JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **190**, 725–737 (1995)

Bi-axial Gegenbauer Functions of the Second Kind

A. K. COMMON*

Institute of Mathematics and Statistics, University of Kent, Canterbury, Kent, CT2 7NF, England

Submitted by George Gasper

Received October 25, 1993; revised March 10, 1994

Bi-axially symmetric monogenic generating functions on $\mathbb{R}^{p+q}$ have been used recently to define generalisations of Gegenbauer polynomials. These polynomials are orthogonal on the unit ball in $\mathbb{R}^p$. Generalised Cauchy transforms of these polynomials are used to define corresponding bi-axial Gegenbauer functions of the second kind. It is demonstrated that these functions of the second kind satisfy second order differential equations related to those satisfied by the corresponding bi-axial Gegenbauer polynomials.

1. INTRODUCTION

We consider here functions on $\mathbb{R}^m$ taking values in a complex Clifford algebra $\mathcal{A}$. The generating vectors of the Clifford algebra $\mathcal{A}$ are $\{e_l;\ l = 1, 2, \dots, m\}$ satisfying the defining relations

$$e_l e_j + e_j e_l = -2\delta_{jl} e_0, \qquad j, l = 1, 2, \dots, m, \tag{1.1}$$

where e_0 is the unit element of the algebra. To every point in $\mathbb{R}^m$ there corresponds a vector in the algebra,

$$\vec{x} = \sum_{l=1}^{m} x_l e_l. \tag{1.2}$$

* E-mail: akc@ukc.ac.uk.

0022-274X/95 $6.00

The function f on an open set Ω of $\mathbb{R}^m$ taking values in $\mathcal{A}$ is said to be left monogenic when

$$\partial_{\vec{x}} f \equiv \sum_{l=1}^{m} e_l \frac{\partial f}{\partial x_l} = 0, \qquad \forall x \in \Omega, \tag{1.3}$$

and right monogenic when

$$f \partial_{\overleftarrow{x}} \equiv \sum_{l=1}^{m} \frac{\partial f}{\partial x_l} e_l = 0, \qquad \forall x \in \Omega. \tag{1.4}$$

Here we will consider a sub-class of such functions defined on bi-axially symmetric domains [1]. The approach is to consider the splitting $\mathbb{R}^m = \mathbb{R}^p + \mathbb{R}^q$ and to denote a general element $\vec{x}$ of $\mathbb{R}^m$ by $\vec{x} = \vec{x_1} + \vec{x_2} = \rho_1 \vec{\omega_1} + \rho_2 \vec{\omega_2}$, where $\rho_1 = |\vec{x_1}|$, $\rho_2 = |\vec{x_2}|$, and $\vec{x_1} \in \mathbb{R}^p$, $\vec{x_2} \in \mathbb{R}^q$. Inner spherical monogenics $P_{k,l}(\vec{x_1}, \vec{x_2})$ have been introduced in [2] for the bi-axial case. They are polynomials which are homogeneous of degrees k in $\vec{x_1}$ and l in $\vec{x_2}$ and satisfy

$$\partial_{\vec{x_1}} P_{k,l}(\vec{x_1}, \vec{x_2}) = \partial_{\vec{x_2}} P_{k,l}(\vec{x_1}, \vec{x_2}) = 0, \tag{1.5}$$

where

$$\partial_{\vec{x_1}} \equiv \sum_{l=1}^{p} e_l \frac{\partial}{\partial x_l}, \qquad \partial_{\vec{x_2}} \equiv \sum_{l=p+1}^{p+q} e_l \frac{\partial}{\partial x_l}. \tag{1.6}$$

Then bi-axial monogenic functions of order (k, l) are monogenic functions of the form

$$\begin{aligned} f_{k,l}(\vec{x}) = [&A_{k,l}(\rho_1, \rho_2) + \vec{\omega_1} B_{k,l}(\rho_1, \rho_2) + \vec{\omega_2} C_{k,l}(\rho_1, \rho_2) \\ &+ \vec{\omega_1}\vec{\omega_2} D_{k,l}(\rho_1, \rho_2)] P_{k,l}(\vec{\omega_1}, \vec{\omega_2}), \end{aligned} \tag{1.7}$$

where $A_{k,l}$, etc. are scalars.

A topic discussed by Sommen [3] for axial monogenics (corresponding essentially to the case when $q = 1$) was the generalisation of Hermite polynomials $H_n(z)$ to polynomials $H_{n,m,k}(\vec{x})$ using an axial monogenic generating function. The set of polynomials $\{H_{n,m,k}(\vec{x}) P_k^{(i)}(\vec{x});\ k, n \in \mathcal{N};\ i = 1, 2, \ldots, \mathcal{H}(m, k)\}$ form an orthogonal basis for $L_2\{\mathbb{R}^m; \exp(-\rho^2/2)\}$, where $\{P_k^{(i)}(\vec{x});\ i = 1, 2, \ldots, \mathcal{H}(m, k)\}$ is an orthonormal basis for the space of inner spherical monogenics of degree k. Subsequently [4, 5] Cnops derived these polynomials from a Rodrigues formula and extended them to Gegenbauer polynomials. He obtained the set of polynomials

$\{\mathbb{C}^{(\alpha)}_{n,m,k}(\vec{x})P^{(i)}_k(\vec{x});\ k,\ n \in \mathcal{N},\ 1 \le i \le \mathcal{H}(m,\ k),\ \alpha > -1\}$ which form an orthogonal basis for $L_2\{\mathbb{B}^m;\ (1 + \vec{x}^2)^\alpha\}$, where $\mathbb{B}^m$ is the unit ball in $\mathbb{R}^m$.

In Section 2 we consider the corresponding generalised Gegenbauer polynomials in the bi-axially symmetric case which have been defined recently by the author and F. Sommen [6]. They have the explicit expressions

$$C^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1) = \frac{(-)^j(\alpha + k + l + p/2 + q/2 - 1)_j(k + p/2)_j}{(l + q/2)_j\Gamma(j + 1)} \times {}_2F_1(\alpha + k + l + p/2 + q/2 + j - 1, -j; k + p/2; -\vec{x}_1^{\,2}) \tag{1.8}$$

$$C^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1) = \frac{(-)^j(\alpha + k + l + p/2 + q/2 - 1)_{j+1}(k + p/2 + 1)_j}{(l + q/2)_{j+1}\Gamma(j + 1)} \times \vec{x}_1\,{}_2F_1(\alpha + k + l + p/2 + q/2 + j, -j; k + 1 + p/2; -\vec{x}_1^{\,2}) \tag{1.9}$$

for $j = 0, 1, 2, \ldots, p, q, k, l \in \mathcal{N}$.

We will show in Section 2 that they are very closely related to the $\{\mathbb{C}^{(\alpha)}_{n,m,k,}(\vec{x}_1)\}$ and hence derive the second order differential equations satisfied by them and also explain how they are given by a Rodrigues-type formula.

The main aim of this work is to extend the concept of Gegenbauer functions of the second kind to bi-axial monogenics. In the standard case this extension is made by considering the Cauchy integral transform of the Gegenbauer polynomials $C^{(\alpha)}_n(z)$ and more generally of Jacobi polynomials $P^{(\alpha,\beta)}_n(z)$; i.e., the Jacobi function of the second kind is defined to be

$$Q^{(\alpha,\beta)}_n(z) = \frac{1}{2}(z - 1)^{-\alpha}(z + 1)^{-\beta}\int_{-1}^{1} \frac{(1 - t)^\alpha(1 + t)^\beta P^{(\alpha,\beta)}_n(t)}{z - t}\,dt \tag{1.10}$$

with the Gegenbauer case corresponding to $\alpha,\ \beta \to \alpha - \frac{1}{2}$. It may be shown [7] that $Q^{(\alpha,\beta)}_n(z)$ satisfies the same second order differential equation as $P^{(\alpha,\beta)}_n(z)$.

In Section 3 we use bi-axially symmetric Cauchy transforms to define corresponding generalised Gegenbauer functions of the second kind. These transforms have been studied previously by F. Sommen and the author [8]. For a given scalar function $f(\lambda)$ one may define

$$\Lambda^{(1)}_{k,l}(f)(\vec{x}_1) = \frac{1}{\omega_p}\int_{\mathbb{R}^p} \frac{[\vec{x}_1 + \vec{x}_2 - \vec{u}]f(\lambda)}{|\vec{x}_1 + \vec{x}_2 - \vec{u}|^{m+2l}} P_{k,l}(\vec{\eta}, \vec{x}_2)\,d^p\vec{u} \tag{1.11}$$

$$\Lambda^{(2)}_{k,l}(f)(\vec{x}_1) = \frac{1}{\omega_p}\int_{\mathbb{R}^p} \frac{[\vec{x}_1 + \vec{x}_2 - \vec{u}]\vec{\eta}f(\lambda)}{|\vec{x}_1 + \vec{x}_2 - \vec{u}|^{m+2l}} P_{k,l}(\vec{\eta}, \vec{x}_2)\,d^p\vec{u}, \tag{1.12}$$

where $\vec{u} = \lambda\vec{\eta}$, $|\vec{\eta}| = 1$, and ω_p is the area of the unit sphere in p-dimensions. These transforms are bi-axial monogenic functions in $\mathbb{R}^m\backslash\mathbb{R}^p$. The generalised Gegenbauer functions of the second kind $Q^{(\alpha)}_{n;p;k,l}(\vec{x_1}, \vec{x_2})$ will be defined by taking the above transforms of the $C^{(\alpha)}_{n;p,q;k,l}(\vec{x_1})P_{k,l}(\vec{x_1}, \vec{x_2})$. Integral expressions for the $Q^{(\alpha)}_{n;p,q;k,l}(\vec{x_1})P_{k,l}(\vec{x_1}, \vec{x_2})$ will be derived by using the Funk–Hecke theorem [9] to perform the angular integrations in (1.11), (1.12) in this case.

Finally, in Section 4, we will derive the second order differential equations satisfied by our generalised Gegenbauer functions. In the standard case $Q^{(\alpha)}_n(z)$ satisfies the same equation as $C^{(\alpha)}_n(z)$. In the bi-axial case this is not quite true although the two sets of differential equations are very closely related.

2. Bi-axial Gegenbauer Polynomials

Generalised Gegenbauer polynomials in the axial monogenic case have been defined by Cnops [4, 5] using a Rodrigues-type formula

$$\mathbb{C}^{(\alpha)}_{n,m,k}(\vec{x})P_k(\vec{x}) \equiv (1 + \vec{x}^2)^{-\alpha}(\partial_{\vec{x}})^n[(1 + \vec{x}^2)^{\alpha+n}P_k(\vec{x})] \tag{2.1}$$

for $\alpha > -1$, n, $m \in \mathcal{N}$. Using this definition, it was shown that these polynomials satisfy the second order differential equation

$$\begin{aligned}(1 + \vec{x}^2)^{-\alpha}\partial_{\vec{x}}\{(1 + \vec{x}^2)^{\alpha+1}\partial_{\vec{x}}[\mathbb{C}^{(\alpha)}_{n,m,k}(\vec{x})P_k(\vec{x})]\}\\ = \beta(n, \alpha, k)\mathbb{C}^{(\alpha)}_{n,m,k}(\vec{x})P_k(\vec{x}),\end{aligned} \tag{2.2}$$

where

$$\beta(n, \alpha, k) = n(2\alpha + n + m + 2k), \qquad n = 0, 2, 4, \dots, \tag{2.3}$$

$$= (2\alpha + n + 1)(n + m + 2k - 1), \qquad n = 1, 3, 5, \dots. \tag{2.4}$$

They were also demonstrated to have the following explicit expression in terms of hypergeometric functions:

$$\begin{aligned}\mathbb{C}^{(\alpha)}_{2j,m,k}(\vec{x}) = 2^{2j}(\alpha + j + 1)_j(m/2 + k)_j\\ {}_2F_1(j + \alpha + m/2 + k, -j; k + m/2; -\vec{x}^2)\end{aligned} \tag{2.5}$$

$$\begin{aligned}\mathbb{C}^{(\alpha)}_{2j+1,m,k}(\vec{x}) = 2^{2j+1}(\alpha + j + 1)_{j+1}(m/2 + k + 1)_j\vec{x}\\ {}_2F_1(j + \alpha + m/2 + k + 1, -j; k + m/2 + 1; -\vec{x}^2).\end{aligned} \tag{2.6}$$

Here it should be noted that factors of $(-)^n$ and $(-)^{n+1}$ on the RHS of Eqs. (9), (10) of Ref. [4] are incorrect and should be deleted.

Comparing these expressions with those given in (1.8), (1.9) for the $C^{(\alpha)}_{n;p,q;k,l}(\vec{x}_1)$ we see that

$$C^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1) = \frac{(-)^j(1/2)_j(\alpha + q/2 + p/2 + k + l - 1)_j}{(l + q/2)_j(\alpha + q/2 + l + j)_j\Gamma(2j+1)} \mathbb{C}^{(\alpha+q/2+l-1)}_{2j,p,k}(\vec{x}_1) \tag{2.7}$$

$$C^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1) = \frac{(-)^j(1/2)_{j+1}(\alpha + q/2 + p/2 + k + l - 1)_{j+1}}{(l + q/2)_{j+1}(\alpha + q/2 + l + j)_{j+1}\Gamma(2j+2)} \mathbb{C}^{(\alpha+q/2+l-1)}_{2j+1,p,k}(\vec{x}_1) \tag{2.8}$$

when $p + q = m$.

It follows that the bi-axial Gegenbauer polynomials satisfy the second order differential equation

$$\begin{aligned}(1 + \vec{x}_1^{\,2})^{-(\alpha+q/2+l-1)} \partial_{\vec{x}_1}\{(1 + \vec{x}_1^{\,2})^{\alpha+q/2+l} \partial_{\vec{x}_1}[C^{(\alpha)}_{n;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)]\} \\ = \beta(n, \alpha + q/2 + l - 1, k)C^{(\alpha)}_{n;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)\end{aligned} \tag{2.9}$$

for $n = 0, 1, 2, \ldots$, where $\beta(n, \alpha + q/2 + l - 1, k)$ is given by (2.3), (2.4) with m replaced by p.

Also as the $C^{(\alpha)}_{n;p,q;k,l}(\vec{x}_1)$ are scalar multiples of $\mathbb{C}^{(\alpha+q/2+l-1)}_{n,p,k}(\vec{x}_1)$ it follows that they may be represented by the corresponding multiples of the Rodrigues formula (2.1) with $\alpha \to \alpha + q/2 + l - 1$.

3. Bi-axial Gegenbauer Functions of the Second Kind

It is seen from (1.8), (1.9) that $C^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)$ is a scalar valued function, while $C^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)$ is vector valued. We may then define the transforms

$$\begin{aligned}\Lambda^{(\alpha)}_{n;p,q;k,l}(\vec{x}) = \frac{1}{\omega_p} \int_{\mathbb{B}^p} \frac{[\vec{x}_1 + \vec{x}_2 - \vec{u}](1 + \vec{u}^{\,2})^{\alpha+q/2+l-1} C^{(\alpha)}_{n;p,q;k,l}(\vec{u})}{|\vec{x}_1 + \vec{x}_2 - \vec{u}|^{m+2l}} \\ P_{k,l}(\vec{u}, \vec{x}_2)\, d^p\vec{u}, \qquad n = 0, 1, 2, \ldots,\end{aligned} \tag{3.1}$$

which are of the form (1.11) for n even and of the form (1.12) for n odd and so are monogenic for $\mathbb{R}^m \backslash \mathbb{B}^p$. We use these transforms to make the following definition:

Definition. The bi-axial Gegenbauer functions of the second kind are given by

$$Q^{(\alpha)}_{n;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2)$$
$$\equiv \lim_{|\vec{x}_2|\to 0}\left\{\frac{[-(\vec{x}_1+\vec{x}_2)^2-1]^{-\alpha+q/2+l}}{|\vec{x}_2|^l\omega_p}\times\int_{\mathbb{B}^p}\frac{(\vec{x}_1+\vec{x}_2-\vec{u})(1+\vec{u}^2)^{\alpha+q/2+l-1}}{|\vec{x}_1+\vec{x}_2-\vec{u}|^{m+2l}}\right.$$
$$\left. C^{(\alpha)}_{n;p,q;k,l}(\vec{u})P_{k,l}(\vec{u},\vec{x}_2)\right\}d^p\vec{u};\, n,p,q,k,l\in\mathcal{N},\qquad \alpha>0. \qquad (3.2)$$

Taking the limit $|\vec{x}_2|\to 0$ and using the Rodrigues formula for the $C^{(\alpha)}_{n;p,q;k,l}(\vec{u})P_{k,l}(\vec{u},\vec{x}_2)$ obtained from (2.1), (2.7), (2.8),

$$Q^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_1)=\frac{(-)^j(1/2)_j(\alpha+q/2+p/2+k+l-1)_j}{(l+q/2)_j(\alpha+q/2+l+j)_j\Gamma(2j+1)}\times\Theta^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2) \qquad (3.3)$$

$$Q^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_1)=\frac{(-)^j(1/2)_{j+1}(\alpha+q/2+p/2+k+l-1)_{j+1}}{(l+q/2)_{j+1}(\alpha+q/2+l+j)_{j+1}\Gamma(2j+2)}\times\Theta^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2) \qquad (3.4)$$

where

$$\Theta^{(\alpha)}_{n;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2)=\frac{[-\vec{x}_1^{\,2}-1]^{-\alpha+q/2+l}}{\omega_p}\times\int_{\mathbb{B}^p}\frac{(\vec{x}_1-\vec{u})(\partial_{\vec{u}})^n[(1+\vec{u}^{\,2})^{(\alpha+q/2+l-1+n)}P_{k,l}(\vec{u},\vec{\omega}_2)]}{|\vec{x}_1-\vec{u}|^{m+2l}}\,d^p\vec{u},\qquad n=0,1,2,\ldots. \qquad (3.5)$$

We show now that the RHS of (3.3), (3.4) are proportional to the spherical monogenic $P_{k,l}(\vec{x}_1,\vec{\omega}_2)$ as implied by this definition (3.2) for $Q^{(\alpha)}_{n;p,q;k,l}(\vec{x}_1)$. We may use Cauchy's theorem [10] to integrate by parts n times, noting that the contributions from the surface of $\mathbb{B}^p$ are zero for $\alpha>0$ when n, l, $q\in\mathcal{N}$. Therefore

$$\Theta^{(\alpha)}_{n;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2)=\frac{[-\vec{x}_1^{\,2}-1]^{-\alpha+q/2+l}(-)^n}{\omega_p}\times\int_{\mathbb{B}^p}\left\{\left[\frac{(\vec{x}_1-\vec{u})}{|\vec{x}_1-\vec{u}|^{m+2l}}\right](\partial_{\vec{u}})^n\right\}(1+\vec{u}^{\,2})^{\alpha+q/2+l-1+n}P_{k,l}(\vec{u},\vec{\omega}_2)\,d^p\vec{u}. \qquad (3.6)$$

Now it may be proved by induction that

$$\left\{\left[\frac{(\vec{x}_1-\vec{u})}{|\vec{x}_1-\vec{u}|^{m+2l}}\right](\partial_{\overleftarrow{u}})^{2j}\right\} = (-4)^j(m/2+l)_j(l+q/2)_j \frac{[\vec{x}_1-\vec{u}]}{|\vec{x}_1-\vec{u}|^{m+2l+2j}} \tag{3.7}$$

$$\left\{\left[\frac{(\vec{x}_1-\vec{u})}{|\vec{x}_1-\vec{u}|^{m+2l}}\right](\partial_{\overleftarrow{u}})^{2j+1}\right\} = \frac{1}{2}(-4)^{j+1}(m/2+l)_j(q/2+l)_{j+1}\frac{1}{|\vec{x}_1-\vec{u}|^{m+2l+2j}} \tag{3.8}$$

for $j = 0, 1, 2, \ldots$. Substituting these expressions in (3.6) and using the result in (3.3), (3.4),

$$\begin{aligned} Q^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2) &= \frac{(1/2)_j(\alpha+m/2+k+l-1)_j 2^{2j}(m/2+l)_j(-\vec{x}_1^{\,2}-1)^{-\alpha+l+q/2}}{(\alpha+q/2+l+j)_j\Gamma(2j+1)\omega_p} \\ &\quad\times \Phi^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2) \end{aligned} \tag{3.9}$$

$$\begin{aligned} &Q^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2) \\ &= \frac{-(1/2)_{j+1}(\alpha+m/2+k+l-1)_{j+1}2^{2j+1}(m/2+l)_j(-\vec{x}^{\,2}{}_1-1)^{-\alpha+l+q/2}}{(\alpha+q/2+l+j)_{j+1}\Gamma(2j+2)\omega_p} \\ &\quad\times \Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} &\Phi^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2) \\ &\equiv \int_{\mathbb{B}^p} \frac{[\vec{x}_1-\vec{u}](1+\vec{u}^{\,2})^{\alpha+q/2+l-1+2j}}{|\vec{x}_1-\vec{u}|^{m+2l+2j}} P_{k,l}(\vec{u},\vec{\omega}_2)\,d^p\vec{u} \end{aligned} \tag{3.11}$$

$$\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1,\vec{\omega}_2) \equiv \int_{\mathbb{B}^p} \frac{(1+\vec{u}^{\,2})^{\alpha+q/2+l+2j}}{|\vec{x}_1-\vec{u}|^{m+;2l+2j}} P_{k,l}(\vec{u},\vec{\omega}_2)\,d^p\vec{u}. \tag{3.12}$$

The angular integrations in these integrals over $\mathbb{B}^p$ may be performed using the Funk–Hecke theorem [9] since $P_{k,l}(\vec{u},\vec{\omega}_2)$ and $\vec{u}P_{k,l}(\vec{u},\vec{\omega}_2)$ are spherical harmonics of orders k and $k+1$, respectively, in $\vec{u}$, while the denominator of the integrand depends only on $\vec{u}$ through its magnitude and scalar product with $\vec{x}$.

The result is that the RHS of (3.11), (3.12) are proportional to $P_{k,l}(\vec{x_1}, \vec{\omega_2})$ as required and specifically

$$\Phi^{(\alpha)}_{2j;p,q;k,l}(\vec{x_1})P_{k,l}(\vec{x_1}, \vec{\omega_2}) = \vec{\omega_1}\omega_{p-1}\int_0^1 (1-\lambda^2)^{\alpha+q/2+l-1+2j}\lambda^{p-1+k} \times\left[\int_{-1}^1 \frac{[\mathbb{P}_{k,p}(s)\rho_1 - \mathbb{P}_{k+1,p}(s)\lambda](1-s^2)^{(p-3)/2}\,ds}{(a^2-2bs)^{m/2+l+j}}\right] d\lambda P_{k,l}(\vec{\omega_1}, \vec{\omega_2}), \quad (3.13)$$

$$\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1})P_{k,l}(\vec{x_1}, \vec{\omega_2}) = \omega_{p-1}\int_0^1 (1-\lambda^2)^{\alpha+q/2+l+2j}\lambda^{p-1+k} \times\left[\int_{-1}^1 \frac{\mathbb{P}_{k,p}(s)(1-s^2)^{(p-3)/2}\,ds}{(a^2-2bs)^{m/2+l+j}}\right] d\lambda P_{k,l}(\vec{\omega_1}, \vec{\omega_2}), \quad (3.14)$$

where $a^2 = \rho_1^2 + \lambda^2$, $b = \lambda\rho_1$, and $p, q, k, l, j \in \mathcal{N}$. Also $\mathbb{P}_{k,p}(s)$ is the Legendre polynomial of order k in p-dimensions.

4. The Differential Equation Satisfied by the Bi-axial Gegenbauer Functions of the Second Kind

We state the result giving the differential equation satisfied by the bi-axial Gegenbauer functions of the second kind.

THEOREM. *The bi-axial Gegenbauer function* $Q^{(\alpha)}_{n,p,q;k,l}(\vec{x_1})P_{k,l}(\vec{x_1}, \vec{\omega_2})$ *defined by* (3.2) *for* $\alpha > 0, p, q, k, l \in \mathcal{N}$, *satisfies the differential equation*

$$(-\vec{x_1}^2 - 1)^{l+q/2-\alpha}\,\partial_{\vec{x_1}}\left\{(-\vec{x_1}^2 - 1)^{\alpha-l-q/2+1}\,\partial_{\vec{x_1}}[Q^{(\alpha)}_{n;p,q;k,l}(\vec{x_1})P_{k,l}(\vec{x_1}, \vec{\omega_2})]\right\} = \gamma(p,q,k,l,n)Q^{(\alpha)}_{n,p,q;k,l}(\vec{x_1})P_{k,l}(\vec{x_1}, \vec{\omega_2}), \qquad n = 0, 1, 2, \ldots, \quad (4.1)$$

where

$$\gamma(p,q,k,l,2j) = -(2\alpha + 2j)(m + 2l + 2k + 2j - 2) \quad (4.2)$$

and

$$\gamma(p,q,k,l,2j+1) = -(2\alpha + p + 2k + 2j)(q + 2l + 2j) \quad (4.3)$$

for $j = 0, 1, 2, \ldots$, *and* $\vec{x_1} \in \mathbb{R}^p \backslash \mathbb{B}^p$.

Proof. We consider first the case when $n = 2j + 1$ as this is slightly easier to deal with. We have from (3.12) for the given values of $\vec{x}_1$ that

$$\partial_{\vec{x}_1}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)] = -(m + 2l + 2j) \times \int_{\mathbb{B}^p} \frac{(\vec{x}_1 - \vec{u})(1 + \vec{u}^2)^{\alpha+q/2+l+2j}}{|\vec{x}_1 - \vec{u}|^{m+2l+2j+2}} P_{k,l}(\vec{u}, \vec{\omega}_2)\, d^p\vec{u} \tag{4.4}$$

and

$$\begin{aligned}(-\vec{x}_1^2 - 1)\, \partial^2_{\vec{x}_1} [\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)] \\ = (m + 2l + 2j)(q + 2l + 2j + 2) \\ \times\left\{\int_{\mathbb{B}^p} \frac{(1 + \vec{u}^2)^{\alpha+q/2+l+2j+1} P_{k,l}(\vec{u}, \vec{\omega}_2)\, d^p\vec{u}}{|\vec{x}_1 - \vec{u}|^{m+2l+2j+2}}\right. \\ \left. + \int_{\mathbb{B}^p} \frac{(\vec{x}^2 - \vec{u}^2)(1 + \vec{u}^2)^{\alpha+q/2+l+2j} P_{k,l}(\vec{u}, \vec{\omega}_2)\, d^p\vec{u}}{|\vec{x}_1 - \vec{u}|^{m+2l+2j+2}}\right\}.\end{aligned} \tag{4.5}$$

We make use of the identify

$$\begin{aligned}0 = \int_{\mathbb{B}^p} \left\{\left[\frac{(\vec{x}_1 - \vec{u})}{|\vec{x}_1 - \vec{u}|^{m+2l+2j+2}}\right] \partial_{\overleftarrow{u}}\right\} (1 + \vec{u}^2)^{\alpha+q/2+l+2j+1} P_{k,l}(\vec{u}, \vec{\omega}_2)\, d^p\vec{u} \\ + \int_{\mathbb{B}^p} \frac{(\vec{x}_1 - \vec{u})}{|\vec{x}_1 - \vec{u}|^{m+2l+2j+2}} \left\{\partial_{\vec{u}}\left[(1 + \vec{u}^2)^{\alpha+q/2+l+2j+1} P_{k,l}(\vec{u}, \vec{\omega}_2)\right]\right\} d^p\vec{u},\end{aligned} \tag{4.6}$$

which follows from Cauchy's theorem [10] since the integrand is zero on the surface of $\mathbb{B}^p$. Hence

$$\begin{aligned}-(q + 2l + 2j + 2) \int_{\mathbb{B}^p} \frac{(1 + \vec{u}^2)^{\alpha+q/2+l+2j+1}}{|\vec{x}_1 - \vec{u}|^{m+2l+2j+2}} P_{k,l}(\vec{u}, \vec{\omega}_2)\, d^p\vec{u} \\ = (2\alpha + q + 2l + 4j + 2)\left\{\int_{\mathbb{B}^p} \frac{(\vec{x}_1^2 - \vec{u}^2)(1 + \vec{u}^2)^{\alpha+q/2+l+2j} P_{k,l}(\vec{u}, \vec{\omega}_2)\, d^p\vec{u}}{|\vec{x}_1 - \vec{u}|^{m+2l+2j+2}}\right. \\ \left. - \vec{x}_1 \int_{\mathbb{B}^p} \frac{(\vec{x}_1 - \vec{u})(1 + \vec{u}^2)^{\alpha+q/2+l+2j} P_{k,l}(\vec{u}, \vec{\omega}_2)\, d^p\vec{u}}{|\vec{x}_1 - \vec{u}|^{m+2l+2j+2}}\right\}.\end{aligned} \tag{4.7}$$

But

$$\int_{\mathbb{B}^p} \frac{(\vec{x_1}^2 - \vec{u}^2)(1+\vec{u}^2)^{\alpha+q/2+l+2j} P_{k,l}(\vec{u}, \vec{\omega_2})\, d^p\vec{u}}{|\vec{x_1} - \vec{u}|^{m+2l+2j+2}}$$

$$\begin{aligned} &= -\omega_{p-1} \int_0^1 \lambda^{p+k-1}(1-\lambda^2)^{\alpha+q/2+l+2j}(\rho_1^2 - \lambda^2) \\ &\quad \times \left[\int_{-1}^{+1} \frac{\mathbb{P}_{k,p}(s)(1-s^2)^{(p-3)/2}\, ds}{[\rho_1^2 + \lambda^2 - 2\lambda\rho_1 s]^{m/2+l+j+1}}\right] d\lambda P_{k,l}(\vec{\omega_1}, \vec{\omega_2}) \end{aligned} \tag{4.8}$$

$$\begin{aligned} &= \Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1}) P_{k,l}(\vec{x_1}, \vec{\omega_2}) \\ &\quad + \frac{2\rho_1}{(m+2l+2j)} \frac{\partial}{\partial\rho_1}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1}) P_{k,l}(\vec{x_1}, \vec{\omega_2})]. \end{aligned} \tag{4.9}$$

Therefore

$$[-\vec{x_1}^2 - 1]\partial^2_{\vec{x_1}}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1}) P_{k,l}(\vec{x_1}, \vec{\omega_2})]$$

$$\begin{aligned} &= -(m+2l+2j)(2\alpha+2j) \times \left\{\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1}) P_{k,l}(\vec{x_1}, \vec{\omega_2})\right. \\ &\quad \left. + \frac{2\rho_1}{(m+2l+2j)} \frac{\partial}{\partial\rho_1}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1}) P_{k,l}(\vec{x_1}, \vec{\omega_2})]\right\} \\ &\quad - (2\alpha+q+2l+4j+2)\vec{x_1}\,\partial_{\vec{x_1}}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1}) P_{k,l}(\vec{x_1}, \vec{\omega_2})]. \end{aligned} \tag{4.10}$$

Now

$$\partial_{\vec{x_1}} = \vec{\omega_1}\left(\frac{\partial}{\partial\rho_1} + \frac{1}{\rho_1}\Gamma_{\vec{x_1}}\right), \qquad \Gamma_{\vec{x_1}} = \vec{x_1} \wedge \partial_{\vec{x_1}}, \tag{4.11}$$

and $\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1})$ is a scalar depending only on $|\vec{x_1}|$ as follows from (3.14). Therefore

$$\begin{aligned} &\rho_1 \frac{\partial}{\partial\rho_1}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1}) P_{k,l}(\vec{x_1}, \vec{\omega_2})] \\ &\qquad = -\vec{x_1}\,\partial_{\vec{x_1}}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1}) P_{k,l}(\vec{x_1}, \vec{\omega_2})] \\ &\qquad\quad + k\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x_1}) P_{k,l}(\vec{x_1}, \vec{\omega_2}) \end{aligned} \tag{4.12}$$

as

$$\Gamma_{\vec{x_1}} P_{k,l}(\vec{x_1}, \vec{\omega_2}) = -kP_{k,l}(\vec{x_1}, \vec{\omega_2}). \tag{4.13}$$

Substituting from (4.12) in (4.10)

$$
\begin{aligned}
&(-\vec{x}_1^{\,2} - 1)\partial^2_{\vec{x}_1}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)] \\
&\quad - [2\alpha - q - 2l - 2]\vec{x}_1\partial_{\vec{x}_1}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)] \\
&\quad + (2\alpha + 2j)(m + 2l + 2k + 2j)\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2) = 0.
\end{aligned} \tag{4.14}
$$

Let us make the definition

$$
\Psi^{(\alpha)}_{n;p,q;k,l}(\vec{x}_1) = (-\vec{x}_1^{\,2} - 1)^{-\alpha + l + q/2}\Phi^{(\alpha)}_{n1;p,q;k,l}(\vec{x}_1). \tag{4.15}
$$

Then

$$
\begin{aligned}
&(-\vec{x}_1^{\,2} - 1)^{-\alpha + l + q/2}\partial_{\vec{x}_1}\{(-1 - \vec{x}_1^{\,2})^{\alpha - l - q/2 + 1}\partial_{\vec{x}_1}[\Psi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)]\} \\
&\quad = (-\vec{x}_1^{\,2} - 1)^{-\alpha + l + q/2}\partial_{\vec{x}_1}\{-2(\alpha - l - q/2)\vec{x}_1 \\
&\qquad \times [\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)] \\
&\qquad + (-1 - \vec{x}_1^{\,2})\partial_{\vec{x}_1}[\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)]\}.
\end{aligned} \tag{4.16}
$$

Using the fact that for any scalar function $f(\rho_1)$

$$
\begin{aligned}
&\partial_{\vec{x}_1}[\vec{x}_1 f(\rho_1)P_k(\vec{x}_1, \vec{\omega}_2)] \\
&\quad = -(p + 2k)f(\rho_1)P_k(\vec{x}_1, \vec{\omega}_2) + \vec{x}_1\partial_{\vec{x}_1}[f(\rho_1)P_k(\vec{x}_1, \vec{\omega}_2)]
\end{aligned} \tag{4.17}
$$

and (4.14), it follows from (4.16) that

$$
\begin{aligned}
&(-\vec{x}_1^{\,2} - 1)^{-\alpha + l + q/2}\partial_{\vec{x}_1}\{(-\vec{x}_1^{\,2} - 1)^{\alpha - l - q/2 + 1}\partial_{\vec{x}_1}[\Psi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)]\} \\
&\quad = \{2(p + 2k)(\alpha - l - q/2) - 2(\alpha + j)(m + 2l + 2k + 2j)\} \\
&\qquad \times \Psi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)
\end{aligned} \tag{4.18}
$$

since $\Phi^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)$ is a scalar valued function of ρ_1. Finally $Q^{(\alpha)}_{2j+1;p,q;k,l}(\vec{x}_1)$ is just proportional to $\Psi^{(\alpha)}_{2j-1;p,q;k,l}(\vec{x}_1)$ so (4.1) follows immediately when n is odd.

To prove (4.1) for even n, we note from (3.11), (3.12) that

$$
\begin{aligned}
&\Phi^{(\alpha)}_{2j+2;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2) \\
&\quad = -\frac{1}{(m + 2l + 2j)}\partial_{\vec{x}_1}[\Phi^{(\alpha+1)}_{2j+1;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)].
\end{aligned} \tag{4.19}
$$

Differentiating (4.14) with $\alpha \to \alpha + 1$,

$$\begin{aligned}\partial_{\vec{x}_1}\{(-\vec{x}_1^{\,2} - 1)\partial_{\vec{x}_1}[\Phi^{(\alpha)}_{2j+2;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)]\\ - (2\alpha - q - 2l)\vec{x}_1\Phi^{(\alpha)}_{2j+2;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)\}\\ + (2\alpha + 2j + 2)(m + 2l + 2k + 2j)\Phi^{(\alpha)}_{2j+2;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2) = 0.\end{aligned} \tag{4.20}$$

Using the fact that $\Phi^{(\alpha)}_{2j+2;p,q;k,l}(\vec{x}_1)$ is equal to $\vec{x}_1$ multiplied into a scalar valued function of ρ_1, it is straightforward to show that

$$\begin{aligned}\partial_{\vec{x}_1}\{\vec{x}_1\Phi^{(\alpha)}_{2j+2;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)\}\\ = \vec{x}_1\partial_{\vec{x}_1}\{\Phi^{(\alpha)}_{2j+2;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)\}\\ + (p + 2k - 2)\Phi^{(\alpha)}_{2j+2;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2).\end{aligned} \tag{4.21}$$

Substituting in (4.20) and setting $j + 2 \to j$, it may be deduced that

$$\begin{aligned}[-(\vec{x}_1^{\,2} + 1)]^{l+q/2-\alpha}\partial_{\vec{x}_1}\{[-(\vec{x}_1^{\,2} + 1)]^{\alpha-l-q/2+1}\partial_{\vec{x}_1}[\Psi^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2)]\}\\ = -(2\alpha + 2j)(m + 2l + 2k + 2j - 2)\Psi^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)P_{k,l}(\vec{x}_1, \vec{\omega}_2),\end{aligned} \tag{4.22}$$

where $\Psi^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)$ is defined in (4.15) and is proportional to $Q^{(\alpha)}_{2j;p,q;k,l}(\vec{x}_1)$. Thus we obtain the differential equation (4.1) for even n. ∎

In the complex scalar case the Gegenbauer functions of the second kind satisfy the same differential equation as the corresponding Gegenbauer polynomials. In the bi-axial case the two sets of differential equations (2.8) and (4.1) are related but not identical. However, from their definitions (2.3), (2.4) and (4.2), (4.3), respectively,

$$\beta(n, \alpha + q/2 + l - 1, k) = -\gamma(p, q, k, l, n) \tag{4.23}$$

for $p = q = 1$, $k = l = 0$ and in this case the equations satisfied by the bi-axial Gegenbauer polynomials are the same as those for the bi-axial Gegenbauer functions of the second kind. Therefore our results do agree with the standard complex variable case since the latter corresponds to taking $p = q = 1$.

Acknowledgments

I thank Frank Sommen for introducing me to the concept of bi-axial monogenic functions and for a fruitful collaboration on the subject.

REFERENCES

1. F. Sommen, Plane elliptic systems and monogenic functions in symmetric domains, *Rend. Circ. Mat. Palermo Suppl.* (2) **6** (1984), 259–269.
2. G. Jank and F. Sommen, Clifford analysis, bi-axial symmetry and pseudo-analytic functions, *Complex Variable* **13** (1990), 195–212.
3. F. Sommen, Special functions in Clifford analysis and axial symmetry, *J. Math. Anal. Appl.* **130** (1988), 100–133.
4. J. Cnops, Relations between orthogonal polynomials in one and several variables, *Bull. Soc. Math. Belg.* **43** (1991), 1–17.
5. J. Cnops, Orthogonal polynomials associated with the Dirac operator in Euclidean space, *Chinese Ann. Math. Ser. B* **13,** No. 1 (1992), 68–79.
6. A. K. Common and F. Sommen, Special bi-axial monogenic functions, *J. Math. Anal. Appl. 185* (1994), 189–206.
7. S. Szego, "Orthogonal Polynomials," Vol. XXIII, AMS, New York, 1939.
8. A. K. Common and F. Sommen, Cauchy transforms and bi-axial monogenic power functions, "Clifford Algebras and their Application in Mathematical Physics" (F. Brackx, R. Delanghe and H. Serras, Eds.), pp. 85–90. Kluwer, Boston, 1993.
9. H. Hochstadt, "The Functions of Mathematical Physics," Pure and Applied Mathematics, Vol. 23, Wiley–Interscience, New York, 1971.
10. F. Brackx, R. Delanghe, and F. Sommen, "Clifford Analysis," Research Notes in Mathematics, Vol. 76 Pitman, London, 1982.