NOTE

A NON-LINEAR ALGEBRAIC MATROID WITH INFINITE CHARACTERISTIC SET

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The algebraic characteristic set $X_{\text{alg}}(M)$ of a matroid $M$ is the set of characteristics of all fields $F$ over which $M$ has an algebraic representation. The characteristic set $X_{\text{lin}}(M)$ for vector representations is defined similarly. P. Vamos has observed that if $X_{\text{lin}}(M)$ is infinite, then $0 \in X_{\text{lin}}(M)$. We show by an example that the corresponding results for algebraic representations does not hold.

An algebraic representation over a field $F$ of a matroid $M$ is a mapping $f$ from the elements of $M$ into a field $K$, which contains $F$ as a subfield, such that a subset $S$ of elements is independent in $M$ if and only if $f(S)$ is algebraically independent over $F$. The image of a non-loop element in $M$ is thus a transcendental over $F$.

A matroid $M$ may possess algebraic representations over fields of different characteristics. The set of all possible characteristics is denoted by $X_{\text{alg}}(M)$. A non-algebraic matroid has $X_{\text{alg}}(M) = \emptyset$, the zero set. The linear characteristic set $X_{\text{lin}}(M)$ is defined analogously with vector representations in place of algebraic representations. By [3, Theorem 11.2.1]: $X_{\text{lin}}(M) \subseteq X_{\text{alg}}(M)$.

P. Vamos proved [2, Corollary 2] that if $X_{\text{lin}}(M)$ is infinite then $0 \in X_{\text{lin}}(M)$. We show by an example that this result can not be extended to algebraic characteristic sets.

The example appeared in [1, Example 8]. Ingleton proved there that the matroid is non-commutative, i.e., $X_{\text{lin}}(M) = \emptyset$. It follows then that $0 \notin X_{\text{alg}}(M)$. For by [1, Corollary], if a matroid is algebraic over a field of characteristic 0 then it is linearly representable. I don't think it has been observed before that this matroid is algebraic over fields of prime characteristic.

Theorem. There is a non-linear matroid of order 8 and rank 4 with $X_{\text{alg}}(M)$ equal to the set of all prime numbers.

Proof. We will denote the elements of the matroid by $1, 2, \ldots, 8$. All subsets of 3 elements are independent. The following subsets with 4 elements are dependent: $\{1, 2, 5, 8\}, \{1, 2, 3, 6\}, \{1, 3, 4, 7\}, \{1, 5, 6, 7\}, \{2, 4, 5, 6\}, \{3, 4, 5, 8\}, \{3, 6, 7, 8\}$. The remaining 4-element subsets are bases of the matroid.
We have tried to depict the matroid in Fig. 1, which is admittedly not a true picture since the rank of the matroid is 4. The matroid is embedded in a matroid of rank 4 and order 15 such that the 7 dependent 4-sets are easily recognized. We show an algebraic representation over GF($p^2$), where $x$, $y$, $z$, $u$ are algebraically independent transcendentals over GF($p^2$) in Fig. 1.

We choose $\lambda \in \text{GF}(p^2) - \text{GF}(p)$, which implies that $\lambda^p \neq \lambda$. Note that \{2, 4, 7, 8\} is independent in $M$. Therefore we must show that $x^p + y + z^p + u$, $x^p + y$, $\lambda(x + y) + z + u$, $\lambda x + z$ are algebraically independent transcendentals over GF($p^2$). If we denote these numbers by $a$, $b$, $c$, $d$ respectively, we find that

$$\lambda^p b - d^p + a - b - c + d = (\lambda^p - \lambda)y.$$

Since $\lambda^p \neq \lambda$ it follows that $y \in \text{GF}(p^2)(a, b, c, d)$, and similarly for $x$, $z$, $u$. We conclude that $a$, $b$, $c$, $d$ are algebraically independent transcendentals over GF($p^2$) since this holds for $x$, $y$, $z$, $u$. The remaining independences can be verified similarly, but we leave these easy verifications for the reader. \qed

References