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Incidence Algebra Antipodes and Lagrange Inversion in One and Several Variables

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1. Introduction

Of the various theoretical frameworks [2, 5, 8, 10] that have been proposed for the purpose of explaining in a unified way the combinatorial significance of algebraic operations on formal power series, regarded as generating functions, the most general and flexible, if not always the most transparent, has been and remains the notion of *incidence algebra* as developed by Doubilet, Rota, and Stanley [4].

The explanatory paradigm based on incidence algebras is this: connected with each special algebraic operation on a "variety" of generating functions (ordinary, exponential, or what have you) is a family of partially ordered sets, hereditary with respect to formation of products and subintervals. In the incidence algebra built from that family, certain "multiplicative" elements are in one-to-one correspondence with individual generating functions. The family is so constructed that *via* this correspondence, the fundamental operation of *convolution* in the incidence algebra reflects the algebraic operation in question on generating functions. In this way the particular algebraic operation acquires a combinatorial interpretation because, as it turns out in every case, the family of partially ordered sets defining the incidence algebra is one with fundamental combinatorial significance.

Despite its many successes, the incidence algebra point of view has not shed much useful combinatorial light on the famous formula of Lagrange for the inverse, under functional composition, of an exponential generating function. It is strange that this should be so because of all operations on generating functions, this very one—functional composition—has one of the most interesting and satisfactory incidence algebra interpretations.

The present work has two major purposes. One is to show how the

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Lagrange inversion formula fits into the incidence algebra framework, thereby implicitly giving it a combinatorial interpretation and also providing a new description of the coefficients appearing in it (Corollary 1). As a side effect, we discover an apparently new bijection involving trees with labelled leaves and partitions (Theorem 5).

Our second major purpose is to show how the description mentioned above of the Lagrange inversion formula admits ready generalization to the case of N generating functions in N variables, for all N. For this purpose we construct the incidence algebra appropriate to composition of such systems of functions.

Remarkably, no really combinatorially satisfactory generalization of the Lagrange inversion formula to several variables has ever been given. This is not to say that there are no satisfactory multi-variate "Lagrange inversion" formulas at all: natural multi-variate formulations from the analytic, algebraic, and umbral-operator-theoretic points of view were found by Good [6], Abhyankar [1], and Joni [9], respectively. In all these formulations, however, the computation of the unknown inverse of a given system of formal power series is a mattter of expanding another power series involving differentiation operators. The formulas do not specify the actual coefficients connecting the inverse system with the original one, let alone offer any combinatorial interpretation of them. Our formulation (Theorem 7, Corollary 2) is new both in offering such a combinatorial interpretation and in deriving it by fundamentally combinatorial means.

Our main tool is the fact that incidence algebras are (co-commutative) Hopf algebras, thus equipped with an operator, the *antipode*, that serves as a universal inverse and powerful generalization of the Möbius function. One of us (Schmitt) showed in his recent doctoral thesis how, by computing this antipode operator, it is possible in principle to obtain inversion formulas for any operation reflected by convolution of multiplicative functions in an incidence algebra. For the case of the Lagrange inversion formulas, we combine his results with a theorem about what we call *filtrations* of trees and the aforementioned bijection between leaf-labelled trees and partitions.

2. The Faà di Bruno Hopf Algebra

In this section we introduce the necessary aspects of the theory of incidence Hopf algebras, particularly the *Faà di Bruno* algebra, which is the one appropriate to the operation of functional composition of exponential generating functions. For the sake of convenience and familiarity, we work over the complex numbers **C**, although in fact we require nothing more of

C than that it is a commutative **Q**-algebra (always, in this paper, with unit).

Let $\mathscr{H}$ be a family of finite partially ordered sets ("posets," in what follows) with $\mathbf{0}$ and $\mathbf{1}$. $\mathscr{H}$ is a *hereditary family* if for every $x \leqslant y \in P \in \mathscr{H}$, $\mathscr{H}$ contains the segment $[x, y]$, and for every $P, Q \in \mathscr{H}$, $\mathscr{H}$ contains the direct product $P \times Q$.

An *order-compatible equivalence relation* $\sim$ on a hereditary family $\mathscr{H}$ of posets is one with the properties that for all $P, Q, R \in \mathscr{H}$:

(i) whenever $P \sim Q$, there is an order-preserving bijection $\alpha_{P,Q}: P \to Q$ such that for all $x \leqslant y \in P$, $[x, y] \sim \alpha_{P,Q}([x, y])$, and

(ii) whenever $P \sim Q$, $P \times R \sim Q \times R$, and

(iii) if P has just one element, $P \times Q \sim Q$.

The equivalence classes of $\sim$ are referred to as *types*. Conditions (ii) and (iii) mean that the direct product induces the structure of commutative monoid on the set of types, with the type ι of the one-element posets as identity. We let $\tilde{\mathscr{H}}$ denote its monoid algebra over **C**. At many places, starting with the next paragraph, we shall silently confuse representatives of types with the types themselves.

Let A be a commutative algebra over **C**. Given a hereditary family $\mathscr{H}$ of finite posets with order-compatible equivalence relation $\sim$, the *reduced incidence algebra* $I(\mathscr{H}, A) = \operatorname{Hom}(\tilde{\mathscr{H}}, A)$ is the vector space of all A-valued functions on the set of types. $I(\mathscr{H}, A)$ is an algebra under the associative operation $*$, called *convolution*, defined by

$$f * g(P) = \sum_{x \in P} f([\mathbf{0}_P, x])\, g([x, \mathbf{1}_P]). \tag{1}$$

Evidently this operation is well-defined, and it is easily seen and well-known to be associative [4]. Its identity element is the function ε taking the value 1 on ι and 0 on all other types.

THEOREM 1 ([12]). *In the reduced incidence algebra $I(\mathscr{H}, \tilde{\mathscr{H}})$, there is a two-sided $*$-inverse S to the identity map $I: \tilde{\mathscr{H}} \to \tilde{\mathscr{H}}$. It is given by the formula*

$$S(P) = \sum_k \sum_{\substack{x_0 < \cdots < x_k \\ x_0 = \mathbf{0}_P \\ x_k = \mathbf{1}_P}} (-1)^k [x_0, x_1] \times [x_1, x_2] \times \cdots \times [x_{k-1}, x_k]. \tag{2}$$

The operator S is called the *antipode*. Theorem 1 follows from the fact that the algebra $\tilde{\mathscr{H}}$ with its "coalgebra" structure induced by the convolution $*$ is a commutative *Hopf algebra* [14]. Formula (2) is the specialization to the present situation of a general formula defining the antipode. The importance of the antipode is its role as a "universal" inverse. More precisely, $f \in I(\mathscr{H}, A) = \mathrm{Hom}(\tilde{\mathscr{H}}, A)$ is called *multiplicative* if it is an *algebra homomorphism*, i.e., if $f(\iota) = 1$ and $f(P \times Q) = f(P)\, f(Q)$ for all $P, Q \in \mathscr{H}$. For multiplicative functions we have:

LEMMA 1 ([14]). *If $f, g \in \mathrm{Hom}_{\mathrm{alg}}(\tilde{\mathscr{H}}, A)$ are multiplicative, then so is $f * g$.*

This lemma was actually known before the Hopf algebra structure of incidence algebras had been observed; cf. [4].

THEOREM 2 ([14]). *If $f \in \mathrm{Hom}_{\mathrm{alg}}(\tilde{\mathscr{H}}, A)$ is multiplicative then $f \circ S$ is its convolution inverse, i.e., $(f \circ S) * f = f * (f \circ S) = \iota$.*

This theorem follows easily from the property of S given in Theorem 1.

EXAMPLE 1. We illustrate the application of Theorems 1 and 2 and indicate how the antipode generalizes the Möbius function. The *zeta-function* in an incidence algebra $I(\mathscr{H}, A)$ is given by $\zeta(P) = 1$ for all $P \in \mathscr{H}$. It is plainly multiplicative. The Möbius function is, by definition, its convolution inverse, which by Theorems 1 and 2 is

$$\mu(P) = \sum_k \sum_{\substack{x_0 < \cdots < x_k \\ x_0 = \mathbf{0}_P \\ x_k = \mathbf{1}_P}} (-1)^k. \tag{3}$$

This formula for the Möbius function is known as Philip Hall's theorem [7].

The *Faà di Bruno Hopf algebra* is the reduced incidence algebra $I(\mathscr{F}, \mathbf{C})$ obtained by taking $\mathscr{F}$ to consist of all finite direct products $\Pi_{n_1} \times \cdots \times \Pi_{n_k}$ of finite partition lattices, with isomorphism for $\sim$. Because the partition lattices are directly indecomposable and finite connected posets have unique direct factorization into indecomposables, the types are in one-to-one correspondence with *type vectors* $\sigma = (\sigma_2, \sigma_3, ...)$ of non-negative integers, where σ_i is the number of direct factors Π_i occurring in a type representative. Thus if $[\pi, \rho] \in \Pi_n$ is a segment, then in its type vector, σ_i is the number of blocks of ρ that contain i blocks of π.

In the Faà di Bruno algebra, a multiplicative function f is determined by its (arbitrary) values on the indecomposable types Π_n $(n \geqslant 2)$. Therefore we may associate to f an exponential generating function, i.e., a formal power series

$$F_f(x) = x + \sum_{n=2}^{\infty} a_n \frac{x^n}{n!}, \tag{4}$$

where $a_n = f(\Pi_n)$ for $n \geqslant 2$.

THEOREM 3 ([4, Theorem 5.2]; cf. also Theorem 6, below). *The mapping $f \mapsto F_f$ defined above is an anti-isomorphism (i.e., an isomorphism onto the opposite group) from the group* $\mathrm{Hom}_{\mathrm{alg}}(\mathscr{F}, \mathbf{C}) \subseteq I(\mathscr{F}, \mathbf{C})$ *of multiplicative functions, under convolution, to the group of one-variable formal power series over* $\mathbf{C}$ *with leading term x, under functional composition.*

Following (roughly) Roman and Rota [11] we shall call a formal power series with leading term x a *delta series.*

3. LAGRANGE INVERSION IN ONE VARIABLE

The *Lagrange inversion formula*, expressed in its most explicit and combinatorial form [3], states that if $f = \Sigma_n a_n x^n/n!$ and $g = \Sigma_n b_n x^n/n!$ are delta series inverse to one another under functional composition, i.e., if $f \circ g(x) = g \circ f(x) = x$, then the coefficients of g are given in terms of those of f by

$$b_n = \sum_{k=1}^{n-1} (-1)^k B_{n+k-1,k}(0, a_2, a_3, \ldots), \tag{5}$$

where $B_{m,k}(x_1, x_2, \ldots)$ are the *exponential Bell polynomials.* Their definition is

$$B_{m,k} = \sum_{\pi \in \Pi(m,k)} x_1^{\pi_1} x_2^{\pi_2} \cdots, \tag{6}$$

where $\Pi(m, k)$ is the set of partitions of $\{1, \ldots, m\}$ into k blocks, and for any such partition π, π_i is the number of blocks of size i. Note that since any invertible power series may be made into a delta series by composition with a linear function, the restriction to delta series involves no essential loss of generality.

Letting X_n stand for the type of Π_n, regarded as an element of $\mathscr{F}$, the Lagrange inversion formula (5) follows, in view of Theorems 2 and 3, from the following fact about the antipode $S_{\mathscr{F}}$ in the Faà d Bruno algebra:

THEOREM 4.

$$S_{\mathscr{F}}(X_n) = \sum_{k=1}^{n-1} (-1)^k B_{n+k-1,k}(0, X_2, X_3, \ldots). \tag{7}$$

The remainder of this section is devoted to a derivation of Theorem 4 from the formula (2) given by Theorem 1 for the antipode.

The formula (2) applied to X_n, i.e., to $P = \Pi_n$, involves a sum over all strict chains from $\mathbf{0}$ to $\mathbf{1}$ of partitions of $\{1, \ldots, n\}$. Now any such chain induces a tree structure with leaves corresponding to the elements $1, \ldots, n$. We will show that for each such tree, there is complete cancellation of all but one of the terms in (2) representing chains that induce that tree. In this way, we reduce (2) to a sum over trees that is termwise equivalent to (7).

DEFINITION 1. A *tree* is a finite poset T with $\mathbf{1}$ in which every element but $\mathbf{1}$ has a unique cover. The *degree* of an element of T is the number of elements it covers. The *leaves* of T are its minimal elements. A *labelling* (resp. *leaf-labelling*) of T is a bijective function from the elements (resp. leaves) of T to a set $\{1, \ldots, n\}$.

DEFINITION 2. Let $C = (\mathbf{0} = x_0 < x_1 < \cdots < x_k = \mathbf{1})$ be a chain in a partition lattice Π_n. The *tree associated with* C, $T(C)$, is the poset of all subsets of $\{1, \ldots, n\}$ that occur as blocks of the partitions $x_i \in C$, ordered by inclusion.

Note that $T(C)$ really is a tree because every two of the blocks comprising it are either comparable or disjoint, and that the leaves of $T(C)$ are the singletons $\{1\}, \{2\}, \ldots, \{n\}$, giving $T(C)$ a natural leaf-labelling.

LEMMA 2. *For a chain $C = (\mathbf{0} = x_0 < x_1 < \cdots < x_k = \mathbf{1})$ in Π_n, the type of $[x_0, x_1] \times [x_1, x_2] \times \cdots \times [x_{k-1}, x_k]$ depends only on $T(C)$, and in fact its type vector σ is given by $\sigma_i =$ (number of elements of degree i in $T(C)$), for all $i \geqslant 2$.*

Proof. Each $[x_{j-1}, x_j]$ contributes a factor Π_d to the type in question for each block of x_j that is the union of d blocks of x_{j-1}. But each such block with $d > 1$ is an element of degree d in $T(C)$. Blocks with $d = 1$ need not be counted since they contribute only a factor $\Pi_1 = \iota$ to the type. ∎

DEFINITION 3. Let P be a finite poset. A *filtration* of P is a chain $G = (I_0 \subset I_1 \subset \cdots \subset I_k)$ of (lower) order-ideals of P such that: (i) $I_0 = \varnothing$, (ii) $I_k = P$, and (iii) for all $1 \leqslant j \leqslant k$, $I_j \backslash I_{j-1}$ is an antichain. The number k is the *length* of the filtration, denoted $l(G)$.

LEMMA 3. *For each leaf-labelled tree T with n leaves and no elements of degree 1, there is a length-preserving bijection α from the set of chains $C = (\mathbf{0} = x_0 < x_1 < \cdots < x_k = \mathbf{1})$ in Π_n having associated tree $T(C) \cong T$, to the set $\mathscr{G}(T')$ of all filtrations of the tree T' obtained by deleting the leaves of T.*

Proof. Let T, C obeying the hypotheses be given. Since the label-preserving isomorphism $T \cong T(C)$ is necessarily unique, we identify T with $T(C)$. Define the filtration $\alpha(C) = (I_0 \subset I_1 \subset \cdots \subset I_k)$ of T' by taking I_j to be the order-ideal whose maximal elements are the non-singleton blocks of x_j, for all $1 \leqslant j \leqslant k$, and $I_0 = \varnothing$. The I_j are strictly increasing because the x_j are. Conditions (i) and (ii) in the definition of filtration hold automatically, and (iii) holds because blocks properly contained in the blocks of x_j are blocks of x_i for some $i < j$, which means $I_j \backslash I_{j-1}$ is contained in the antichain of maximal elements of I_j.

On the other hand, given T and a filtration $G = (I_0 \subset I_1 \subset \cdots \subset I_k)$ of T', define a chain $\beta(G) = (\mathbf{0} = x_0 < x_1 < \cdots < x_k = \mathbf{1})$ by taking x_j to be the partition of $\{1, ..., n\}$ whose non-singleton blocks are the sets of labels on leaves below each maximal element of I_j, for all $1 \leqslant j \leqslant k$, and $x_0 = \mathbf{0}$. $x_k = \mathbf{1}$ because $I_k = T$, and the x_j are strictly increasing because the I_j are, and because the tree T has no elements of degree 1, which means that the sets of leaves below distinct elements are distinct.

It is clear from their construction that, for any given T, the maps α and β are inverse to one another and hence are bijections. It is also clear that they preserve length. ∎

LEMMA 4. *Let P be a finite poset, and let $\mathscr{G}(P)$ be the set of all filtrations of P. Then*

$$\sum_{G \in \mathscr{G}(P)} (-1)^{l(G)} = (-1)^{|P|}. \tag{8}$$

Proof. Let $J(P)$ be the distributive lattice of order-ideals of P, ordered by inclusion. Let μ and ζ denote the Möbius and zeta-function, respectively, in the reduced incidence algebra $I(\mathscr{D}, \mathbf{C})$ associated with the hereditary class $\mathscr{D}$ of finite distributive lattices, with isomorphism for $\sim$. It is well-known (cf. [13, Example 3.9.6]) that this Möbius function is

$$\mu(L) = \begin{cases} (-1)^n & \text{if } L \text{ is a Boolean algebra of rank } n \\ 0 & \text{otherwise.} \end{cases}$$

Since a segment $[I_1, I_2] \subseteq J(P)$ is a Boolean algebra of rank n if and only if $I_2 \backslash I_1$ is an antichain of n elements, it follows that

$$\mu \circ S(J(P)) = \sum_k \sum_{\substack{I_0 \subset \cdots \subset I_k \\ I_0 = \varnothing \\ I_k = P}} (-1)^k \mu([I_0, I_1])\mu([I_1, I_2]) \cdots \mu([I_{k-1}, I_k])$$

$$= \sum_{G \in \mathscr{G}(P)} (-1)^{l(G)}(-1)^{|P|}.$$

By Theorem 2, this is equal to $\zeta(J(P)) = 1$, which proves (8). ∎

COROLLARY 1. *For the Faà di Bruno Hopf algebra, the antipode is given by*

$$S_{\mathscr{F}}(X_n) = \sum_T (-1)^{|T| - n} \Omega(T), \tag{9}$$

where the sum ranges over trees with n labelled leaves and no elements of degree 1, *and $\Omega(T)$ is the type $X_2^{\sigma_2} X_3^{\sigma_3} \cdots$ with type vector $(\sigma_2, \sigma_3, ...)$, where σ_i is the number of elements of degree i in T, for all $i \geqslant 2$.*

Proof. Immediate from (2) and Lemmas 2, 3, and 4. ∎

Formula (9) is the new description of the Lagrange inversion formula promised in the Introduction. It is formally similar to (7) and the equivalent (5), except that it is a sum of "weight" monomials corresponding to trees, rather than to partitions. To prove Theorem 4, we now show that there is in fact a weight-preserving bijection between leaf-labelled trees and partitions which establishes the equality of the sums (9) and (7) term by term.

DEFINITION 4. Let T be a tree. A labelling λ of T is *utterly increasing* if:

(i) for all $v \leqslant w \in T$, $\lambda(v) \leqslant \lambda(w)$, and

(ii) for all non-leaves $v, w \in T$, $\lambda(v) \leqslant \lambda(w)$ if and only if $\max\{\lambda(v') \mid v' \in c(v)\} \leqslant \max\{\lambda(w') \mid w' \in c(w)\}$, where $c(x)$ denotes the set of elements covered by x in T.

LEMMA 5. *Let T be a tree with leaf-labelling λ. There is a unique utterly increasing labelling λ^* of T extending λ.*

Proof. Say T has n leaves and $n+k$ elements altogether. Construct λ^* as follows: for $1 \leqslant i \leqslant n$, let λ^* agree with λ. Then assign labels $n+1$ through $n+k$ inductively by giving label i to that element x for which $\max\{\lambda^*(y) \mid y \in c(x)\}$ is least, among all unlabelled elements which cover only previously labelled elements. Plainly λ^* constructed this way is utterly increasing.

Suppose λ^{**} is another utterly increasing labelling that agrees on leaves with λ and λ^*. Let v, w be two non-leaves of T and let I be the order-ideal they generate. We assume by induction that for every order-ideal $J \subset I$, the ordering of the λ^*-labels of its elements is the same as that of its λ^{**}-labels. Then condition (ii) for an utterly increasing labelling implies that $\lambda^*(v) < \lambda^*(w)$ if and only if $\lambda^{**}(v) < \lambda^{**}(w)$. This shows that the orderings of λ^*-labels and λ^{**}-labels are the same on every order-ideal I, including $I = T$, which implies $\lambda^* = \lambda^{**}$. ▮

THEOREM 5. *There exists a weight-preserving bijection ϕ from the set $T(n, k)$ of (non-isomorphic) leaf-labelled trees with n leaves and $n+k$ elements total, to the set $\Pi(n+k-1, k)$ of partitions of $\{1, ..., n+k-1\}$ into k parts. Here the weight of a tree T is the monomial $w(T) = x_1^{\sigma_1} x_2^{\sigma_2} \cdots$, where σ_i is the number of elements of degree i in T. The weight of a partition π is the monomial $w(\pi) = x_1^{\pi_1} x_2^{\pi_2} \cdots$, where π_i is the number of blocks of π with i elements, just as in* (6).

Proof. Let $T \in T(n, k)$ and let λ^* be the unique utterly increasing labelling of T that extends the leaf-labelling. Define $\phi(T)$ to be the partition of $\{1, ..., n+k-1\}$ whose blocks are the sets of labels $\{\lambda^*(y) \mid y \in c(x)\}$ as x ranges over the non-leaf elements of T. Clearly $\phi(T) \in \Pi(n+k-1, k)$ and $w(\phi(T)) = w(T)$.

We show that ϕ is a bijection by constructing its inverse ψ. For this, let $\pi \in \Pi(n+k-1, k)$. Let the blocks of π be $B_1, ..., B_k$ and assume that they are numbered so that $m_1 > \cdots > m_k$, where $m_i = \max B_i$. We define a sequence $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k$ of trees with underlying sets $\{x \in T_j\} = \{u\} \cup \bigcup_{i=1}^{j} B_i$, where $u = n+k$ serves as the root $\mathbf{1}$ in each tree: let $T_0 = \{u\}$ and build T_j from T_{j-1} for each $1 \leqslant j \leqslant k$ by adding the elements of B_j as new leaves less than the leaf l of T_{j-1} which is greatest among all leaves of T_{j-1}. Let $\psi(\pi) = T_k$, labelled by the identity map.

One easily verifies by induction on the sequence $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k$ the following facts for each T_j. First, T_j contains $\{n+k-j, ..., n+k\}$. Second, $n+k-j$ is the greatest leaf of T_j. Hence, third, the non-leaves of T_j are exactly the set $\{n+k-j+1, ..., n+k\}$. Finally, fourth, the elements of T_j satisfy conditions (i) and (ii) for an utterly increasing labelling. Thus, in particular, $\psi(\pi) = T_k$ has leaf-labels $\{1, ..., n\}$ and its full labelling is utterly increasing. From this it is easy to see that ϕ and ψ inverse to one another, hence bijections. ▮

Proof of Theorem 4. By Theorem 5 and Corollary 1, $S_{\mathscr{F}}(X_n)$ is the sum over all partitions $\pi \in \Pi(n+k-1, k)$ without one-element blocks of $(-1)^k w(\pi)$, for all possible k. The ban on one-element blocks restricts k to the range $1, ..., n-1$, yielding (7). ▮

4. Lagrange Inversion in Several Variables

In this section, we generalize the results (and the methods) of the previous section to obtain a formula analogous to (9) for computing the functional-composition inverse of a system of N exponential generating functions in N variables.

Let N be fixed throughout the following discussion.

A *multicolored set* is a finite set X together with a map $\theta: X \to \{1, ..., N\}$. The value $\theta(x)$ is called the *color* of the element $x \in X$. The *size* $|X|$ of a multicolored set X is the vector $(|X_1|, ..., |X_N|)$, where $X_r = \{x \in X \mid \theta(x) = r\}$.

A *multicolored partition* of a multicolored set X is a partition π of X whose set of blocks (also denoted by π) is multicolored, subject to the condition that for each singleton block $\{x\} \in \pi$, $\theta(\{x\}) = \theta(x)$. Each block of π is of course itself a multicolored set, as a subset of X.

The multicolored partitions of a multicolored set X with $|X| = (n_1, ..., n_N) = \mathbf{n}$ form a poset $\Pi_{\mathbf{n}}$, where we define $\pi \leqslant \rho$ if $\pi \leqslant \rho$ as partitions, and for each block B of π that is also a block of ρ, $\theta_\pi(B) = \theta_\rho(B)$. Because of this restriction, ρ induces a multicolored partition $\rho \mid \pi$ of the blocks of π. The poset $\Pi_{\mathbf{n}}$ has a $\mathbf{0}$ but for $|\mathbf{n}| = \sum n_i > 1$, lacks a $\mathbf{1}$. Rather it has N maximal elements, corresponding to the partition $\mathbf{1}_X$ taken with each possible color for its unique block. The posets we actually wish to work with are those obtained by deleting all but one of these maximal elements. We denote them by $\Pi^r_{\mathbf{n}}$, where r is the color corresponding to the maximal element not deleted.

Let $\pi \leqslant \rho$ be multicolored partitions. From the above definitions it is easily seen that the segment $[\pi, \rho]$ is isomorphic to

$$\prod_{B \in \rho \mid \pi} \Pi^{\theta(B)}_{|B|}. \tag{10}$$

Let $\mathscr{F}^N$ be the hereditary class of segments of multicolored partition posets $\Pi^r_{\mathbf{n}}$; it really is a hereditary class by (10). We take $\sim$ to be the relation of color-isomorphism: $[\pi, \rho] \sim [\pi', \rho']$ when $\rho \mid \pi$ and $\rho' \mid \pi'$, restriced to their non-singleton blocks, are alike as multicolored partitions of multicolored sets. It is easy to see that $\sim$ is order-compatible.

The *multicolored Faà di Bruno Hopf algebra* is the incidence algebra $I(\bar{\mathscr{F}}^N, \mathbf{C})$. As with the one-variable Faà di Bruno algebra, the types form a free commutative monoid. In the present case, the generators are the types of the $\Pi^r_{\mathbf{n}}$ for $1 \leqslant r \leqslant N$ and $|\mathbf{n}| > 1$; a multiplicative function f is determined by its (arbitrary) values on these types. Accordingly, we associate to f a vector of N exponential generating functions in N variables:

$$F_f^r(x_1, ..., x_N) = x_r + \sum_{|\mathbf{n}|>1} a_{\mathbf{n}}^r \frac{x^{\mathbf{n}}}{\mathbf{n}!},$$

where $a_{\mathbf{n}}^r = f(\Pi_{\mathbf{n}}^r)$, $x^{\mathbf{n}} = x_1^{n_1} \cdots x_N^{n_N}$, and $\mathbf{n}! = n_1! \cdots n_N!$.

THEOREM 6. *The mapping* $f \mapsto (F_f^1, ..., F_f^N)$ *defined above is an anti-isomorphism from the group* $\mathrm{Hom}_{\mathrm{alg}}(\mathscr{F}^N, \mathbf{C}) \subseteq I(\mathscr{F}^N, \mathbf{C})$ *of multiplicative functions, under convolution, to the group of systems of formal power series in N variables over* $\mathbf{C}$ *with zero constant term and identity linear term, under functional composition.*

Proof. Let $f, g \in \mathrm{Hom}_{\mathrm{alg}}(\mathscr{F}^N, \mathbf{C})$ satisfy $f(\Pi_{\mathbf{n}}^r) = a_{\mathbf{n}}^r$ and $g(\Pi_{\mathbf{n}}^r) = b_{\mathbf{n}}^r$ for $|\mathbf{n}| > 1$. For $\mathbf{n} = (0, ..., 1, ..., 0)$ with 1 in the jth position and 0 elsewhere, define $a_{\mathbf{n}}^r = b_{\mathbf{n}}^r = 1$, if $j = r$, and 0 otherwise. Thus we can write

$$F_f^r(x_1, ..., x_N) = \sum_{|\mathbf{n}| \geqslant 1} a_{\mathbf{n}}^r \frac{x^{\mathbf{n}}}{\mathbf{n}!},$$

and similarly for $F_g^r(x_1, ..., x_N)$.

The mapping referred to in the theorem is obviously bijective, so we need only to show $F_g^r \circ (F_f^1, ..., F_f^N) = F_{f*g}^r$. By definition of functional composition we have

$$F_g^r \circ (F_f^1, ..., F_f^N) = \sum_{|\mathbf{n}| \geqslant 1} \frac{b_{\mathbf{n}}^r}{\mathbf{n}!} (F_f^1)^{n_1} \cdots (F_f^N)^{n_N}.$$

The coefficient of $x^{\mathbf{d}}/\mathbf{d}!$ in the above expansion is

$$\sum_{|\mathbf{n}| \leqslant |\mathbf{d}|} \sum_{\substack{\mathbf{k}_1 + \cdots + \mathbf{k}_{|\mathbf{n}|} = \mathbf{d} \\ |\mathbf{k}_i| \geqslant 1 \, \forall i}} \frac{b_{\mathbf{n}}^r \mathbf{d}! \, a_{\mathbf{k}_1}^1 a_{\mathbf{k}_2}^1 \cdots a_{\mathbf{k}_{n_1}} a_{\mathbf{k}_{n_1+1}}^2 \cdots a_{\mathbf{k}_{n_2}}^2 \cdots a_{\mathbf{k}_{n_1+\cdots n_{N-1}}}^N \cdots a_{\mathbf{k}_{|\mathbf{n}|}}^N}{\mathbf{n}! \, \mathbf{k}_1! \cdots \mathbf{k}_{|\mathbf{n}|}!}. \tag{11}$$

On the other hand, the coefficient of $x^{\mathbf{d}}/\mathbf{d}!$ in $F_{f*g}^r(x_1, ..., x_N)$ is given by

$$f * g(\Pi_{\mathbf{d}}^r) = \sum_{\pi \in \Pi_{\mathbf{d}}^r} b_{|\pi|}^r \prod_{B \in \pi} a_{|B|}^{\theta(B)} \tag{12}$$

$$= \sum_{|\mathbf{n}| \leqslant |\mathbf{d}|} \sum_{(B_1, ..., B_{|\mathbf{n}|})} \frac{b_{\mathbf{n}}^r}{\mathbf{n}!} a_{|B_1|}^1 \cdots a_{|B_{n_1}|}^1 a_{|B_{n_1+1}|}^2 \cdots a_{|B_{n_2}|}^2$$

$$\cdots a_{|B_{n_1+\cdots+n_{N-1}+1}|}^N \cdots a_{|B_{|\mathbf{n}|}|}^N, \tag{13}$$

where the inner sum ranges over all *ordered* partitions $(B_1, ..., B_{|\mathbf{n}|})$ of a multicolored set X of size $\mathbf{d}$. Now there are exactly $\mathbf{d}!/(\mathbf{b}_1! \cdots \mathbf{b}_{|\mathbf{n}|}!)$ ordered partitions $(B_1, ..., B_{|\mathbf{n}|})$ of X having $|B_i| = \mathbf{b}_i$ for $1 \leqslant i \leqslant |\mathbf{n}|$; hence the last expression above is equal to (11). ∎

For the remainder of this section, we follow a development for the N-variate case parallel to that in Section 3 for one variable.

Let $X^r_{\mathbf{n}}$ stand for the type of $\Pi^r_{\mathbf{n}}$, regarded as an element of $\mathscr{F}^N$. For each $\mathbf{n}$ we fix a "standard" multicoloring of $\{1, ..., |\mathbf{n}|\}$. Exactly how we choose to do this is immaterial, but for definiteness let us agree to make $\theta(1) = \cdots = \theta(n_1) = 1$, $\theta(n_1+1) = \cdots = \theta(n_2) = 2$, and so forth. We aim to prove:

THEOREM 7. *For the multicolored Faà di Bruno Hopf algebra, the antipode is given by*

$$S_{\mathscr{F}^N}(X^r_{\mathbf{n}}) = \sum_T (-1)^{|T|-|\mathbf{n}|}\, \Omega(T), \tag{14}$$

where the sum ranges over multicolored trees (i.e., trees with a multicolored set of elements) having $|\mathbf{n}|$ labelled leaves colored according to the standard $\mathbf{n}$-coloring, no elements of degree 1, *and $\theta(\mathbf{1}_T) = r$. $\Omega(T)$ is the type*

$$\prod_{\substack{v \in T\\ \text{non-leaf}}} X^{\theta(v)}_{|c(v)|}.$$

Note that Theorem 7 reduces to Corollary 1 in the case $N = 1$, and that by virtue of Theorem 6, Theorem 7 is a full-fledged inversion formula for systems of N exponential formal power series in N variables, which we state explicitly as

COROLLARY 2 (N-variable Lagrange inversion). *Let $F = (f_1, ..., f_N)$ and $G = (g_1, ..., g_N)$ be systems of N formal power series in N variables having zero constant term and identity linear term which are mutually inverse under functional composition: $f_r(g_1, ..., g_N) = g_r(f_1, ..., f_N) = x_r$ for $1 \leqslant r \leqslant N$. Writing $f_r = \Sigma_{\mathbf{n}} a^r_{\mathbf{n}} x^{\mathbf{n}}/\mathbf{n}!$ and $g_r = \Sigma_{\mathbf{n}} b^r_{\mathbf{n}} x^{\mathbf{n}}/\mathbf{n}!$, one gives the coefficients of G in terms of those of F by*

$$b^r_{\mathbf{n}} = \sum_T (-1)^{|T|-|\mathbf{n}|} F(T),$$

where the sum ranges over the set of multicolored trees specified in Theorem 7, *and for each tree T,*

$$F(T) = \prod_{\substack{v \in T\\ \text{non-leaf}}} a^{\theta(v)}_{|c(v)|}.$$

DEFINITION 5. Let $C = (\mathbf{0} = x_0 < x_1 < \cdots < x_k = \mathbf{1})$ be a chain in a multicolored partition poset $\Pi^r_{\mathbf{n}}$. Since a multicolored partition is a partition, the x_i form a chain in $\Pi_{|\mathbf{n}|}$, which has an associated tree $T(C)$ as per Definition 2. By the definition of $<$ in $\Pi^r_{\mathbf{n}}$, blocks common to more than one of the x_i have the same color in all of them; thus $T(C)$ is a multicolored tree, the *multicolored tree associated with* C.

LEMMA 6. *For a chain* $C = (\mathbf{0} = x_0 < x_1 < \cdots < x_k = \mathbf{1})$ *in* $\Pi^r_{\mathbf{n}}$, *the type of* $[x_0, x_1] \times [x_1, x_2] \times \cdots \times [x_{k-1}, x_k]$ *is* $\Omega(T(C))$.

Proof. Essentially, the proof of Lemma 2, but with the obvious modifications to take the coloring into account. ∎

LEMMA 7. *For each multicolored leaf-labelled tree* T *with* $|\mathbf{n}|$ *leaves having the standard* $\mathbf{n}$*-coloring, no elements of degree* 1, *and* $\theta(\mathbf{1}_T) = r$, *there is a length-preserving bijection* α *from the set of chains* $C = (\mathbf{0} = x_0 < x_1 < \cdots < x_k = \mathbf{1})$ *in* $\Pi^r_{\mathbf{n}}$ *having associated multicolored tree* $T(C) = T$, *to the set* $\mathscr{G}(T')$ *of all filtrations of the tree* T' *obtained by deleting the leaves of* T.

Proof. Regard $\Pi^r_{\mathbf{n}}$ as a poset of multicolored partitions of $\{1, ..., |\mathbf{n}|\}$ with the standard $\mathbf{n}$-coloring.

The chains C having $T(C) = T$, regarded as chains in $\Pi_{|\mathbf{n}|}$, consist of exactly one instance of each chain whose (non-multicolored) associated tree is the underlying non-multicolored tree of T. This is so because each such chain in $\Pi_{|\mathbf{n}|}$ can clearly have the non-singleton blocks of its constituent partitions colored in precisely one way to agree with the coloring of T. (The singleton blocks already agree with T since they and the leaves of T both have the standard coloring.)

Now apply Lemma 3. ∎

Proof of Theorem 7. Immediate from (2) and Lemmas 6, 7, and 4. ∎

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