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Radii Properties for Subclasses of Convex Functions*

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A function $f(z) = z + \cdots$ is said to be in $\mathcal{D}$ if $\operatorname{Re} f'(z) \geq |zf''(z)|$, $|z| < 1$. Using extreme point theory, the authors determine the largest disks $|z| \leq \beta = \beta(\alpha)$ for which $f(\beta z)/\beta \in \mathcal{D}$ when f is convex of order α or when $\operatorname{Re} f'(z) > \alpha$, $0 \leq \alpha < 1$.

1. INTRODUCTION

Denote by $\mathcal{S}$ the family of functions f, normalized by $f(0) = f'(0) - 1 = 0$, that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$ and by K the subfamily of convex functions. A function f is in K if and only if $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$, $z \in \Delta$. In [6], Ruscheweyh introduced the subfamily $\mathcal{D}$ of K, consisting of functions f for which

$$\operatorname{Re} f'(z) \geq |zf''(z)|, \qquad z \in \Delta. \tag{1}$$

Further work on $\mathcal{D}$, including some interesting convolution conjectures that would generalize the former Bieberbach conjecture (de Branges' theorem [2]), may be found in [3].

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Denote by $K(\alpha)$, $0 \le \alpha < 1$, the subfamily of K consisting of functions f that satisfy $\text{Re}\,(1 + zf''(z)/f'(z)) > \alpha$, $z \in \Delta$. Characterizing the extreme points of the closed convex hull of various compact families enables us to apply the Krein–Milman theorem to solve many linear extremal problems. In [1], it is shown that f is in the closed convex hull of $K(\alpha)$, $f \in \overline{\text{co}}\, K(\alpha)$, if and only if

$$f(z) = \int_X f_x(z)\, d\mu(x), \tag{2}$$

where $|x| = 1$, μ is a probability measure defined on the unit circle X, and

$$f_x(z) = \begin{cases} \dfrac{1}{(1-2\alpha)x}\left[\dfrac{1}{(1-xz)^{1-2\alpha}} - 1\right], & \alpha \ne \dfrac{1}{2} \\ -\bar{x}\log(1-xz), & \alpha = \dfrac{1}{2} \end{cases} \tag{3}$$

are the extreme points to $\overline{\text{co}}\, K(\alpha)$.

While $\mathfrak{D}$ is a convex family, $K(\alpha)$ is not. In fact, a convex linear combination of functions in $K(\alpha)$ need be in $\mathcal{S}$ only when $\alpha \ge \frac{1}{2}$. See [7]. Although $\mathfrak{D}$, which is contained in K, is a considerably smaller family than K, $\mathfrak{D} \not\subset K(\alpha)$ for any $\alpha > 0$. This can be illustrated with the function $z + z^2/4$, which is in $\mathfrak{D} - K(\alpha)$ for each $\alpha > 0$. On the other hand, we will see that $K(\alpha) \not\subset \mathfrak{D}$ for any α, $0 \le \alpha < 1$.

In Section 2, we will find the largest disk $|z| \le \beta = \beta(\alpha)$ in which (1) is satisfied for $f \in K(\alpha)$. This is equivalent to finding the largest β for which $f(\beta z)/\beta \in \mathfrak{D}$. When $\alpha \ge (3 - \sqrt{5})/4$, we will show that $\beta(\alpha) = 1/(3 - 2\alpha)$. For $0 \le \alpha < (3 - \sqrt{5})/4$, the sharp result is less aesthetically pleasing. Finally, in Section 3, we will find the largest $\beta = \beta(\alpha)$ for which $f(\beta z)/\beta \in \mathfrak{D}$ when $f \in R(\alpha)$, the subfamily of $\mathcal{S}$ consisting of functions f for which $\text{Re}\, f'(z) > \alpha$, $z \in \Delta$.

2. Main Results

It is convenient to characterize the family $\mathfrak{D}$ by $f \in \mathfrak{D} \Leftrightarrow \text{Re}\,\{f'(z) + e^{i\gamma} z f''(z)\} > 0$ for all $z \in \Delta$ and $\gamma \in (-\pi, \pi]$. Then, from (2) and (3), we have for $f \in \overline{\text{co}}\, K(\alpha)$ that

$$f'(z) + e^{i\gamma} z f''(z) = \int_X [f_0'(xz) + e^{i\gamma} xz f_0''(xz)]\, d\mu(x), \tag{4}$$

where

$$f_0(z) = \begin{cases} \dfrac{1}{(1-2\alpha)}\left[\dfrac{1}{(1-z)^{1-2\alpha}} - 1\right], & \alpha \neq \dfrac{1}{2} \\ -\log(1-z), & \alpha = \dfrac{1}{2}. \end{cases} \tag{5}$$

It follows from (4) that $f(\beta z)/\beta \in \mathfrak{D}$ whenever $f_0(\beta z)/\beta \in \mathfrak{D}$. Thus, it suffices to prove our results for $f_0(z)$ given by (5). We consider separately the cases $\alpha = 0$ and $\alpha = \frac{1}{2}$.

THEOREM 1. *If* $f \in \overline{\text{co}}\, K$, *then* $f(\beta z)/\beta \in \mathfrak{D}$ for $\beta \approx 0.329$, *the smallest positive root of*

$$1 - \beta^2 - 2\beta\cos\theta + 2\beta^2\cos^2\theta - 2\beta\sqrt{1+\beta^2-2\beta\cos\theta} = 0,$$

where

$$\cos\theta = \frac{-(11-6\beta^2) + \sqrt{117 - 72\beta^2 + 36\beta^4}}{4\beta^3}.$$

The result is sharp.

Proof. From (2) and (3) with $\alpha = 0$, we may write $f(z) = \int_X (z/(1-xz))\, d\mu(x)$. Then, from (4), (5), and the superharmonicity of $\operatorname{Re} f' - |zf''|$, it suffices to show that

$$p(z) = f_0'(\beta z) + e^{i\gamma}\beta z f_0''(\beta z) = \frac{1}{(1-\beta z)^2} + \frac{e^{i\gamma}(2\beta z)}{(1-\beta z)^3}$$

satisfies $\operatorname{Re} p(z) \geq 0$ for $|z| = 1$. When $z = e^{i\theta}$,

$$\begin{aligned}\operatorname{Re} p(z) &\geq \operatorname{Re}\frac{1}{(1-\beta z)^2} - \frac{2\beta}{|1-\beta z|^3} \\ &= \frac{1-\beta^2-2\beta\cos\theta+2\beta^2\cos^2\theta}{|1-\beta z|^4} - \frac{2\beta}{|1-\beta z|^3} := g(\beta,\theta).\end{aligned}$$

A calculation shows that

$$\begin{aligned}&|1-\beta e^{i\theta}|^6 \frac{\partial g(\beta,\theta)}{\partial\theta} \\ &\quad = -2\beta\sin\theta\,[2\beta^3\cos\theta + (1-3\beta^2) - 3\beta\sqrt{1+\beta^2-2\beta\cos\theta}] = 0\end{aligned}$$

for $\theta = 0, \pi$, and θ_0, where θ_0 is a zero of the equation

$$\cos^2\theta + \frac{11 - 6\beta^2}{2\beta^3}\cos\theta + \frac{1 - 15\beta^2}{4\beta^6} = 0.$$

We have $g(\beta, \theta) \geq \min\{g(\beta, 0), g(\beta, \pi), g(\beta, \theta_0)\}$. Now $g(\frac{1}{3}, 0) = g(1, \pi) = 0$. On the other hand, $g(\beta, \theta_0) = 0$ for $\beta \approx 0.329$ and $\cos\theta_0 \approx 0.841$. Since $\theta = \theta_0$ gives the minimum β for which $g(\beta, \theta) = 0$, the proof is complete.

The case $\alpha = \frac{1}{2}$ provides a simpler solution.

THEOREM 2. *If $f \in \overline{\text{co}}\, K(\frac{1}{2})$, then $2f(z/2) \in \mathfrak{D}$. The result is sharp.*

Proof. In view of (5), for $f_0(z) = -\log(1 - z)$, we want to find the largest β for which

$$q(z) = f_0'(\beta z) + e^{i\gamma}\beta z f_0''(\beta z) = \frac{1}{1 - \beta z} + e^{i\gamma}\frac{\beta z}{(1 - \beta z)^2} \tag{6}$$

satisfies $\text{Re}\, q(z) \geq 0$ for $z = e^{i\theta}$, $-\pi < \theta \leq \pi$. But

$$\text{Re}\, q(z) \geq \frac{1 - \beta\cos\theta}{|1 - \beta z|^2} - \frac{\beta}{|1 - \beta z|^2} \geq 0$$

as long as $1 - 2\beta \geq 0$. Thus, we have $\beta = \frac{1}{2}$, as needed.

Remark. In [3], it is essentially shown that partial sums f_n of $f \in \overline{\text{co}}\, K(\frac{1}{2})$ also satisfy $2f_n(z/2) \in \mathfrak{D}$. It suffices to consider the partial sums $g_n(z) = z + \sum_{k=2}^{n} (z^k/k)$ of $g(z) = -\log(1 - z)$ for which it was proved that $2g_n(z/2) \in \mathfrak{D}$.

In the proof of Theorem 2, (6) for $z = e^{i\theta}$ yielded the sharp result when $\theta = 0$. However, in the more computationally involved proof of Theorem 1, $\theta \approx \text{Arccos}\,(0.841)$ led to the sharp value for β. Next, we will show that $\theta = 0$ is extremal for α sufficiently large.

THEOREM 3. *If $f \in \overline{\text{co}}\, K(\alpha)$, $\alpha \geq (3 - \sqrt{5})/4$, then $f(\beta z)/\beta \in \mathfrak{D}$ for $\beta = \beta(\alpha) = 1/(3 - 2\alpha)$. The result is sharp.*

Proof. Again, we need consider only $f_0(z)$ given by (5). We will show that

$$h(z) = f_0'(\beta z) + e^{i\gamma}\beta z f_0''(\beta z) = \frac{1}{(1 - \beta z)^{2-2\alpha}} + \frac{e^{i\gamma}(2 - 2\alpha)\beta z}{(1 - \beta z)^{3-2\alpha}}$$

satisfies $\text{Re}\, h(z) \geq 0$ for $|z| = 1$ and $\beta \leq \beta(\alpha) = 1/(3 - 2\alpha)$. We have

$$\operatorname{Re} h(z) \geq \frac{\operatorname{Re}(1-\beta\bar{z})^{2-2\alpha}}{|1-\beta z|^{4-4\alpha}} - \frac{2(1-\alpha)\beta|1-\beta z|^{1-2\alpha}}{|1-\beta z|^{4-4\alpha}} \geq 0$$

whenever

$$\operatorname{Re}(1-\beta\bar{z})^{2-2\alpha} - 2(1-\alpha)\beta|1-\beta z|^{1-2\alpha} \geq 0. \tag{7}$$

Upon noting that $|\operatorname{Arg}(1-\beta e^{-i\theta})| < \pi/2$, we set

$$\rho(\beta, \theta) = |1-\beta e^{-i\theta}| = \sqrt{1+\beta^2-2\beta\cos\theta}$$

and

$$\Phi(\beta, \theta) = \operatorname{Arg}(1-\beta e^{-i\theta}) = \operatorname{Arctan}\left(\frac{\beta\sin\theta}{1-\beta\cos\theta}\right). \tag{8}$$

For $z = e^{i\theta}$, $-\pi < \theta \leq \pi$, (7) can be rewritten as

$$[\rho(\beta, \theta)]^{1-2\alpha}\{\rho(\beta, \theta)\cos(2(1-\alpha)\Phi(\beta, \theta)) - 2\beta(1-\alpha)\} \geq 0.$$

Consequently, $f(\beta z)/\beta \in \mathcal{D}$ as long as

$$\rho(\beta, \theta)\cos(2(1-\alpha)\Phi(\beta, \theta)) - 2\beta(1-\alpha) \geq 0. \tag{9}$$

For the sharp result, first we will find the θ that minimizes

$$L(\beta, \theta) = \rho(\beta, \theta)\cos(2(1-\alpha)\Phi(\beta, \theta)) \tag{10}$$

and, then, we will determine the largest β for which (9) holds. When $\theta = 0$, (9) becomes $1-\beta-2\beta(1-\alpha) \geq 0$, that is, $\beta \leq 1/(3-2\alpha)$. Consequently, the proof will be complete when we show that $\min_{\theta\in(-\pi,\pi]} L(\beta, \theta) = L(\beta, 0) = 1-\beta$ for $\alpha \geq (3-\sqrt{5})/4$ and all admissible β.

In view of Theorem 2, $\beta = \beta(\alpha) \leq \frac{1}{2}$ if and only if $\alpha \leq \frac{1}{2}$. When $\alpha \geq \frac{1}{2}$, $2(1-\alpha)|\Phi(\beta, \theta)| \leq |\Phi(\beta, \theta)| < \pi/2$. When $\alpha < \frac{1}{2}$, $2(1-\alpha)|\Phi(\beta, \theta)| \leq |\Phi(\frac{1}{2}, \theta)| \leq 2\operatorname{Arctan}(1/\sqrt{3}) = \pi/3$. In either case, we have

$$2(1-\alpha)|\Phi(\beta, \theta)| < \pi/2 \tag{11}$$

and $\min_{\theta\in(-\pi,\pi]} L(\beta, \theta) > 0$. Since

$$\frac{\partial \rho}{\partial \theta} = \frac{\beta \sin \theta}{\rho(\beta, \theta)} \quad \text{and} \quad \frac{\partial \Phi}{\partial \theta} = \frac{\beta(\cos \theta - \beta)}{(\rho(\beta, \theta))^2},$$

it follows that

$$\frac{\partial L}{\partial \theta} = \frac{\beta}{\rho(\beta, \theta)}\{(\sin \theta)\cos(2(1-\alpha)\Phi(\beta, \theta)) \\ -2(1-\alpha)(\cos \theta - \beta)\sin(2(1-\alpha)\Phi(\beta, \theta))\}$$

vanishes whenever

$$M(\theta) = (\sin \theta)\cos(2(1-\alpha)\Phi(\beta, \theta)) \\ - 2(1-\alpha)(\cos \theta - \beta)\sin(2(1-\alpha)\Phi(\beta, \theta))$$

vanishes. From (8), we see that $M(\theta) = 0$ at least when $\sin \theta = 0$, i.e., for $\theta = 0$ and $\theta = \pi$. But $L(\beta, \pi) = 1 + \beta > L(\beta, 0) = 1 - \beta$. In addition, when $\sin\theta \neq 0$, $\sin\theta$ and $\sin(2(1-\alpha)\Phi(\beta, \theta))$ have the same sign. Consequently, from (11), we see that, if $\theta \neq 0, \pi$, then $M(\theta)$ can vanish only where $\cos \theta > \beta$. Finally, since $M(\theta)$ is an odd function, it suffices to show that $M(\theta) > 0$ when $\theta \in (0, \operatorname{Arccos} \beta)$. Taking (11) into account, we conclude that $M(\theta) > 0$, for $\theta \in (0, \operatorname{Arccos} \beta)$, if and only if

$$\frac{\sin \theta}{2(1-\alpha)(\cos \theta - \beta)} - \tan(2(1-\alpha)\Phi(\beta, \theta)) > 0.$$

In view of (8), this is equivalent to showing

$$G(\theta) := \operatorname{Arctan}\left(\frac{\sin \theta}{2(1-\alpha)(\cos \theta - \beta)}\right) \\ - 2(1-\alpha)\operatorname{Arctan}\left(\frac{\beta \sin \theta}{1 - \beta \cos \theta}\right) > 0. \tag{12}$$

Since $G(0) = 0$, we will be done if we can show that $G'(\theta) > 0$. We have

$$G'(\theta) = 2(1-\alpha)\left[\frac{1 - \beta \cos \theta}{4(1-\alpha)^2(\cos \theta - \beta)^2 + \sin^2\theta} - \frac{\beta(\cos \theta - \beta)}{1 - 2\beta \cos \theta + \beta^2}\right]$$

and

$$G''(\theta) = \left[\frac{-4\beta(1-\beta^2)(1-\alpha)^2 - (4(1-\alpha)^2-1)(\beta+\beta\cos^2\theta - 2\cos\theta)}{(4(1-\alpha)^2(\cos\theta-\beta)^2+\sin^2\theta)^2}\right. \tag{13}$$
$$\left. + \frac{\beta(1-\beta^2)}{(1+\beta^2-2\beta\cos\theta)^2}\right](2(1-\alpha)\sin\theta).$$

If $\frac{1}{2} \le \alpha < 1$, then

$$G'(\theta) \ge 2(1-\alpha)\left[\frac{1-\beta\cos\theta}{(\cos\theta-\beta)^2+\sin^2\theta} - \frac{\beta(\cos\theta-\beta)}{1-2\beta\cos\theta+\beta^2}\right]$$
$$= 2(1-\alpha) > 0$$

and we are done. If $0 \le \alpha < \frac{1}{2}$ and $0 < \theta < \operatorname{Arccos}\beta$, then

$$4\beta(1-\beta^2)(1-\alpha)^2 + (4(1-\alpha)^2-1)(\beta+\beta\cos^2\theta-2\cos\theta)$$
$$< 4\beta(1-\beta^2)(1-\alpha)^2 + (4(1-\alpha)^2-1)(-\beta(1-\beta^2)) = \beta(1-\beta^2).$$

Consequently,

$$G''(\theta) > 2(1-\alpha)\sin\theta\left[\frac{-\beta(1-\beta^2)}{(1+\beta^2-2\beta\cos\theta)^2} + \frac{\beta(1-\beta^2)}{(1+\beta^2-2\beta\cos\theta)^2}\right] = 0.$$

Thus, G' is strictly increasing for $\theta \in (0, \operatorname{Arccos}\beta)$ and $G(\theta) > 0$ as long as $G'(0) \ge 0$. Whenever

$$G'(0) = \frac{2(1-\alpha)}{(1-\beta)}\left[\frac{1}{4(1-\alpha)^2} - \beta\right] \ge 0,$$

we may take $\beta = \beta(\alpha) = 1/(3-2\alpha)$. Since $1/(4(1-\alpha)^2) \ge 1/(3-2\alpha)$ if and only if $\alpha \ge (3-\sqrt{5})/4$, the proof is complete.

In the proof of Theorem 3, the restriction $\alpha \ge (3-\sqrt{5})/4$ was made in order to obtain $G'(0) \ge 0$. Although $G'(0) < 0$ when $0 \le \alpha < (3-\sqrt{5})/4$, we have, from (13), that G is concave upward. Since $G(0) = 0$ and $G(\operatorname{Arccos}\beta) > 0$, there must be a unique $\theta = \theta(\alpha) \in (0, \operatorname{Arccos}\beta)$ for which $G(\theta(\alpha)) = 0$. This θ minimzies $L(\beta, \theta)$ defined by (10) and leads to a sharp result from (9). We summarize this with

THEOREM 4. *If* $f \in \overline{\text{co}}\, K(\alpha)$, $0 \le \alpha < (3-\sqrt{5})/4$, *then* $f(\beta z)/\beta \in \mathfrak{D}$ *for* $\beta = \beta(\alpha)$, *where* $\beta(\alpha)$ *is the unique* β *for which both* (i) $G(\theta(\alpha)) = 0$ ($0 < \theta(\alpha) < \operatorname{Arccos}\beta$, *$G$ defined by* (12)) *and* (ii) *equality holds in* (9) *when* $\theta = \theta(\alpha)$.

Remark. When $\alpha = \frac{1}{2}$ and $\alpha = 0$, Theorems 3 and 4 are seen to be special cases of Theorems 2 and 1, respectively. The $\beta = \beta(\alpha)$ that give sharp results in Theorem 4 satisfy

$$0.329 < \beta(0) \leq \beta(\alpha) < \beta\left(\frac{3-\sqrt{5}}{4}\right) = \frac{1}{3 - 2((3-\sqrt{5})/4)} = \frac{3-\sqrt{5}}{2} < 0.382,$$

and $1/(3 - 2\alpha) - \beta(\alpha) \to 0$ as $\alpha \to (3 - \sqrt{5})/4$.

3. A Subclass of S

Denote by $R(\alpha)$ the subfamily of S consisting of functions f for which $\operatorname{Re} f'(z) > \alpha$, $z \in \Delta$. Hallenbeck [4] showed that $f \in R(\alpha)$ if and only if

$$f'(z) = \int_X \frac{1 + (1-2\alpha)xz}{1 - xz}\, d\mu(x), \tag{14}$$

where $|x| = 1$ and μ is a probability measure defined on the unit circle X.

THEOREM 5. *If $f \in R(\alpha) = \overline{\text{co}}\, R(\alpha)$, then $f(\beta z)/\beta \in \mathfrak{D}$ for*

$$\beta = \begin{cases} \dfrac{-1 + \sqrt{2(1-\alpha)}}{1 - 2\alpha}, & \alpha \neq \dfrac{1}{2} \\[2ex] \dfrac{1}{2}, & \alpha = \dfrac{1}{2}. \end{cases}$$

The result is sharp.

Proof. From (14), it suffices to consider f for which $f'(z) = (1 + (1 - 2\alpha)z)/(1 - z)$. We wish to find the largest β for which

$$p(z) = f'(\beta z) + e^{i\gamma}\beta z f''(\beta z) = \frac{1 + (1-2\alpha)\beta z}{1 - \beta z} + \frac{e^{i\gamma}2(1-\alpha)\beta z}{(1-\beta z)^2}$$

satisfies $\operatorname{Re} p(z) \geq 0$ for $z = e^{i\theta}$. But

$$\operatorname{Re} p(z) \geq \frac{1 - 2\alpha\beta\cos\theta - (1 - 2\alpha)\beta^2}{|1 - \beta z|^2} - \frac{2(1 - \alpha)\beta}{|1 - \beta z|^2} \geq 0$$

if $1 - 2\beta - (1 - 2\alpha)\beta^2 \geq 0$. Solving for β yields the result.

Remark 1. Since $\mathfrak{D} \subset K$, the $\beta = \beta(\alpha)$ given in Theorem 5 also furnishes us with a disk $|z| < \beta$ in which $f \in R(\alpha)$ is convex. Silverman [8] showed the radius of convexity of $R(\alpha)$ to be

$$\begin{cases} \dfrac{1}{1 - 2\alpha + \sqrt{4\alpha^2 - 6\alpha + 2}}, & 0 \leq \alpha \leq 1/10 \\ \left(1 + \sqrt{\dfrac{1 - \alpha}{\alpha}}\right)^{-1/2}, & 1/10 < \alpha < 1. \end{cases}$$

When $\alpha = 0$, this bound agrees with the one given in Theorem 5.

Remark 2. A function $f \in S$ is starlike, $f \in St$, if and only if $\operatorname{Re}\{zf'(z)/f(z)\} > 0$, $z \in \Delta$. Hamilton and Tuan [5] showed that the radius of starlikeness of $\overline{\text{co}}\ St$ is $r_0 \approx 0.4035$, the positive root of the equation $r^6 + 5r^4 + 79r^2 - 13 = 0$. Since $f \in K$ if and only if $zf' \in St$, this is equivalent to saying that the radius of convexity of $\overline{\text{co}}\ K$ is r_0. The $\beta = \beta(\alpha)$ in Theorems 3 and 4 give lower bounds on the radius of convexity of $\overline{\text{co}}\ K(\alpha)$, $0 \leq \alpha < 1$.

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