note that they imply similar bounds for their adjoints $e^{-tK^*}b^*$ and $e^{-tK^*}a^*$ since $B^{2\prime} = B^2$. The proof is exactly the same for be^{-tK^*} and ae^{-tK^*} since the sign in front of X_0 has essentially no importance in the proof. Taking the adjoints again give the result for $e^{-tK}b^*$ and $e^{-tK}a^*$.

Now for the bound on b^*e^{-tK} we simply have to note that for $f_0 \in B^2$ given and $f(t) = e^{-tK} f_0$ we have

$$\|b^* f(t)\|^2 = \big(bb^* f(t), f(t)\big) = \big(b^* b f(t), f(t)\big) + d\|f(t)\|^2 = \|bf(t)\|^2 + d\|f(t)\|^2$$

and we get the result. For a^*e^{-tK} we similarly write

$$\|a^* f(t)\|^2 = \big(aa^* f(t), f(t)\big) = \big(a^* a f(t), f(t)\big) + \big(\Delta V f(t), f(t)\big) \leqslant \|af(t)\|^2 + C\|f(t)\|^2$$

since V is with second derivatives bounded. This gives the result. For the other terms, we repeat the procedure followed in preceding paragraph and the proof of the last assertion in Proposition 3.1 is complete.

Let us come back now to the bounds on ae^{-tK} and be^{-tK}. We note that the operators are well defined since e^{-tK} is defined from $\mathcal{M}^{1/2}\mathbb{H}^{-k,-k}$ to $\mathcal{M}^{1/2}\mathbb{H}^{k,k}$. For the proof we shall need a series of results.

The first thing we do it to change the function f by the standard conjugation tool: we pose

$$u = f/\mathcal{M}^{1/2}, \qquad u_0 = f_0/\mathcal{M}^{1/2}. \tag{14}$$

After this conjugation, operator K is replaced by the following:

$$K = v\partial_x - \partial_x V \partial_v + \gamma(-\partial_v + v/2)(\partial_v + v/2) \tag{15}$$

acting on u, and defined in the flat space L^2. We recall also that the Witten Laplacian in both variables v and x reads after conjugating

$$\Lambda^2 = \gamma(-\partial_x + \partial_x V/2).(\partial_x + \partial_x V/2) + \gamma(-\partial_v + v/2)(\partial_v + v/2), \tag{16}$$

and that operators a, b and their adjoints are now

$$\begin{aligned} a &= \gamma^{1/2}(\partial_x + \partial_x V/2), \qquad a^* = \gamma^{1/2}(-\partial_x + \partial_x V/2), \quad \text{and} \\ b &= \gamma^{1/2}(\partial_v + v/2), \qquad b^* = \gamma^{1/2}(-\partial_v + v/2). \end{aligned} \tag{17}$$

Operator $X_0 = v\partial_x - \partial_x V \partial_v$ is unchanged and we have again

$$K = X_0 + b^* b, \qquad \Lambda^2 = a^* a + b^* b. \tag{18}$$

For all this conjugated operators we keep the same notations as for the unconjugated ones. No confusion is possible since they act in L^2 on the conjugated function u instead of f. The norm is the standard one associated with the L^2 space.

We work now in the L^2 setting we just defined. We recall that for any $l \in \mathbb{R}$ and $u_0 \in \Lambda^l L^2$, Proposition 2.2 implies that for all $t > 0$ and $k \in \mathbb{R}$, $u(t) \in \Lambda^k L^2$. We now choose u_0 such that $\Lambda^2 u_0 \in L^2$ and we pose for $t \geqslant 0$,

$$A(t) = t^3 \|au\|^2 + Et^2 \operatorname{Re}(au, bu) + Dt\|bu\|^2 + C\|u\|^2.$$

A is a $\mathcal{C}^0(\mathbb{R}^+, \mathbb{R}) \cap \mathcal{C}^1(\mathbb{R}^{+*}, \mathbb{R})$ function, and we can compute its time derivative for $t > 0$.

Derivative of $\|u\|^2$. We have

$$\partial_t \|u\|^2 = -2 \operatorname{Re}(Ku, u) = -2\|bu\|^2. \tag{19}$$

Derivative of $t\|bu\|^2$. We write

$$\partial_t t\|bu\|^2 = \|bu\|^2 + t\partial_t \big(b^* bu, u\big).$$

Let us compute separately the second derivative. We have

$$\begin{aligned}\partial_t (b^* bu, u) &= -\operatorname{Re}\big(b^* bKu, u\big) - \operatorname{Re}(b^* bu, Ku)\\ &= -2\big\|b^* bu\big\|^2 - \operatorname{Re}\big(b^* bX_0 u, u\big) + \operatorname{Re}\big(X_0 b^* bu, u\big)\\ &= -2\big\|b^* bu\big\|^2 - \operatorname{Re}\big(\big[b^* b, X_0\big]u, u\big).\end{aligned}$$

Using that $a = [b, X_0]$, we get

$$\begin{aligned}\partial_t \big(b^* bu, u\big) &= -2\big\|b^* bu\big\|^2 - \operatorname{Re}\big(b^* au, u\big) - \operatorname{Re}\big(a^* bu, u\big)\\ &= -2\big\|b^* bu\big\|^2 - 2\operatorname{Re}(au, bu).\end{aligned}$$

As a consequence we can write that

$$\partial_t \big(t\|bu\|^2\big) = \|bu\|^2 - 2t\big\|b^* bu\big\|^2 - 2\operatorname{Re} t(au, bu). \tag{20}$$

Derivative of $t^2 \operatorname{Re}(au, bu)$. We write

$$\partial_t t^2 \operatorname{Re}(au, bu) = 2t \operatorname{Re}(au, bu) + t^2 \partial_t \operatorname{Re}(au, bu). \tag{21}$$

Let us compute again separately the second derivative:

$$\begin{aligned}\partial_t \operatorname{Re}(au, bu) &= -\operatorname{Re}(aKu, bu) - \operatorname{Re}(au, bKu)\\ &= -\operatorname{Re}\big(ab^* bu, bu\big) - \operatorname{Re}\big(au, bb^* bu\big) - \operatorname{Re}(aX_0 u, bu) - \operatorname{Re}(au, bX_0 u).\end{aligned}$$

We can commute the field X_0 in the last two terms and we get

$$\partial_t \operatorname{Re}(au, bu) = -\operatorname{Re}\big(ab^* bu, bu\big) - \operatorname{Re}\big(au, bb^* bu\big) - \operatorname{Re}\big([a, X_0]u, bu\big) - \operatorname{Re}\big(au, [b, X_0]u\big)$$
$$+ \underbrace{\operatorname{Re}(X_0 au, bu) + \operatorname{Re}(au, X_0 bu)}_{=0 \text{ since } X_0 \text{ is skew-adjoint}}.$$

Now use the facts that $[b, X_0] = a$ and $-\operatorname{Hess} V b = [a, X_0]$. This yields

$$\partial_t \operatorname{Re}(au, bu) = -\operatorname{Re}(bau, bbu) - \operatorname{Re}\big(b^* au, b^* bu\big) + (\operatorname{Hess} V bu, bu) - \|au\|^2$$

and using (21) we get

$$\partial_t \big(t^2 \operatorname{Re}(au, bu)\big) = 2t \operatorname{Re}(au, bu) - t^2 \|au\|^2 + t^2 (\operatorname{Hess} V bu, bu)$$
$$- t^2 \operatorname{Re}(bau, bbu) - t^2 \operatorname{Re}\big(b^* au, b^* bu\big).$$

Using eventually the fact that $b^* b = bb^* - \gamma d$ yields

$$\partial_t \big(t^2 \operatorname{Re}(au, bu)\big) = 2t \operatorname{Re}(au, bu) - t^2 \|au\|^2 + t^2 (\operatorname{Hess} V bu, bu)$$
$$- 2t^2 \operatorname{Re}(bau, bbu) - t^2 \gamma d \operatorname{Re}(au, bu). \tag{22}$$

Derivative of $t^3 \|au\|^2$. We write

$$\partial_t t^3 \|au\|^2 = 3t^2 \|au\|^2 + t^3 \partial_t \|au\|^2. \tag{23}$$

We study separately the second term:

$$\partial_t \operatorname{Re}(au, au) = -\operatorname{Re}(aKu, au) - \operatorname{Re}(au, aKu)$$
$$= -\operatorname{Re}\big(ab^* bu, au\big) - \operatorname{Re}\big(au, ab^* bu\big) - \operatorname{Re}(aX_0 u, au) - \operatorname{Re}(au, aX_0 u).$$

We shall again commute the field X_0 in the last terms

$$\partial_t \operatorname{Re}(au, au) = -2\|bau\|^2 - 2\operatorname{Re}(au, aX_0 u)$$
$$= -2\|bau\|^2 - \operatorname{Re}\big([a, X_0]u, au\big) + \underbrace{\operatorname{Re}(X_0 au, au)}_{=0 \text{ since } X_0 \text{ is skew-adjoint}}.$$

Now since $-\operatorname{Hess} V b = [a, X_0]$. We get

$$\partial_t \operatorname{Re}(au, au) = -2\|bau\|^2 + 2\operatorname{Re}(\operatorname{Hess} V bu, au).$$

From (23) we can therefore write

$$\partial_t t^3 \|au\|^2 = 3t^2 \|au\|^2 - 2t^3 \|bau\|^2 + 2t^3 \operatorname{Re}(\operatorname{Hess} V bu, au). \tag{24}$$

Derivative of A. We put together the results of (19), (20), (22), (24) and we get the following formula for the derivative of A, where we have put the similar terms on the same lines:

$$\begin{aligned}
&\partial_t A(t) \\
&\quad = -2C\|bu\|^2 - 2tD\|b^*bu\| - t^2E\|au\|^2 - 2t^3\|bau\|^2 && \boxed{1} \\
&\qquad + D\|bu\|^2 + t^2E\,\mathrm{Re}(\mathrm{Hess}\,V bu, bu) && \boxed{2} \\
&\qquad + 2tD\,\mathrm{Re}(au, bu) + 2tE\,\mathrm{Re}(au, bu) + 2tE\,\mathrm{Re}(\mathrm{Hess}\,V bu, au) - t^2D\gamma d\,\mathrm{Re}(au, bu) && \boxed{3} \\
&\qquad + 3t^2\|au\|^2 && \boxed{4} \\
&\qquad - 2Et^2\,\mathrm{Re}(bau, bbu). && \boxed{5}
\end{aligned}$$

We bound now each terms on the lines $\boxed{2}$–$\boxed{5}$ by terms appearing in $\boxed{1}$. We suppose that $t \in]0, 1]$. Now since the Hessian of V is bounded by a constant, say C_V, we have

$$\boxed{2} \leqslant (D + EC_V)\|bu\|^2 \ll 2C\|bu\|^2 \quad \text{if } D, E \ll C. \tag{25}$$

For the term $\boxed{3}$, we write for $\eta > 0$,

$$\begin{aligned}
\boxed{3} &\leqslant (2D + 2E + 2EC_V + D\gamma n)t\|au\|\|bu\| \\
&\leqslant \eta t^2\|au\|^2 + \frac{Cte(D, E, \gamma d)}{\eta}\|bu\|^2.
\end{aligned}$$

We therefore get that for a given E, we have to choose η sufficiently small and then C big enough to get

$$\boxed{3} \ll Et^2\|au\|^2 + 2C\|bu\|^2. \tag{26}$$

Now we treat the term $\boxed{4}$. This is easy since we only need to take $E \gg 3$ in order to get

$$\boxed{4} \ll -3t^2\|au\|^2. \tag{27}$$

For the last term $\boxed{5}$ we write:

$$\begin{aligned}
\boxed{5} &= -2Et^2\,\mathrm{Re}(bau, bbu) \leqslant 2Et^2\|bau\|\|bbu\| \\
&\leqslant E\left(\eta' t^3\|bau\|^2 + \frac{t}{\eta'}\|bbu\|\right) \\
&\leqslant E\eta' t^3\|bau\|^2 + \frac{Et}{\eta'}\|b^*bu\|,
\end{aligned}$$

where in the last estimate we use the fact that for w s.t. $\Lambda_b w \in L^2$, we have $\|bw\| \leqslant \|b^*w\|$. Now for a given E we have to choose first η' small enough, and then D sufficiently large to write

$$\boxed{5} \ll 2tD\|b^*bu\|^2 + 2t^3\|bau\|^2. \tag{28}$$

Synthesis. We checked that each line $\boxed{2}$–$\boxed{5}$ can separately be bounded by a term appearing in $\boxed{1}$. In order to get the fact that $t \mapsto A(t)$ is decreasing, we choose the constants as follows: first E so that (27), and then η, η', C and D such that (28) and (26). Eventually increasing C so that (25) holds yields the result. In particular since $A(t)$ is right-continuous in 0 we get that for all $t \in [0, 1]$

$$A(t) = t^3\|au\|^2 + Et^2\operatorname{Re}(au, bu) + Dt\|bu\|^2 + C\|u\|^2 \leqslant C\|u_0\|^2.$$

In particular we have for $t \in [0, 1]$,

$$\|au(t)\| \leqslant C^{1/2}t^{-3/2}\|u_0\|, \qquad \|bu(t)\| \leqslant (C/D)^{1/2}t^{-1/2}\|u_0\|.$$

This is the short time estimate ($t \in]0, 1]$) in Proposition 3.1 for $\Lambda^2 u_0 \in L^2$. For $t \geqslant 1$ we simply write that

$$\|be^{-tK}u_0\| = \|be^{-K/2}e^{-(t-1/2)K}u_0\| \leqslant C_2\|e^{-(t-1/2)K}u_0\| \leqslant C_2\|u_0\|,$$

where we used first the short time estimate (with $t = 1/2$) and then the fact that K is maximal accretive. The result for $u_0 \in L^2$ follows then by density. □

4. Exponential time decay

4.1. An abstract Hilbert result

Let us first state a general lemma about semigroup of operators. Let $\mathcal{K}$ be the infinitesimal generator of a semigroup of contraction on a Hilbert space H (in particular $\overline{D(\mathcal{K})} = H$). We want to extend the following basic result:

If $\exists\alpha > 0$ such that $\alpha\|\varphi\|^2 \leqslant \operatorname{Re}(\mathcal{K}\varphi, \varphi)$ for all $\varphi \in D(\mathcal{K})$ then $\forall\varphi_0 \in H$, $t \geqslant 0$ we have $\|e^{-t\mathcal{K}}\varphi_0\| \leqslant e^{-\alpha t}\|\varphi_0\|$.

Of course the converse is true applying the Lummer–Phillips theorem to the operator $\mathcal{K} - \alpha\,\mathrm{Id}$ (see for example [21]). We want to extend the right sense.

Lemma 4.1. *Let $\mathcal{K}$ be the infinitesimal generator of a semigroup of contraction on a Hilbert space H and suppose that there exist a constant $\alpha > 0$ and a bounded operator $\mathcal{L}$ with norm bounded by $C \geqslant 1$ such that*

$$\forall\varphi \in D(\mathcal{K}) \quad \alpha\|\varphi\|^2 \leqslant \operatorname{Re}(\mathcal{K}\varphi, \varphi) + \operatorname{Re}\big(\mathcal{K}\varphi, \big(\mathcal{L} + \mathcal{L}^*\big)\varphi\big). \tag{29}$$

Then for all $\varphi_0 \in H$ and $t \geqslant 0$ we have

$$\|e^{-t\mathcal{K}}u_0\| \leqslant 3e^{-\frac{\alpha t}{3C}}\|\varphi_0\|. \tag{30}$$

Proof. We write for $\varphi_0 \in D(\mathcal{K})$, $\varphi(t) = e^{-t\mathcal{K}}\varphi_0 \in D(\mathcal{K})$. Using (29) and since $\mathcal{K}$ is accretive we get

$$3C\frac{\alpha}{3C}\|\varphi\|^2 \leqslant 4C\operatorname{Re}(\mathcal{K}\varphi, \varphi) + \operatorname{Re}\big(\mathcal{K}\varphi, \big(\mathcal{L} + \mathcal{L}^*\big)\varphi\big)$$

and since $|\operatorname{Re}(\mathcal{L}\varphi,\varphi)| \leqslant C\|\varphi\|^2$ we have

$$\frac{\alpha}{3C}\big(2C\|\varphi\|^2 + \operatorname{Re}(\mathcal{L}\varphi,\varphi)\big) \leqslant 4C\operatorname{Re}(\mathcal{K}\varphi,\varphi) + \operatorname{Re}\big(\mathcal{K}\varphi, \big(\mathcal{L}+\mathcal{L}^*\big)\varphi\big).$$

Now $\partial_t\|\varphi\|^2 = -2\operatorname{Re}(\mathcal{K}\varphi,\varphi)$ and $\partial_t \operatorname{Re}(\mathcal{L}\varphi,\varphi) = -\operatorname{Re}(\mathcal{K}\varphi,(\mathcal{L}+\mathcal{L}^*)\varphi)$ therefore

$$\frac{\alpha}{3C}\big(2C\|\varphi\|^2 + \operatorname{Re}(\mathcal{L}\varphi,\varphi)\big) + \frac{\partial}{\partial_t}\big(2C\|\varphi\|^2 + \operatorname{Re}(\mathcal{L}\varphi,\varphi)\big) \leqslant 0.$$

Integrating between 0 and t gives

$$2C\|\varphi\|^2 + \operatorname{Re}(\mathcal{L}\varphi,\varphi) \leqslant e^{-\frac{\alpha t}{3C}}\big(2C\|\varphi_0\|^2 + \operatorname{Re}(\mathcal{L}\varphi_0,\varphi_0)\big).$$

Using twice the fact that $\|\mathcal{L}\|$ is bounded by C we get

$$C\|\varphi\|^2 \leqslant 3Ce^{-\frac{\alpha t}{3C}}\|\varphi_0\|^2.$$

This gives (30) since $\|\varphi\| \leqslant \|\varphi_0\|$ and the proof is complete. □

4.2. The case of the Fokker–Planck operator

We want now to apply the preceding abstract result to the linear Fokker–Planck equation and prove Theorem 1.2. A part of the proof is very close to the one given in [14] and in particular uses Kohn's type arguments about hypoellipticity developed there. We work from now on with a potential V satisfying both conditions of type (H1) and (H2).

We shall work in the following in the orthogonal in the B^2 sense of the Maxwellian. For the following we call

$$\mathcal{B}^2_\perp = \mathcal{M}^\perp \cap B^2 = \left\{ f \in B^2 \text{ s.t. } \int f\,dx\,dv = 0 \right\},$$

endowed with the norm of B^2, where $\perp$ stands for the orthogonal with respect to the scalar product in B^2 (recall that $B^{2\prime}$ was identified with B^2 according to the measure $\mathcal{M}^{-1}\,dx\,dv$ in (8)).

We note that $\mathcal{B}^2_\perp$ is stable for K. Indeed for all $f \in \mathcal{B}^2_\perp$, we have

$$\langle Kf, \mathcal{M}\rangle = \int Kf\mathcal{M}\mathcal{M}^{-1}\,dx = \int f\big(K^*\mathcal{M}\big)\mathcal{M}^{-1} = 0.$$

Since K restricted to B^2 generates a semigroup of contraction, we have the same property in $\mathcal{B}^2_\perp$. Anyway restricted to $\mathcal{B}^2_\perp$ the semigroup has a much better property at infinity:

Proposition 4.2. *Suppose V satisfies hypotheses* (H1) *and* (H2). *Then there exists a constant A depending only on $\|V''\|_{L^\infty}$ such that for all $t \geqslant 0$*

$$\big\|e^{-tK}\big\|_{\mathcal{B}^2_\perp} \leqslant 3e^{-\alpha t/A},$$

where α was defined in (H2). *The same bound occurs for K^*.*

Proof. Again we work in L^2 after conjugation by the square root of the Maxwellian, and the new unknown function is u. We therefore keep for the proof of the proposition the notations K, a, b, X_0 and Λ^2 introduced in (14)–(18). Note that after the conjugation, the space B^2 becomes the orthogonal of the square root of the Maxwellian and we denote it by $H = (\mathcal{M}^{1/2})^\perp$. We note that H is stable for both K and Λ^2 and that $\mathcal{M}^{1/2}$ is a single eigenfunction for K and Λ^2 with eigenvalue 0. We also introduce the following operator on (the flat space) L^2:

$$\Lambda_\delta^2 \stackrel{\text{def}}{=} \delta^2 + a^*a + b^*b,$$

where $\delta^2 \leqslant \gamma$ is to be fixed later.

Let us take $u \in \mathcal{S}$. We first quote the result of Proposition 2.5 (case $\varepsilon = 0$ there) in [14], which is true under our assumptions on V:

$$\begin{aligned}\|u\|^2 \leqslant{} & \operatorname{Re}\big(Ku, \big(L + L^*\big)u\big) - 2\operatorname{Re}\big(b^*bu, Lu\big) - \operatorname{Re}\big(\mathcal{A}^*bu, u\big) \\ & + (1+\gamma)\delta^{-1}\|bu\|\|u\| + \delta^2\big(\Lambda_\delta^{-2}u, u\big),\end{aligned} \tag{31}$$

where $L = \Lambda_\delta^{-2}a^*b$ and $\mathcal{A}^* = [\Lambda_\delta^{-2}a^*, X_0]$. From Proposition 5.4 in [14] we have an explicit bound for the bounded operator $\mathcal{A}^*$, and this is also easy to get bounds for L and $a\Lambda_\delta^{-2}b^*$,

$$\big\|\mathcal{A}^*\big\| \leqslant C_V\delta^{-1}, \qquad \|L\| \leqslant \sqrt{2d\gamma}\,\delta^{-1}, \qquad \big\|a\Lambda_\delta^{-2}b^*\big\| \leqslant 1.$$

(For the second one we simply observe that if a_j, b_j denote the components of a and b we have

$$\big\|b^*au\big\| \leqslant \sum_j \big\|b_j^*a_ju\big\| \quad \text{and} \quad \big\|b_j^*a_ju\big\|^2 = \big(b_jb_j^*u, a_j^*a_ju\big) = \big(b_j^*b_ju, a_j^*a_ju\big) + \gamma\big(a_j^*a_ju, u\big)$$

and since $\delta^2 \leqslant \Lambda_\delta^2$ we get

$$\big\|b_j^*a_ju\big\|^2 \leqslant 2\gamma\delta^{-2}\big\|\Lambda_\delta^2u\big\|^2$$

therefore

$$\big\|b^*au\big\|^2 \leqslant 2d\gamma\delta^{-2}\big\|\Lambda_\delta^2u\big\|^2 \quad \text{and} \quad \big\|b^*a\Lambda^{-2}u\big\|^2 \leqslant 2d\gamma\delta^{-2}\|u\|^2.$$

Taking the adjoint and the square root gives the result.)

We can then write from (31) that

$$\begin{aligned}\|u\|^2 \leqslant{} & \operatorname{Re}\big(Ku, \big(L + L^*\big)u\big) + 2\big|\big(a\Lambda_\delta^{-2}b^*bu, bu\big)\big| + \big|\operatorname{Re}\big(\mathcal{A}^*bu, u\big)\big| \\ & + (1+\gamma)\delta^{-1}\|bu\|\|u\| + \delta^2\big(\Lambda_\delta^{-2}u, u\big) \\ \leqslant{} & \operatorname{Re}\big(Ku, \big(L + L^*\big)u\big) + 2\|bu\|^2 + C_V\delta^{-1}\|bu\|\|u\| \\ & + (1+\gamma)\delta^{-1}\|bu\|\|u\| + \delta^2\big(\Lambda_\delta^{-2}u, u\big).\end{aligned}$$

Using first the inequality $|xy| \leqslant x^2 + 4^{-1}y^2$ and then the fact that $\operatorname{Re}(Ku, u) = \|bu\|^2$ we get

$$\|u\|^2 \leqslant \operatorname{Re}\big(Ku, \big(L+L^*\big)u\big) + C'_V \delta^{-2}\|bu\|^2 + \frac{1}{4}\|u\|^2 + \delta^2\big(\Lambda_\delta^{-2}u, u\big)$$

$$\leqslant \operatorname{Re}\big(Ku, \big(L+L^*\big)u\big) + C'_V \delta^{-2}\operatorname{Re}(Ku, u) + \frac{1}{4}\|u\|^2 + \delta^2\big(\Lambda_\delta^{-2}u, u\big). \tag{32}$$

Now we suppose that $u \in H$ therefore $\delta^2(\Lambda_\delta^{-2}u, u) \leqslant \frac{\delta^2}{\alpha+\delta^2}\|u\|^2$ and we choose $\delta^2 = \alpha$ (which is lower than γ because of the harmonic part of Λ_0^2). This gives

$$\delta^2\big(\Lambda_\delta^{-2}u, u\big) \leqslant \frac{1}{2}\|u\|^2$$

and putting this in (32) we get

$$\frac{1}{4}\|u\|^2 \leqslant \operatorname{Re}\big(Ku, \big(L+L^*\big)u\big) + C'_V \delta^{-2}\operatorname{Re}(Ku, u).$$

As a consequence

$$\frac{\alpha}{4C'_V}\|u\|^2 \leqslant \operatorname{Re}\big(Ku, \big(\widetilde{L}+\widetilde{L}^*\big)u\big) + \operatorname{Re}(Ku, u),$$

where $\widetilde{L} = \frac{\delta^2}{C'_V}L$ satisfies $\|\widetilde{L}\| \leqslant \sqrt{2d\gamma}\,\delta^{-1}\delta^2/C'_V \leqslant 1$ since $\delta\sqrt{2d\gamma} \leqslant \sqrt{2d}\gamma \leqslant C'_V$. The result of the lemma is then a direct consequence of Lemma 4.1. Taking $A = 12C'_V$ completes the proof of the proposition. □

Remark 4.3. We can point out that the gain with respect to the estimate in [14] is first that the constant in front of the exponential is universal (= 3) and in particular does not depend on V or α. It makes sense in the study of the Vlasov–Poisson–Fokker–Planck system with small data in Section 4 since this constant has to be compared with the size of the initial data. The second remark is that no assumption about the increasing of $\partial_x V$ is made, and we can understand this fact by saying that the existence of a spectral gap for the Witten Laplacian implies a (generalized) spectral gap for the Fokker–Planck operator, without assumptions on the remaining part of the spectrum (implied for example by the compacity of the resolvent).

Remark 4.4. Note to the end that in the preceding study is also valid even for V'' not in L^∞ since the only real needed assumption is that the constant C_V in (32) is not infinite (see [14] for its expression). Anyway in this case one has to be careful when defining the commutators, and some additional assumptions on V may be needed.

Putting together Propositions 4.2 and 3.1 we can complete the proof of Theorem 1.2. In fact it is included in the following proposition:

Proposition 4.5. *Suppose V satisfies hypotheses* (H1) *and* (H2). *Then there exist constants C and A such that for all $t > 0$,*

(i) be^{-tK}, $e^{-tK}b^*$, be^{-tK^*} *and* $e^{-tK^*}b^*$ *are bounded by* $C(1+t^{-1/2})e^{-\alpha t/A}$,
(ii) ae^{-tK}, $e^{-tK}a^*$, ae^{-tK^*} *and* $e^{-tK^*}a^*$ *are bounded by* $C(1+t^{-3/2})e^{-\alpha t/A}$

as bounded operators in B^2, where C and A depend only on $\|V''\|_{L^\infty}$.

Proof of Proposition 4.5 and Theorem 1.2. We simply use the following fact: for a given $f_0 \in \mathcal{M}^{1/2}\mathcal{S} \subset B^2$, we have

$$\left(b^* f_0, \mathcal{M}\right)_{B^2} = (f_0, b\mathcal{M})_{B^2} = 0$$

i.e. $b^* f_0$ is orthogonal to the square root of the Maxwellian. Of course it is also the case for $e^{-K} b^* f_0$ since

$$\left(e^{-K} b^* f_0, \mathcal{M}\right)_{B^2} = \left(b^* f_0, e^{-K^*} \mathcal{M}\right)_{B^2} = \left(b^* f_0, \mathcal{M}\right)_{B^2} = 0.$$

Now for $t \geqslant 1$ we can apply Proposition 4.2 and we get the following bound:

$$\begin{aligned} \left\| e^{-tK} b^* f_0 \right\| = \left\| e^{-(t-1)K} e^{-K} b^* f_0 \right\| &\leqslant C e^{-\alpha(t-1)/A} \left\| e^{-K} b^* f_0 \right\| \\ &\leqslant 3 e^{\gamma/A} e^{-\alpha t} \left\| e^{-K} b^* f_0 \right\|, \end{aligned} \tag{33}$$

since $\alpha \leqslant \gamma$ because of the harmonic part of Λ^2. Now from Proposition 3.1 applied with $t = 1$ we get

$$\left\| e^{-K} b^* f_0 \right\|_{B^2} \leqslant C' \left(1 + t^{-1/2}\right) \| f_0 \|_{B^2}.$$

This inequality together with (33) give the bound

$$\left\| e^{-tK} b^* f_0 \right\|_{B^2} \leqslant C_2 \left(1 + t^{-1/2}\right) e^{-\alpha t/A} \| f_0 \|_{B^2}$$

for an initial data $f_0 \in \mathcal{M}^{1/2}\mathcal{S}$. It can be clearly extended to $f_0 \in B^2$ by density. The proof of the estimates about $e^{-tK} a^*$ can be done exactly in the same way. The assertions concerning K^* are immediate since the structure of the operator is the same. Eventually the ones concerning $a e^{-tK}$ and $b e^{-tK}$ are immediate using the adjoint of the preceding ones. The proof of Proposition 4.5 is complete. □

Remark 4.6. In the particular case of $V = x^2$ or more generally for quadratic external potentials, one can compute explicitly the Green function of e^{-tK} using the method of characteristics (see e.g. [17]). Anyway if it gives after some work the short time behavior, the exponential decay of $e^{-tK} b^*$ is not clear on the formulas. In fact the short time decays in Proposition 3.1 can be viewed as consequences of the Lie group structure of the operator (if one assimilates b^* and b) whereas the long time behavior is deeply linked with the spectral properties of K.

5. Applications

5.1. Strong solutions for a given interaction potential

In this section we work again in a linear setting and study the following Fokker–Planck equation:

$$\begin{cases} \partial_t f + v.\partial_x f - (E + \partial_x V).\partial_v f - \gamma \partial_v.(\partial_v + v) f = U, \\ f|_{t=0} = f_0, \end{cases} \tag{34}$$

where $E(t,x)$ is a given time-dependent potential satisfying $E \in L^\infty([0,T[\times\mathbb{R}^d)$ and V is again a potential satisfying hypothesis (H1). We shall prove existence and uniqueness in the space $\mathbb{H}^{l,k}$ based on B^2 and defined in (12). In the following propositions we will assume:

$$\text{(H3)}\qquad \begin{cases} V \text{ satisfies (H1)},\\ E \in L^\infty\big([0,T[\times\mathbb{R}^d\big),\\ f_0 \in B^2\big(\mathbb{R}^{2n}\big),\\ U \in L^2\big([0,T[,\mathbb{H}^{0,-1}\big(\mathbb{R}^d_x\times\mathbb{R}^d_v\big)\big).\end{cases}$$

The first proposition gives existence and uniqueness of a unique mild solution of the system (34).

Proposition 5.1. *Under hypothesis* (H3) *there exists a unique mild solution of* (34)*, where by definition a mild solution is a solution* $f \in \mathcal{C}([0,T[,B^2)$ *satisfying*

$$f(t) = e^{-tK} f_0 + \int_0^t e^{-(t-s)K} E(s)\partial_v f(s) + \int_0^t e^{-(t-s)K} U(s)\,ds.$$

Proof. This is obtained via a standard fixed point theorem in $L^\infty([0,T[,B^2)$. We only sketch the proof. To simplify the notations we suppose $\gamma = 1$ which implies $\partial_v = -b^*$. Let F be the following operator from $L^\infty([0,T[,B^2)$ into itself given by

$$\begin{aligned} F(f) &= e^{-tK} f_0 + \int_0^t e^{-(t-s)K} E(s)\partial_v f(s) + \int_0^t e^{-(t-s)K} U(s)\\ &= e^{-tK} f_0 - \int_0^t e^{-(t-s)K} b^* E(s) f(s) + \int_0^t e^{-(t-s)K}\big(1+b^*b\big)\Lambda_b^{-1}\Lambda_b^{-1} U(s). \end{aligned} \tag{35}$$

According to the diffusion estimates given in Proposition 3.1 and using the fact the $b\Lambda_b^{-1}$ is bounded by 1 as an operator in B^2, we get that for all $0 < t < T$

$$\big\| e^{-tK}\big(1+b^*b\big)\Lambda_b^{-1}\big\|_{B^2} \leqslant Ct^{-1/2}.$$

We therefore get for $f \in B^2$, $F(f) \in B^2$ and for all $t > 0$,

$$\big\|F(f)\big\|_{L^\infty([0,t[,B^2)} \leqslant Ct^{1/2}\|f\|_{L^\infty([0,t[,B^2)}.$$

Using a standard fixed point theorem we get that f is the unique limit of the following iteration scheme:

$$\partial_t f^{n+1} + K f^{n+1} + E b^* f^n = U, \qquad f^0 = f_0,$$

and the continuity is clear from formula (35). □

5.2. A mollified Vlasov–Poisson–Fokker–Planck system

In this section we study the following nonlinear problem, to be understood as a modified Vlasov–Poisson–Fokker–Planck system, where the nonlinear coupling is mollified:

$$\begin{cases} \partial_t f + v.\partial_x f - (E + \partial_x V).\partial_v f - \gamma \partial_v.(\partial_v + v) f = 0, \\ E(t,x) \stackrel{\text{def}}{=} \partial_x V_{\text{nl}}(t,x) = -\zeta * \dfrac{\kappa}{|S^{d-1}|} \dfrac{x}{|x|^d} *_x \rho(t,x), \\ \quad \text{where } \rho(t,x) = \displaystyle\int f(t,x,v)\,dv, \\ f|_{t=0} = f_0, \end{cases} \tag{36}$$

where $\zeta \in \mathcal{S}$ (depending only on x). We shall write in the following

$$\varphi = -\zeta * \frac{1}{|S^{d-1}|} \frac{x}{|x|^d} \tag{37}$$

so that the field reads $E = \kappa\varphi * \rho$. In fact the following result and the ones in the next section work as well for any $\varphi \in L^\infty$. We shall prove the following proposition.

Proposition 5.2. *Suppose that V satisfies hypothesis* (H1) *and that $f_0 \in B^2$. Then for all $T > 0$, the approximate problem* (36) *admits a unique strong solution in $\mathcal{C}([0,T[, B^2)$.*

Proof. We suppose $\gamma = 1$ ($\partial_v = -b^*$) in the proof for convenience. The existence is given by a fixed point theorem. We study the following family of linear problems where f^0 is fixed and on an interval of time $[0,T]$ for T finite and fixed:

$$\begin{cases} \partial_t f^{n+1} + v.\partial_x f^{n+1} - \big(E^n + \partial_x V\big).\partial_v f^{n+1} - \gamma \partial_v.(\partial_v + v) f^{n+1} = 0, \\ E^n = \partial_x V_{\text{nl}}^n = \kappa\varphi *_x \rho^n \quad \text{with } \rho^n(t,x) = \displaystyle\int f^n(t,x,v)\,dv, \\ f|_{t=0} = f_0 \quad \text{and} \quad f^0 = f_0. \end{cases}$$

In the following we call C any constant independent of n (but perhaps depending on T). Proposition 5.1 yields that for each $n \geqslant 0$ this problem admits a mild solution f^{n+1} since

$$\|E^n\|_{L^\infty(dt\,dx)} \leqslant \|\varphi\|_{L^\infty} \|\rho^n\|_{L^1(dx)} = \|\varphi\|_{L^\infty} = C,$$

from Young inequality. This solution is given by

$$f^{n+1}(t) = e^{-tK} f_0 - \int_0^t e^{-(t-s)K} b^* E^n f^{n+1}(s)\,ds,$$

and we observe using a Gronwall inequality and the diffusion estimate from Proposition 3.1 that there exists a constant C independent of n such that $\|f^n\|_{B^2} \leqslant C_T$. Now for all $0 \leqslant t \leqslant T$

$$\begin{aligned}\|f^{n+1}-f^n\|_{B^2} &\leqslant \left\|\int_0^t e^{-(t-s)K} b^* E^n\big(f^{n+1}(s)-f^n(s)\big)\,ds\right\|_{B^2} \\ &\quad+\left\|\int_0^t e^{-(t-s)K} b^* f^n\big(E^n(s)-E^{n-1}(s)\big)\,ds\right\|_{B^2} \\ &\leqslant C'_T\sqrt{t}\big(\|f^{n+1}-f^n\|_{B^2}-\|f^n-f^{n-1}\|_{B^2}\big)\end{aligned}$$

since

$$\|E^n(s)-E^{n-1}(s)\|_{L^\infty} \leqslant \|\varphi\|_{L^\infty}\|f^n-f^{n-1}\|_{L^1} \leqslant \|\varphi\|_{L^\infty}\|f^n-f^{n-1}\|_{B^2}.$$

Now a standard fixed point theorem give that on any interval $[0, c_T[\, \in [0, T[$ the scheme converges in $L^\infty([0, c_T[, B^2)$ where c_T is independent of n. We can apply the same procedure on any interval of type $[t, t + c_T[\, \subset [0, T[$ for t arbitrary and we get that f^n converges (strongly) in $L^\infty([0, T[, B^2)$ toward a function f, and that this is also the case for E^n toward E in $L^\infty([0, T[, L^\infty)$ where E is given by

$$E = \kappa\varphi *_x \int f(t, x, v)\,dv.$$

The function f is therefore a mild solution of the problem $\partial_t f + Kf + Eb^* f = 0$, $f|_{t=0} = f_0$. Since by Proposition 5.1 the solution is unique we get the result. □

5.3. Exponential time decay for small nonlinear coupling

In this subsection we continue the study of the mollified Vlasov–Poisson–Fokker–Planck equation defined in (36). The aim of this section is to prove Theorem 1.3 and Corollary 1.4 about the exponential decay for small charge. Let us now define as in the introduction the Fokker–Planck operator corresponding to the stationary Vlasov–Poisson–Fokker–Planck equation

$$K_\infty = v\partial_x - \partial_x(V + V_\infty)\partial_v - \gamma\partial_v(\partial_v + v).$$

We know that $V_\infty \in W^{\infty,\infty}$ so that total potential $V + V_\infty$ satisfies hypothesis (H1), and we suppose that it also satisfies hypothesis (H2bis). The Maxwellian associated to this operator is

$$\mathcal{M}_\infty(x, v) = \frac{e^{-(v^2/2+V(x)+V_\infty(x))}}{\int e^{-(v^2/2+V(x)+V_\infty(x))}\,dx\,dv}$$

and is in $\mathcal{S} \subset L^1$ with norm 1 in L^1. We define also the associated spaces $\mathcal{B}^2_\infty = \{f \in \mathcal{D}'$ s.t. $f/\mathcal{M}_\infty \in L^2(\mathcal{M}_\infty\,dx\,dv)\}$. Since $V + V_\infty$ satisfies the hypotheses (H1) and (H2bis) we can apply all the results obtained for a generic potential V. We recall that α_∞ is the smallest positive real part of the eigenvalues of the corresponding Witten Laplacian

$$\Lambda^2_\infty = -\gamma\partial_v(\partial_v + v) - \gamma\partial_x\big(\partial_x + \partial_x(V + V_\infty)\big)$$

in $\mathcal{B}^2_\infty$. We denote by the same symbols Λ^2_∞ and K_∞ the closure from $\mathcal{C}_0^\infty$ of the corresponding operators in $\mathcal{B}^2_\infty$, and recall that they are maximal accretive from Proposition 2.1. We then follow Section 4.2 by defining in our context the space

$$\mathcal{B}^2_{\infty,\perp} = \mathcal{M}^\perp \cap \mathcal{B}^2_\infty = \left\{ f \in \mathcal{B}^2_\infty \text{ s.t. } \int f\,dx\,dv = 0 \right\}$$

endowed with the norm of $\mathcal{B}^2_\infty$, where $\perp$ stands for the orthogonal with respect to the scalar product. We note that $\mathcal{B}^2_{\infty,\perp}$ is stable for K_∞. The following proposition is a direct consequence of Propositions 4.5 and 4.2 for $V + V_\infty$.

Proposition 5.3. *There exist constants C_∞ and A_∞ such that for all $t > 0$,*

(i) $e^{-tK_\infty} b^*$ *is bounded by* $C_\infty(1 + t^{-1/2})e^{-\alpha_\infty t/A_\infty}$ *on* $\mathcal{B}^2_\infty$,
(ii) $e^{-tK_\infty} a^*$ *is bounded by* $C_\infty(1 + t^{-3/2})e^{-\alpha_\infty t/A_\infty}$ *on* $\mathcal{B}^2_\infty$,
(iii) e^{-tK_∞} *is bounded by* $3e^{-\alpha_\infty t/A_\infty}$ *on* $\mathcal{B}^2_{\infty,\perp}$,

where C_∞ and A_∞ depend only on $\|(V_\mathrm{e} + V_\infty)''\|_{L^\infty}$ and the physical constants (*uniformly in κ varying in a fixed compact set*).

We work now in the Hilbert space $\mathcal{B}^2_\infty$ which we recall is norm-equivalent to B^2 since $V_\infty \in W^{\infty,\infty}$. For convenience we again suppose $\gamma = 1$. For $t, x \in \mathbb{R}^+ \times \mathbb{R}^d$ we denote

$$V_{\mathrm{diff}}(t,x) = V_{\mathrm{nl}}(t,x) - V_\infty(x).$$

We can write the Cauchy problem associated to the VPFP system as follows:

$$\begin{cases} \partial_t f + K_\infty f = -b^* \partial_x V_{\mathrm{diff}} f, \\ f|_{t=0} = f_0. \end{cases}$$

Using the a priori bounds for the solution f given by Proposition 5.2, the unique solution satisfies the following Duhamel formula written in terms of K_∞ in $\mathcal{B}^2_\infty$:

$$f(t,x,v) = e^{-tK_\infty} f_0(x,v) - \int_0^t e^{-(t-s)K_\infty} b^* \partial_x V_{\mathrm{diff}}(s,x) f(s,x,v)\,ds. \tag{38}$$

We know that $\partial_x V_{\mathrm{diff}} \in L^\infty(\mathbb{R}_t \times \mathbb{R}^d_x)$ and we recall that $\varphi(x) = -\zeta * \frac{1}{|S^{d-1}|} \frac{x}{|x|^d}$, so that $\partial_x V_{\mathrm{diff}}$ reads

$$\partial_x V_{\mathrm{diff}}(t,x) = \partial_x V_{\mathrm{nl}}(t,x) - \partial_x V_\infty(x) = \kappa \varphi(x) *_x \big(\rho(t,x) - \rho_\infty(x)\big),$$

where $\rho_\infty(t,x) = \int f_\infty(t,x,v)\,dv$ with $f_\infty = \mathcal{M}_\infty$. This is clear that $f_\infty = \mathcal{M}_\infty$ is the projection in the Hilbert space $\mathcal{B}^2_\infty$ of the Cauchy data f_0 on the fundamental space $\mathrm{Span}(\mathcal{M}_\infty)$ since

$$f_\infty = (f_0, \mathcal{M}_\infty)_{\mathcal{B}^2_\infty} \mathcal{M}_\infty = \left(\iint f_0 \mathcal{M}_\infty \mathcal{M}_\infty^{-1} \, dx \, dv \right) \mathcal{M}_\infty = \mathcal{M}_\infty.$$

Let us denote by $g(t,x,v) = f(t,x,v) - f_\infty(x,v)$. Since $f_\infty \in \operatorname{Ker}(K_\infty)$ we have

$$e^{-tK_\infty} f_\infty = f_\infty.$$

The Duhamel formula (38) therefore reads

$$g(t) = e^{-tK_\infty} g_0 + \int_0^t e^{-(t-s)K_\infty} b^* \partial_x V_{\text{diff}}(s) f(s) \, ds$$

and we have $\partial_x V_{\text{diff}}(t,x) = \kappa \varphi *_x \int g(t,x,v)\, dv$. In fact we shall use the following representation:

$$g(t) = e^{-tK_\infty} g_0 + \int_0^t e^{-(t-s)K_\infty} b^* \partial_x V_{\text{diff}}(s) \big(g(s) + f_\infty \big) \, ds. \tag{39}$$

Now we take the $\mathcal{B}^2_\infty$ norm in this formula. We first note that $g_0 \in \mathcal{B}^2_{\infty,\perp}$ which gives from Proposition 5.3 that for all $t \geqslant 0$,

$$\big\| e^{-tK_\infty} g_0 \big\|_{\mathcal{B}^2_\infty} \leqslant 3 e^{-\alpha_\infty t / A_\infty} \|g_0\|_{\mathcal{B}^2_\infty}. \tag{40}$$

In order to estimate the integrals in (39), we first estimate the $L^\infty(dx)$ norm of $\partial_x V_{\text{diff}}(s)$ for all $s \in (0,t)$. In the following $\|\varphi\|$ stands for $\|\varphi\|_{L^\infty}$. First we note that

$$\big\| \partial_x V_{\text{diff}}(s) \big\|_{L^\infty(dx)} \leqslant \kappa \|\varphi\| \left\| \int g(s) \, dv \right\|_{L^1(dx)} = \kappa \|\varphi\| \big\| g(s) \big\|_{L^1(dx\,dv)} \leqslant \kappa \|\varphi\| \big\| g(s) \big\|_{\mathcal{B}^2_\infty}$$

which gives

$$\big\| \partial_x V_{\text{diff}}(s) f_\infty(s) \big\|_{\mathcal{B}^2_\infty} \leqslant \big\| \partial_x V_{\text{diff}}(t) \big\|_{L^\infty} \|f_\infty\|_{\mathcal{B}^2_\infty} \leqslant \kappa \|\varphi\| \big\| g(s) \big\|_{\mathcal{B}^2_\infty}. \tag{41}$$

Now we estimate the norm of $\partial_x V_{\text{diff}}(s)$ in an another way

$$\big\| \partial_x V_{\text{diff}}(s) w \big\|_{L^\infty} \leqslant \kappa \|\varphi\| \big\| g(s) \big\|_{L^1(dx\,dv)} \leqslant \kappa \|\varphi\| \big(\big\| f(s) \big\|_{L^1(dx\,dv)} + \|f_\infty\|_{L^1(dx\,dv)} \big) \leqslant 2\kappa \|\varphi\|$$

since f and f_∞ are L^1-normalized. This gives

$$\big\| \partial_x V_{\text{diff}}(s) g(s) \big\|_{\mathcal{B}^2_\infty} \leqslant \big\| \partial_x V_{\text{diff}}(s) \big\|_{L^\infty} \big\| g(s) \big\|_{\mathcal{B}^2_\infty} \leqslant 2\kappa \|\varphi\| \|g\|_{\mathcal{B}^2_\infty}. \tag{42}$$

Putting together (41), (42) we get

$$\big\| \partial_x V_{\text{diff}}(s) \big(g(s) + f_\infty \big) \big\|_{\mathcal{B}^2_\infty} \leqslant 3\kappa \|\varphi\| \|g\|_{\mathcal{B}^2_\infty}.$$

Now applying Proposition 5.3 to the operator K_∞ with the associated rate α_∞. We can write for $t-s>0$

$$\begin{aligned}
&\big\|e^{-(t-s)K_\infty} b^* \partial_x V_{\mathrm{diff}}(s)\big(g(s)+f_\infty\big)\big\|_{\mathcal{B}^2_\infty} \\
&\quad \leqslant C_2\big(1+(t-s)^{-1/2}\big)e^{-\alpha_\infty(t-s)/A_\infty}\big\|\partial_x V_{\mathrm{diff}}(s)\big(g(s)+f_\infty\big)\big\|_{\mathcal{B}^2_\infty} \\
&\quad \leqslant \kappa C\big(1+(t-s)^{-1/2}\big)e^{-\alpha_\infty(t-s)/A_\infty}\|g\|_{\mathcal{B}^2_\infty}.
\end{aligned} \tag{43}$$

Putting (40)–(43) in the Duhamel formula (39) and calling from now on C any constant depending on $\|\varphi\|$ and the derivatives of $V+V_\infty$, we get

$$\big\|g(t)\big\|_{\mathcal{B}^2_\infty} \leqslant 3e^{-\alpha_\infty t/A_\infty}\|g_0\|_{\mathcal{B}^2_\infty} + C\kappa\int_0^t \big(1+(t-s)^{-1/2}\big)e^{-\alpha_\infty(t-s)/A_\infty}\big\|g(s)\big\|_{\mathcal{B}^2_\infty}\,ds.$$

Let us define for $t\geqslant 0$, $\psi(t)=e^{\alpha_\infty t/(2A_\infty)}\|g(t)\|_{\mathcal{B}^2_\infty}$. We get for $t\geqslant 0$,

$$\psi(t)\leqslant 3\psi(0)+C\kappa\int_0^t \big(1+(t-s)^{-1/2}\big)e^{-\alpha_\infty(t-s)/(2A_\infty)}\psi(s)\,ds.$$

With an other constant C we get

$$\psi(t)\leqslant 3\psi(0)+(C\kappa/\alpha_\infty)\sup_{s\in[0,t]}\psi(s).$$

Note here that contrary to A_∞ the constant α_∞ cannot be absorbed in the constant C since not controlled by semi-norms of $(V_{\mathrm{e}}+V_\infty)''$. Under the assumption

$$C\kappa/\alpha_\infty\leqslant 1/2$$

we get that for all $t\geqslant 0$, $\sup_{s\in[0,t]}\psi(s)\leqslant 6\psi(0)$. This reads in terms of g:

$$\big\|g(t)\big\|_{\mathcal{B}^2_\infty}\leqslant 6\big\|g(0)\big\|_{\mathcal{B}^2_\infty}e^{-\alpha_\infty t/(2A_\infty)},$$

and the proof of Theorem 1.3 is complete.

Proof of Corollary 1.4. First recall that $0\leqslant H(f(t)|\mathcal{M}_\infty)$ since f and $\mathcal{M}_\infty$ are L^1-normalized. Using the inequality $\ln(s)\leqslant s-1$ we get

$$\begin{aligned}
H\big(f(t)|\mathcal{M}_\infty\big) &= \iint \frac{f(t)}{\mathcal{M}_\infty}\ln\bigg(\frac{f(t)}{\mathcal{M}_\infty}\bigg)\mathcal{M}_\infty\,dx\,dv \\
&\leqslant \iint \frac{f(t)}{\mathcal{M}_\infty}\bigg(\frac{f(t)-\mathcal{M}_\infty}{\mathcal{M}_\infty}\bigg)\mathcal{M}_\infty\,dx\,dv \\
&\leqslant \big\|f(t)\big\|_{\mathcal{B}^2_\infty}\big\|f(t)-\mathcal{M}_\infty\big\|_{\mathcal{B}^2_\infty}.
\end{aligned}$$

Now applying Theorem 1.3 we first notice that

$$\|f(t)\|_{\mathcal{B}^2_\infty} \leqslant \|g(t)\|_{\mathcal{B}^2_\infty} + \|f_\infty\|_{\mathcal{B}^2_\infty} \leqslant 7\|f_0\|_{\mathcal{B}^2_\infty}$$

since $\|f_\infty\|_{\mathcal{B}^2_\infty} = 1 \leqslant \|f_0\|_{\mathcal{B}^2_\infty}$. Using again Theorem 1.3 yields

$$H\big(f(t)|f_\infty\big) \leqslant C\|f_0\|_{\mathcal{B}^2_\infty}\|f_0 - f_\infty\|_{\mathcal{B}^2_\infty} e^{-\alpha_\infty t/2A_\infty}.$$

The proof of Corollary 1.4 is complete. □

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