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Note

# Degree sequence and supereulerian graphs

Suohai Fan[a], Hong-Jian Lai[b,*], Yehong Shao[c], Taoye Zhang[d], Ju Zhou[b]

[a] Department of Mathematics, Jinan University Guangzhou 510632, PR China
[b] Department of Mathematics, West Virginia University, Morgantown, WV 26506, United States
[c] Arts and Science, Ohio University Southern, Ironton, OH 45638, United States
[d] Department of Mathematics, Penn State Worthington Scranton, Dunmore, PA 18512, United States

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**Abstract**

A sequence $d = (d_1, d_2, \ldots, d_n)$ is graphic if there is a simple graph $G$ with degree sequence $d$, and such a graph $G$ is called a realization of $d$. A graphic sequence $d$ is line-hamiltonian if $d$ has a realization $G$ such that $L(G)$ is hamiltonian, and is supereulerian if $d$ has a realization $G$ with a spanning eulerian subgraph. In this paper, it is proved that a nonincreasing graphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a supereulerian realization if and only if $d_n \geq 2$ and that $d$ is line-hamiltonian if and only if either $d_1 = n - 1$, or $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2}(d_j - 2)$.


*Keywords:* Degree sequence; Collapsible graphs; Hamiltonian line graphs; Supereulerian graphs

## 1. Introduction

We consider finite graphs in this note. Undefined terms can be found in [1]. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex $v \in V(G)$ is called a *pendent vertex* if $d(v) = 1$. Let $D_1(G)$ denote the set of all pendent vertices of $G$. An edge $e \in E(G)$ is called a *pendent edge* if one of its endpoints is a pendent vertex. If $v \in V(G)$, then $N_G(v) = \{u : uv \in E(G)\}$; and if $T \subseteq V(G)$, then $N_G(T) = \{u \in V(G) \setminus T : uv \in E(G) \text{ and } v \in T\}$. When the graph $G$ is understood in the context, we may drop the subscript $G$.

A *circuit* is a connected 2-regular graph. A *cycle* is a graph such that the degree of each vertex is even. A cycle $C$ of $G$ is a *spanning eulerian subgraph* of $G$ if $C$ is connected and spanning. A graph $G$ is *supereulerian* if $G$ contains a spanning eulerian subgraph.

If $G$ has vertices $v_1, v_2, \ldots, v_n$, the sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a *degree sequence* of $G$. A sequence $d = (d_1, d_2, \ldots, d_n)$ is *nonincreasing* if $d_1 \geq d_2 \geq \cdots \geq d_n$. A sequence $d = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a simple graph $G$ with degree sequence $d$. Furthermore, such a simple graph $G$ is called a *realization* of $d$. Let $\mathcal{G}$ denote the set of all graphic degree sequences. A sequence $d \in \mathcal{G}$ is *line-hamiltonian* if $d$ has a realization $G$ such that $L(G)$ is hamiltonian.

* Corresponding author.
*E-mail address:* hjlai@math.wvu.edu (H.-J. Lai).

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The sequence $S$ is called a bipartite graphic sequence if there is a bipartite graph $G$ with bipartition $\{X, Y\}$ such that $\{d(x_1), \ldots, d(x_m)\} = \{s_1, \ldots, s_m\}$, and $\{d(y_1), \ldots, d(y_n)\} = \{t_1, \ldots, t_n\}$ where $X = \{x_1, \ldots, x_m\}$ and $Y = y_1, \ldots, y_n$ and $d(v)$ is the degree of a vertex $v$; the graph $G$ is called a realization of $S$. In [9], Luo et al. proved the following theorem.

**Theorem 1.1** (*Luo, Zang, and Zhang [9]*). *Every bipartite graphic sequence with the minimum degree $\delta \geq 2$ has a realization that admits a nowhere-zero 4-flow.*

In this paper, the following result is obtained.

**Theorem 1.2.** *Let $d = (d_1, d_2, \ldots, d_n) \in \mathcal{G}$ be a nonincreasing sequence. Then $d$ has a supereulerian realization if and only if either $n = 1$ and $d_1 = 0$, or $n \geq 3$ and $d_n \geq 2$.*

In [7], Jaeger proved the following result.

**Theorem 1.3** (*Jaeger [7]*). *Every supereulerian graph admits a nowhere-zero 4-flow.*

Theorem 1.2, together with 1.3, implies a result analogous to Theorem 1.1.

**Theorem 1.4** (*Luo, Zang, and Zhang [9]*). *Let $d = (d_1, d_2, \ldots, d_n) \in \mathcal{G}$ be a nonincreasing sequence. Then $d$ has a realization that admits a nowhere-zero 4-flow if and only if $d_n \geq 2$.*

The following characterization on line-hamiltonian graphic sequences is also obtained.

**Theorem 1.5.** *Let $d = (d_1, d_2, \ldots, d_n) \in \mathcal{G}$ be a nonincreasing sequence with $n \geq 3$. The following are equivalent.*

(i) *$d$ is line-hamiltonian.*
(ii) *either $d_1 = n - 1$, or*

$$\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2} (d_j - 2). \qquad (1)$$

(iii) *$d$ has a realization $G$ such that $G - D_1(G)$ is supereulerian.*

## 2. Collapsible sequences

Let $X \subseteq E(G)$. The *contraction* $G/X$ is the graph obtained from $G$ by identifying the endpoints of each edge in $X$ and then deleting the resulting loops. Note that multiple edges may arise.

Let $O(G)$ denote the set of vertices of odd degree in $G$. A graph $G$ is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, $G$ has a connected spanning subgraph $H_R$ such that $O(H_R) = R$. A sequence $d = (d_1, d_2, \ldots, d_n)$ is *collapsible* if $d$ has a simple collapsible realization.

**Theorem 2.1.** *Let $G$ be a connected graph. Each of the following holds.*

(i) *(Catlin, Corollary of Lemma 3, [2]) If $H$ is a collapsible subgraph of $G$, then $G$ is collapsible if and only if $G/H$ is collapsible.*
(ii) *(Catlin, Corollary 1, [2]) If $G$ contains a spanning tree $T$ such that each edge of $T$ is contained in a collapsible subgraph of $G$, then $G$ is collapsible.*
(iii) *(Caltin, Theorem 7, [2]) $C_2$, $K_3$ (circuits of 2 or 3 edges) are collapsible.*
(iv) *(Caltin, Theorem 2, [2]) If $G$ is collapsible, then $G$ is supereulerian.*

Theorem 2.1(ii) and (iii) imply Corollary 2.2(i); Theorem 2.1(i) and (iii) imply Corollary 2.2(ii).

**Corollary 2.2.** (i) *If every edge of a spanning tree of $G$ lies in a $K_3$, then $G$ is collapsible.*
(ii) *If $G - v$ is collapsible and if $v$ has degree at least 2 in $G$, then $G$ is collapsible.*

**Corollary 2.3.** *If $d = (d_1, d_2, \ldots, d_n)$ is a nonincreasing graphic sequence with $d_1 = n - 1$ and $d_n \geq 2$, then every realization of $d$ is collapsible.*

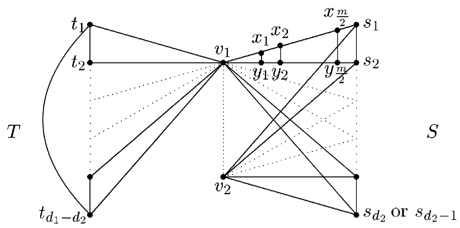


Fig. 1. $G$.

**Proof.** Let $G$ be a realization of $d$ with $N(v_1) = \{v_2, \ldots, v_n\}$ and let $T$ be the spanning tree with $E(T) = \{v_1 v_k : 2 \le k \le n\}$. Since $d_n \ge 2$ and $N(v_1) = \{v_2, \ldots, v_n\}$, for any $v_i \in \{v_k : 2 \le k \le n\}$, there is $v_j \in \{v_k : 2 \le k \le n\} \setminus \{v_i\}$ such that $v_i v_j \in E(G)$. It follows that every edge of $T$ lies in a $K_3$, and so by Corollary 2.2(i), $G$ is collapsible. $\square$

**Lemma 2.4.** *If $d = (d_1, d_2, \ldots, d_n)$ is a nonincreasing graphic sequence with $d_3 = \cdots = d_n = 3$, then $d$ is collapsible.*

**Proof.** Let $v_1, v_2$ be two vertices and let

$$S = \begin{cases} \{s_1, s_2, \ldots, s_{d_2}\} & \text{if } d_2 \text{ is even} \\ \{s_1, s_2, \ldots, s_{d_2 - 1}\} & \text{if } d_2 \text{ is odd} \end{cases}$$

be a set of vertices other than $\{v_1, v_2\}$ and let $T = \{t_1, t_2, \ldots, t_{d_1 - d_2}\}$ be a set of $d_1 - d_2$ vertices not in $S \cup \{v_1, v_2\}$. Let $H$ denote the graph obtained from $\{v_1, v_2\} \cup S \cup T$ by joining $v_2$ to each vertex of $S$ and joining $v_1$ to each vertex of $S \cup T$ (if $d_2$ is odd, then we also join $v_1$ and $v_2$). Note that $d_H(v_1) = d_2 + d_1 - d_2 = d_1$, $d_H(v_2) = d_2$, $d_H(s) = 2$ for $s \in S$ and $d_H(t) = 1$ for $t \in T$.

*Case* 1 $d_1 - d_2 \ge 3$. Let $C = t_1 t_2 \cdots t_{d_2 - d_1} t_1$ be a circuit passing through all vertices of $T$ and let $H' = H \cup E(C)$. As $|S|$ is even, we join all vertices of $S$ in pairs (i.e., $s_1 s_2, s_3 s_4, \ldots$) in $H'$ and denote the resulting graph by $H''$. Note that $d_{H''}(v_1) = d_1$, $d_{H''}(v_2) = d_2$ and $d_{H''}(v) = 3$ for $v \in S \cup T$.

Also note that

$$|V(H'')| = \begin{cases} 2 + d_1 & \text{if } d_2 \text{ is even} \\ 1 + d_1 & \text{if } d_2 \text{ is odd.} \end{cases}$$

Let $m = n - |V(H'')|$. Then

$$m = \begin{cases} n - (2 + d_1) & \text{if } d_2 \text{ is even} \\ n - (1 + d_1) & \text{if } d_2 \text{ is odd} \end{cases}$$

is even as $n$ and $d_1$ have the same parity if $d_2$ is even while $n$ and $d_1$ have different parity if $d_2$ is odd. By the construction of $H''$, $H''$ contains a triangle $v_1 s_1 s_2$. We subdivide $v_1 s_1$ and $v_1 s_2$ $\frac{m}{2}$ times, respectively, and let $x_1, x_2, \ldots, x_{\frac{m}{2}}$ and $y_1, y_2, \ldots, y_{\frac{m}{2}}$ be the new vertices resulted in subdividing $v_1 s_1$ and $v_1 s_2$, respectively. Then for $1 \le j \le \frac{m}{2}$, we join $x_j y_j$ and denote the resulting graph by $G$ (see Fig. 1). Hence, by the construction of $G$, $G$ is a realization of $d$.

*Case* 2 $d_1 - d_2 = 2$. Let $G$ be the construction as in Case 1 except that we join $t_1$ to $s_1$, $t_1$ to $t_2$, $t_2$ to $s_2$, and delete $s_1 s_2$.

*Case* 3 $d_1 - d_2 = 1$. Let $G$ be the construction as in Case 1 except that we join $t_1$ to both $s_1$ and $s_2$, and delete $s_1 s_2$.

By Theorem 2.1(iii), $K_3$ is collapsible. If we contract $v_1 x_1 y_1$, then we get a triangle $v_1 x_2 y_2$ in the contraction, and if we contract $v_1 x_2 y_2$, then we get a triangle $v_1 x_3 y_3$ in the contraction. Repeat this process by contracting a triangle $v_1 x_i y_i$ for each $i$ with $1 \le i \le \frac{m}{2}$ in the subsequent contraction. In Case 2 and Case 3, this process results in a graph in which each edge lies in a triangle. In Case 1, this process eventually results in a triangle $v_1 s_1 s_2$. After contracting $v_1 t_1 t_2$ we obtained a graph in which each edge lies in a triangle. Since 2-circuit is collapsible, the contraction of a maximally collapsible graph will result in a simple graph. By Corollary 2.2(i) and (ii), $G$ is collapsible in each case. $\square$

**Theorem 2.5** (*Havel [6], Hakimi [4]*). *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence. Then $d$ is graphic if and only if $d' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphic.*

**Theorem 2.6** (*Kleitman and Wang [8]*). *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence. Then d is graphic if and only if $d' = (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n)$ is graphic.*

**Lemma 2.7.** *If $d = (d_1, d_2, \ldots, d_n)$ is a nonincreasing sequence with $n \geq 4$ and $d_n = 3$, then d is graphic if and only if $d' = (d_1 - 1, d_2 - 1, d_3 - 1, d_4, \ldots, d_{n-1})$ is graphic.*

**Proof.** Theorem 2.6 implies Lemma 2.7 by letting $k = n$ and $d_k = 3$. □

**Theorem 2.8.** *If $d = (d_1, d_2, \ldots, d_n)$ is a nonincreasing graphic sequence with $n \geq 4$ and $d_n \geq 3$, then d has a collapsible realization.*

**Proof.** We argue by induction on $n$. If $n = 4$, then the assumption that $d_n \geq 3$ forces that the only realization of $d$ is $K_4$, and by Theorem 2.1(i), (iii), $K_4$ is collapsible.

Next we assume that $n \geq 5$. If $d_n \geq 4$, then $d_2 - 1 \geq d_3 - 1 \geq \cdots \geq d_{d_1+1} - 1 \geq 3$ and $d_{d_1+2} \geq \cdots \geq d_n \geq 3$. By Theorem 2.5 and the induction hypothesis, $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ has a collapsible realization $H$. Assume that $V(H) = \{v_2, v_3, \ldots, v_n\}$ such that $v_2, v_3, \ldots, v_{d_1+1}$ have degrees $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1$ in $H$, respectively, and such that $v_{d_1+2}, \ldots, v_n$ have degrees $d_{d_1+2}, \ldots, d_n$ in $H$, respectively. Then obtain a realization $H'$ of $d$ from $H$ by adding a new vertex $v_1$ and joining $v_1$ to $v_2, v_3, \ldots, v_{d_1+1}$, respectively. By Corollary 2.2(ii) $H'$ is collapsible.

Therefore, we may assume that $d_n = 3$. If $d_3 = 3$, then by Lemma 2.4, $(d_1, d_2, 3, \ldots, 3)$ is collapsible. Hence we assume further that $d_3 \geq 4$.

In this case, $d_1 - 1 \geq d_2 - 1 \geq d_3 - 1 \geq 3$ and $d_4 \geq \cdots \geq d_n = 3$. By Lemma 2.7, $(d_1 - 1, d_2 - 1, d_3 - 1, d_4, \ldots, d_{n-1})$ is graphic. By the induction hypothesis, $(d_1 - 1, d_2 - 1, d_3 - 1, d_4, \ldots, d_{n-1})$ has a collapsible realization $K$ with $V(K) = \{u_1, u_2, \ldots, u_{n-1}\}$ such that $u_1, u_2, u_3$ have degrees $d_1 - 1, d_2 - 1, d_3 - 1$ in $K$, respectively, and such that $u_4, u_5, \ldots, u_{n-1}$ have degrees $d_4, \ldots, d_{n-1}$ in $K$, respectively. We obtain a realization $K'$ of $d$ from $K$ by adding a new vertex $u_n$ and joining $u_n$ to $u_1, u_2, u_3$, respectively. By Corollary 2.2(ii) $K'$ is collapsible. □

## 3. Supereulerian sequence and Hamiltonian line graph

Let $X$ and $Y$ be two sets. Then $X \triangle Y = (X \cup Y) - (X \cap Y)$ denotes the *symmetric difference* of $X$ and $Y$. We start with the following observation (Lemma 3.1) and a few other lemmas. Throughout this section, we assume that $n \geq 3$.

**Lemma 3.1** (*Edmonds [3]*). *If $d = (d_1, d_2, \ldots, d_n)$ is a nonincreasing graphic sequence with $d_n \geq 2$, then there exists a 2-edge-connected realization of d.*

**Lemma 3.2.** *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence with $d_1 \leq n - 2$ and $d_n = 2$. Then d is graphic if and only if either of the following holds.*

(i) *$d' = (d_1, d_2, \ldots, d_{n-1})$ is graphic, or*
(ii) *$d'' = (d_1, d_2, \ldots, d_i - 1, \ldots, d_j - 1, \ldots, d_{n-1})$ is graphic for some $d_i \geq 3$ and $d_j \geq 3$, or*
(iii) *both $d_{n-1} = d_n = 2$, and for some $j$ with $1 \leq j < n - 1$ and with $d_j \geq 4$, $d''' = (d_1, d_2, \ldots, d_{j-1}, d_j - 2, d_{j+1}, \ldots, d_{n-2})$ is graphic, or*
(iv) *$n = 3$ and $d = (2, 2, 2)$.*

**Proof.** Suppose that $d = (d_1, d_2, \ldots, d_n)$ is graphic. Then there exists a 2-edge-connected realization $G$ of $d$ with $d(v_i) = d_i$ for $1 \leq i \leq n$. Suppose that $N(v_n) = \{v_i, v_j\}$. If $v_i v_j \notin E(G)$, then $G - v_n + \{v_i v_j\}$ is a realization of $(d_1, d_2, \ldots, d_{n-1})$, and so (i) holds. Thus we assume that $v_i v_j \in E(G)$.

If both $v_i, v_j$ have degree at least 3 in $G$, then $d''$ is graphic and so (ii) must hold. Thus we may assume further that $v_i$ has degree 2. If $v_j$ also has degree 2 in $G$, then $n = 3$ and (iv) must hold. Therefore, we may assume that $v_j$ has degree at least 3, and so $v_j$ is a cut-vertex of $G$. Since $G$ is 2-edge-connected and since $v_j$ is a cut-vertex, $d_j = d(v_j) \geq 4$. In this case, $d'''$ is the degree sequence of $G - \{v_n, v_i\}$, and so $d'''$ is graphic. The sufficiency can be proved by reversing the arguments above. □

**Proof of Theorem 1.2.** If a nonincreasing graphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a supereulerian realization, then we must have $d_n \geq 2$ as every supereulerian graph is 2-edge-connected.

We argue by induction on $n$ to prove the sufficiency. If $n = 3$, then since $d_n \geq 2$, $K_3$, a supereulerian graph, is the only realization of $d$.

Suppose that $n \geq 4$ and that the theorem holds for all such graphic sequences with fewer than $n$ entries. Let $d = (d_1, d_2, \ldots, d_n) \in \mathcal{G}$ be a nonincreasing sequence with $d_n \geq 2$. If $d_n \geq 3$, then by Theorem 2.8, $d$ has a collapsible realization $G$. By Theorem 2.1(vi), $G$ is supereulerian. If $d_1 = d_2 = \cdots = d_n = 2$, then $C_n$ is a supereulerian realization of $d$.

In the following, we assume that $d_1 > d_n = 2$. If $d_1 = n - 1$, then by Corollary 2.3, $d$ has a realization $G$ such that $G$ is collapsible. By Theorem 2.1(iv), $G$ is supereulerian. Thus $d$ in this case must be supereulerian.

Thus we may assume that $2 < d_1 \leq n - 2$. By Lemma 3.2, one of the conclusions of Lemma 3.2 (except Lemma 3.2(iv)) must hold.

If Lemma 3.2(i) holds, then $d' = (d_1, d_2, \ldots, d_{n-1})$ is graphic. By induction, there is a supereulerian realization $G'$ of $d'$. Let $C'$ be a spanning eulerian subgraph of $G'$ and $e = uv$ be an edge of $C'$. Then by subdividing $e$ of $G'$ into $uv_n$, $v_n v$, we obtain a supereulerian realization of $d$ as $d_n = 2$.

If Lemma 3.2(ii) holds, then for some $i, j$, $d'' = (d_1, d_2, \ldots, d_i - 1, \ldots, d_j - 1, \ldots, d_{n-1})$ is graphic, with $d_i \geq 3$ and $d_j \geq 3$. By induction, there is a supereulerian realization $G''$ of $d''$. Let $C''$ be a spanning eulerian subgraph of $G''$. If $v_i v_j \in E(G'')$, then let $C_1 = v_i v_j v_n$ and so $G = G'' + \{v_i v_n, v_j v_n\}$ is a supereulerian realization of $d$. If $v_i v_j \notin E(G'')$, then we can get a realization $G$ of $d$ from $G'' + \{v_i v_j\}$ by subdividing an edge $e = uv$ of $C'$ into $uv_n$ and $v_n v$.

If Lemma 3.2(iii) holds, then both $d_{n-1} = d_n = 2$, and for some $j$ with $1 \leq j < n - 1$ and with $d_i \geq 4$, $d''' = (d_1, d_2, \ldots, d_{j-1}, d_j - 2, d_{j+1}, \ldots, d_{n-2})$ is graphic. By induction, there is a supereulerian realization $G'''$ of $d'''$. Let $C'''$ be a spanning eulerian subgraph of $G'''$. Obtain $G$ from $G'''$ by adding two new vertices $v_{n-1}$ and $v_n$ and three new edges $v_j v_n, v_n v_{n-1}, v_{n-1} v_j$. Then $G$ is a realization of $d$, and $E(C''') \cup \{v_j v_n, v_n v_{n-1}, v_{n-1} v_j\}$ is a spanning eulerian subgraph of $G$. □

In order to prove Theorem 1.5, we need the following result which shows the relationship between hamiltonian circuits in the line graph $L(G)$ and eulerian subgraph in $G$. A subgraph $H$ of $G$ is dominating if $E(G - V(H)) = \emptyset$.

**Theorem 3.3** (*Harary and Nash-Williams, [5]*). *Let $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if $G$ has a dominating eulerian subgraph.*

**Proof of Theorem 1.5.** (i) ⇒ (ii). Let $G$ be a realization of $d$ such that $L(G)$ is hamiltonian. By Theorem 3.3, $G$ has a dominating eulerian subgraph $H$. If $d_1 = n - 1$, then we are done. Suppose that $d_1 \leq n - 2$. Then $|V(H)| \geq 2$. For any $v_i$ with $d(v_i) = 1$, $v_i$ must be adjacent to a vertex $v_j$ in $H$ and so $d_{G-E(H)}(v_j)$ is no less than the number of degree 1 vertices adjacent to $v_j$. Furthermore, since $H$ is eulerian and nontrivial, $d_H(v_j) \geq 2$ and so (1) must hold.

(ii) ⇒ (iii) Suppose $d \in \mathcal{G}$ is a nonincreasing sequence such that $d_n \geq 1$ and $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2}(d_j - 2)$. If $d_n \geq 2$, then by Theorem 1.2, $d$ has a supereulerian realization. So we assume that $d_n = 1$.

**Claim 3.4.** *Any realization of $d$ contains a nontrivial circuit.*

Suppose that there exists a realization $G$ of $d$ such that $G$ is a tree. We may assume that $d_i \geq 2$ for $1 \leq i \leq k$ and $d_j = 1$ for $k + 1 \leq j \leq n$. Then

$$\sum_{i=1}^{k} d_i + (n - k) = \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n} d_i = \sum_{i=1}^{n} d_i = 2|E(G)| = 2(n - 1),$$

and so

$$\sum_{i=1}^{k}(d_i - 2) + (n - k) = 2(n - 1) - 2k.$$

Hence

$$\sum_{d_j \geq 2}(d_j - 2) = \sum_{i=1}^{k}(d_i - 2) = 2(n - 1) - 2k - (n - k) = n - k - 2 < n - k = \sum_{d_i=1} d_i,$$

contrary to (1). This completes the proof of the claim.

Thus we assume that $G$ is a realization of $d$ containing a nontrivial circuit $C$.

**Claim 3.5.** *There is a realization $G$ of $d$ such that $\delta(G - D_1(G)) \geq 2$.*

As $G$ contains a nontrivial circuit $C$, $G - D_1(G)$ is not empty. Let $S = N(D_1(G))$. It suffices to show that for each $s \in S$, $N_{G-D_1(G)}(s) \geq 2$. Suppose, to the contrary, that there is $s \in S$ such that $N_{G-D_1(G)}(s) = 1$. Choose $G$ to be a graph such that $P(G) = \{s : s \in S$ with $d_G(s) = d_t \geq 2$ such that $N_{G-D_1(G)}(s) = 1\}$ is as small as possible. Let $x \in P(G)$. Then $x \notin C$. Choose $e \in E(C)$ and we subdivide $e$ and let $v_e$ denote the subdivision vertex. And we delete $d_t - 1$ pendent edges of $x$, add $d_t - 2$ pendent edges to $v_e$ and denote the resulting graph $G_x$. (Note that if $d_t - 2 = 0$, then we subdivide $e$ without adding any pendent edges.) Let $N_1(x)$ be the set of pendent vertices adjacent to $x$. So $d_{G_x}(v_e) = 2+d_t-2 = d_t$ and $|D_1(G_x)| = |(D_1(G)-N_1(x))\cup\{x\}|+d_t-2 = |D_1(G)|-(d_t-1)+1+d_t-2 = |D_1(G)|$ but $|P(G_x)| < |P(G)|$, contradicting the choice of $G$.

(iii) $\Rightarrow$ (i) If $G$ is a realization of $d$ such that $G - D_1(G)$ is supereulerian, then by Theorem 3.3, $L(G)$ is hamiltonian. □

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