On optimality conditions of relaxed non-convex variational problems

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Abstract

Non-convex variational problems in many situations lack a classical solution. Still they can be solved in a generalized sense, e.g., they can be relaxed by means of Young measures. Various sets of optimality conditions of the relaxed non-convex variational problems can be introduced. For example, the so-called “variations” of Young measures lead to a set of optimality conditions, or the Weierstrass maximum principle can be the base of another set of optimality conditions. Moreover the second order necessary and sufficient optimality conditions can be derived from the geometry of the relaxed problem. In this article the sets of optimality conditions are compared. Illustrative examples are included.

Keywords: Young measures; Weierstrass maximum principle; Second order optimality conditions

Introduction

Non-convex optimization problems (e.g., optimal control, variational calculus problems, and non-cooperative games), except special situations (cf. Refs. [1–6]), have no classical solution. This so-called non-attainment usually happens due to oscillations of approximate solutions. Then it is suitable to redefine the problem and solve it in a general-
ized sense. Generally speaking, there are two possibilities. The minimized functional can be substituted by its weak lower semi-continuous envelope, which requires a convexification of the involved integrand. This convexification may be difficult to obtain analytically. The numerical approach to the convexification can be found in Refs. [7,8]. The other possibility is to extend the original problem continuously on a hull of the original function spaces (cf. Refs. [9–15]). This article concentrates on the relaxation done by means of so-called Young measures. This relaxation is a special case of the extension of the original problem mentioned above.

A set of necessary optimality conditions for the relaxed problem can be obtained using the so-called “variations” of Young measures, if the dimension of the variational problem is one (cf. Refs. [16,17]). Another set of necessary optimality conditions can be derived directly from the geometry of the relaxed problem. The set then contains a condition known as the Weierstrass maximum principle (cf. Ref. [15]). It is shown in this article that the second set of conditions is more selective than the first set, i.e., the first set of conditions is always satisfied, if the second set of conditions is satisfied. An analytically solvable example is included to illustrate this evidence. Another interesting set of necessary optimality conditions can be found in Ref. [18]. We prove that this set of optimality conditions is equivalent to the set of optimality conditions containing the Weierstrass maximum principle.

Second order necessary optimality conditions for the relaxed problem can be obtained by application of the general form of second order necessary optimality conditions derived for the optimization problems in a Banach space (cf. Ref. [19, Section 3.2.2]).

A set of sufficient conditions for existence of a local extreme is presented at the end of the article. This set can be obtained by completing the set of optimality conditions containing the Weierstrass maximum principle by a suitable second order condition.

Examples of the application of the second order conditions to non-convex relaxed problems are included.

1. Relaxation of the non-convex variational problem

The class of non-convex variational problems studied in the sequel is

$$\text{minimize } \Phi(u) := \int_{\Omega} F(x, u(x), \nabla u(x)) \, dx,$$

where $u \in W^{1,p}(\Omega)$, $u|_{\partial\Omega} = u_D$, (1)

and $W^{1,p}(\Omega)$ is a Sobolev space, so it is natural that $u_D$ belongs to a Sobolev space $W^{1-1/p,p}(\partial\Omega)$, provided $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. It is assumed that $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying inequalities

$$c_1 |s|^p - c_2 |u|^p_1 - C \leq F(x, u, s) \leq a_0(x) + C(|u|^q + |s|^p), \quad (2)$$

$$|F(x, u_1, s) - F(x, u_2, s)| \leq (a_1(x) + b |u_1|^{q-1} + b |u_2|^{q-1} + b |s|^{p(a-1)/q}) |u_1 - u_2| \quad (3)$$
for some \( c_1 > 0, c_2, C \in \mathbb{R}, p > 1, p_1 < p, p_1 < q, a_0 \in L^1(\Omega), a_1 \in L^{q/(q-1)}(\Omega), b > 0, 1 \leq q \leq pn/(n - p) \) for \( 1 < p < n, 1 \leq q < \infty \) for \( p \geq n \). Under these assumptions \( \Phi \) is coercive and continuous. Non-convexity of \( F(x, u, \cdot) \) can cause that \( \Phi \) is not weakly lower semi-continuous, which is related with possible non-existence of a solution due to oscillations of gradients of minimizing sequences (cf. Ref. [4]). To obtain a generalized solution, the problem can be extended by means of \( L^p \)-Young measures. The relaxed problem is

\[
\begin{align*}
\text{minimize } & \tilde{\Phi}(u, \nu) := \int\int_{\Omega \times \mathbb{R}^n} F(x, u(x), s) \nu_x(ds) dx, \\
\text{subject to } & \int_{\mathbb{R}^n}s \nu_x(ds) = \nabla u(x) \quad \text{for a.a. } x \in \Omega, \\
& u \in W^{1,p}(\Omega), \quad \nu \in \mathcal{YP}(\Omega; \mathbb{R}^n), \quad u|_{\partial \Omega} = u_D.
\end{align*}
\]

The set of \( L^p \)-Young measures is defined as follows:

\[
\mathcal{YP}(\Omega; \mathbb{R}^n) := \left\{ \nu = \{v_x\}_{x \in \Omega} \in L^\infty(\Omega; \text{rc}a(\mathbb{R}^n)); \ v_x \in \text{rc}a_1^+(\mathbb{R}^n) \text{ for a.a. } x \in \Omega, \right. \\
& \left. \int\int_{\Omega \times \mathbb{R}^n} |s|^p v_x(ds) dx < \infty \right\},
\]

where \( \text{rc}a(\mathbb{R}^n) \) stands for Radon measures, \( \text{rc}a_1^+(\mathbb{R}^n) \) denotes the set of probability Radon measures, and \( L^\infty(\Omega; \text{rc}a(\mathbb{R}^n)) \) is a Bochner space of bounded weakly measurable mappings from \( \Omega \) into \( \text{rc}a(\mathbb{R}^n) \). The set of \( L^p \)-Young measures forms a convex \( \sigma \)-compact hull of \( L^p(\Omega) \) thanks to a dense and (norm, weak\( \ast \))-continuous embedding \( y \mapsto \nu := \{\delta_{y(x)}\}_{x \in \Omega} \), where \( \delta_s \) means a Dirac measure supported at \( s \in \mathbb{R}^n \) (cf. Ref. [15]). The functional \( \tilde{\Phi} \) in problem (4) can be convex even for non-convex \( F \). Problem (4) then becomes minimization of a convex functional over a convex set.

**Theorem 1** (Kinderlehrer and Pedregal [20], here modified). *Assumptions (2), (3) imply*

(i) problem (4) has a solution,

(ii) \( \inf(1) = \min(4) \),

(iii) every minimizing sequence of the problem (1) contains a subsequence, whose embedding \( u \mapsto (u, \{\delta_{\nabla u(x)}\}_{x \in \Omega}) \) converges to a solution of the problem (4),

(iv) conversely, any solution to the problem (4) can be attained by a minimizing sequence of the problem (1).

Existence can be shown by coercivity and continuity arguments (cf. Refs. [15,20,21]). The solution in \( L^p \)-Young measures exists only for \( p > 1 \), as assumed (cf. Eq. (2)) otherwise the concentration of energy cannot be avoided in the sense that the set \( \{x \mapsto F(x, u_i(x), \nabla u_i(x)) : i \in \mathbb{N} \} \) is not relatively weakly compact in \( L^1(\Omega) \), provided \( \{u_i\}_{i \in \mathbb{N}} \) is the minimizing sequence of the problem (1). For a more detailed proof cf. Ref. [20].
2. Optimality conditions of the first and the second order

The optimality conditions of the problem (4) can be formulated if the function $F$ is regular enough. Regularity of $F$ is the subject of these lemmas.

**Lemma 2.** Let $q \geq 3$, $p \geq 3$, $F(x, \cdot, s) \in C^2(\mathbb{R})$, and let there exist constants $b, C \in \mathbb{R}$ and functions $a_1 \in L^{q/(q-1)}(\Omega)$, $a_2 \in L^{q/(q-2)}(\Omega)$, $a_3 \in L^{q/(q-3)}(\Omega)$ such that

$$
\left| \frac{\partial F}{\partial u}(x, u, s) \right| \leq a_1(x) + C\left(|u|^{q-1} + |s|^{p(q-1)/q}\right),
$$

$$
\left| \frac{\partial^2 F}{\partial u^2}(x, u, s) \right| \leq a_2(x) + C\left(|u|^{q-2} + |s|^{p(q-2)/q}\right),
$$

$$
\left| \frac{\partial^2 F}{\partial u^2}(x, u_1, s) - \frac{\partial^2 F}{\partial u^2}(x, u_2, s) \right| \leq (a_3(x) + b|u_1|^{q-3} + b|u_2|^{q-3} + b|s|^{p(q-3)/q})|u_1 - u_2|.
$$

Let assumption (2) hold, too. Then the functional $N_\nu : L^q(\Omega) \to \mathbb{R}$, defined by $N_\nu(u) := \tilde{\Phi}(u, \nu)$, is twice continuously differentiable.

**Proof.** The functional $N_\nu'(u) : L^q(\Omega) \to \mathbb{R}$ defined by

$$
N_\nu'(u)[v] := \int_\Omega \int_{\mathbb{R}^n} \frac{\partial F(x, u(x), s)}{\partial u} v(x) \nu_x(ds) dx
$$

is the Gâteaux differential of $N_\nu$ at $u$. Indeed, by definition, it is enough to show that

$$
\lim_{t \to 0} \frac{N_\nu(u + tv) - N_\nu(u)}{t} = N_\nu'(u)[v].
$$

The differentiability of $F$ together with the mean value theorem assure that for each $u, v \in L^q(\Omega)$, $t \in \mathbb{R}$, there exists $t^* \in \mathbb{R}$ such that $0 < t^* < t$, and

$$
\frac{F(x, u(x) + tv(x), s) - F(x, u(x), s)}{t} = \frac{\partial F(x, u(x) + t^*v(x), s)}{\partial u} v(x).
$$

Then it follows from the assumption (5) and Hölder’s inequality that

$$
\int_\Omega \int_{\mathbb{R}^n} \frac{F(x, u(x) + tv(x), s) - F(x, u(x), s)}{t} v_x(ds) dx
$$

$$
\leq \left[ \|a_1\|_{L^{q/(q-1)}(\Omega)} + C\left(\|u + t^*v\|_{L^q(\Omega)}^{q-1} + \int_{\mathbb{R}^n} |s|^{p(q-1)/q} v_x(ds) \right)\right]
\times \|v\|_{L^q(\Omega)}.
$$

The definition of $L^p$-Young measures and Jensen’s inequality lead to the inequality

$$
\int_{\mathbb{R}^n} |s|^{p(q-1)/q} v_x(ds) \leq \int_{\mathbb{R}^n} |s|^p v_x(ds),
$$
thus
\[
\left\| \int_{\mathbb{R}^n} |s|^{p(q-1)/q} v_s(x) \, ds \right\|_{L^{q/(q-1)}(\Omega)} \leq \left( \int_{\Omega} \int_{\mathbb{R}^n} |s|^p v_s(x) \, dx \right)^{(q-1)/q} < \infty.
\]

Then Eq. (8) follows from the Lebesgue’s dominated convergence theorem. Similar arguments based on assumption (6) assure that the functional
\[ N''_\nu (u) : L^q(\Omega) \times L^q(\Omega) \rightarrow \mathbb{R} \]
defined by
\[ N''_\nu (u)[v, w] := \int_{\Omega} \int_{\mathbb{R}^n} \frac{\partial^2 F(x, u(x), s)}{\partial u^2} v(x) w(x) v_s(x) \, dx \]
is the second Gâteaux differential of \( N_\nu \) at \( u \). From the assumption (7) it follows that the mapping \( u \mapsto N''_\nu (u) \) is continuous. It holds that
\[
\left\| \frac{\partial F}{\partial u}(x, u_1(x), s) - \frac{\partial F}{\partial u}(x, u_2(x), s) \right\| \leq \int_0^1 \frac{\partial^2 F(x, tu_1 + (1-t)u_2, s)}{\partial u^2} \, dt |u_1 - u_2|,
\]
which is sufficient for continuity of the mapping \( u \mapsto N''_\nu (u) \) (cf. Eq. (6)). Continuity of \( N_\nu \) can be proved in the same manner as the continuity of the mapping \( u \mapsto N'_\nu (u) \).

**Lemma 3.** Let \( n = 1, p \geq 2, q \geq 2, F(x, \cdot, \cdot) \in C^2(\mathbb{R} \times \mathbb{R}^n) \), and let there exist constants \( b \in \mathbb{R}, c \in \mathbb{R} \) and functions \( a_4 \in L^{p/(p-1)}(\Omega), a_5 \in L^{p/(p-2)}(\Omega) \) such that
\[
\left| \frac{\partial F}{\partial s}(x, u, s) \right| \leq a_4(x) + C \left( |u|^{p-1} + |s|^{p-1} \right), \tag{9}
\]
\[
\left| \frac{\partial^2 F}{\partial s^2}(x, u, s) \right| \leq a_5(x) + C \left( |u|^{p-2} + |s|^{p-2} \right), \tag{10}
\]
\[
\left| \frac{\partial^2 F}{\partial u \partial s}(x, u, s) \right| \leq a_5(x) + C \left( |u|^{p-2} + |s|^{p-2} \right). \tag{11}
\]
Let assumptions (2), (5), and (6) hold, too. Then the original functional \( \Phi : W^{1,p}(\Omega) \rightarrow \mathbb{R} \) (cf. Eq. (1)), is twice Gâteaux differentiable.

The proof of this lemma is similar to the proof of the previous lemma.

A set of the first and the second order optimality conditions can be derived using the “variations” of Young measures (cf. Ref. [17]). This method gives practically usable optimality conditions only for \( n = 1 \). The necessary optimality conditions of the problem (4) based on the first and second order optimality conditions obtained by the “variations” of Young measures are summarized in this theorem.
Theorem 4 (Pedregal [17]). Let \( n = 1 \). Let assumptions (2), (5), (6), (9)–(11) be satisfied. Let \((u, \nu)\) be a solution of the problem (4), and let

\[
\lambda(x) := \int_{\mathbb{R}} \frac{\partial F}{\partial s}(x, u(x), s) \nu_x(ds).
\]

Then

\[
\int_{\mathbb{R}} \frac{\partial F}{\partial u}(x, u(x), s) \nu_x(ds) = \frac{d\lambda}{dx}(x) \text{ in } W^{-1,p}(\Omega)
\]

and

\[
\text{supp} (\nu_x) \subset \left\{ s \in \mathbb{R}; \frac{\partial F(x, u(x), s)}{\partial s} = \lambda(x) \right\} \cap \left\{ s \in \mathbb{R}; \frac{\partial^2 F(x, u(x), s)}{\partial^2 s} \geq 0 \right\}
\]

for a.a. \( x \in \Omega \).

Another set of necessary optimality conditions of the relaxed problem (4) can be derived by means of the natural Young measure geometry and results to the Weierstrass maximum principle. The necessary optimality conditions derived directly from the relaxed problem (4) are the subject of the following theorem. It involves only the conditions of the first order, still the conditions included in this theorem turn out to be more selective than the conditions in the theorem above.

Theorem 5 (Roubíček [15]). Let assumptions (2) and (5) be satisfied, and let \((u, \nu)\) be a solution of the problem (4). Then there exists \( \lambda \in L^{p/(p-1)}(\Omega, \mathbb{R}^n) \) such that

\[
\int_{\mathbb{R}^n} \frac{\partial F}{\partial u}(x, u(x), s) \nu_x(ds) = \text{div} \lambda(x) \text{ in } W^{-1,p}(\Omega)
\]

and

\[
\max_{s \in \mathbb{R}^n} \{ \lambda(x) \cdot s - F(x, u(x), s) \} = \int_{\mathbb{R}^n} \{ \lambda(x) \cdot s - F(x, u(x), s) \} \nu_x(ds)
\]

for a.a. \( x \in \Omega \). Conversely, if \( \Phi \) is convex (cf. Eq. (4)), functions \((u, \nu)\) of \( W^{1,p}(\Omega) \times \mathcal{Y}^p(\Omega; \mathbb{R})\), \( u|_{\partial \Omega} = u_D\), \( \lambda \in L^{p/(p-1)}(\Omega)\), satisfy the conditions (15) and (16), and it holds that

\[
\int_{\mathbb{R}^n} s \nu_x(ds) = \nabla u(x),
\]

then \((u, \nu)\) solves the problem (4).

For \( n = 1 \) there exists a correspondence between the two sets of optimality conditions above. It is expressed in the following theorem.

Theorem 6. Let \( n = 1 \). Let assumptions (2), (5), (6), (9)–(11) be satisfied. Then the optimality conditions (15) and (16) are satisfied, only if conditions (13) and (14) are satisfied.
Proof. Let the conditions (15) and (16) be satisfied. If \( n = 1 \), and \( F \) is smooth, the condition (16) implies that the function \( \lambda \) from Theorem 5 must be equal to the function \( \lambda \) defined in Eq. (12). Then the condition (13) follows from the condition (15). If the condition (16) holds, then the validity of the condition (14) follows from the smoothness of \( F \). \( \square \)

Another set of necessary optimality conditions for the relaxed variational problems is the subject of this theorem. Let

\[
F'_s := \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial s_n} \right).
\]

Theorem 7 (Demoulini [18], here modified for stationary case). Let assumptions (2) and (5) be satisfied. Let \( F(x, u, \cdot) \in C^1(\mathbb{R}^n) \) for a.a. \( x \in \Omega \), and for all \( u \in \mathbb{R} \). Let \( (u, v) \) be a solution of the problem (4). Then it holds that

\[
\int_{\mathbb{R}^n} \frac{\partial F}{\partial u}(x, u(x), s) v_x(ds) = \text{div} \int_{\mathbb{R}^n} F'_s(x, u(x), s) v_x(ds), \quad \text{in} \ W^{-1,p}(\Omega), \quad (18)
\]

\[
\text{supp}(v_x) \subset \{ s \in \mathbb{R}^n; F(x, u(x), s) = F^{**}(x, u(x), s) \} \quad \text{a.e. in} \ \Omega, \quad (19)
\]

and

\[
\int_{\mathbb{R}^n} F'_s(x, u(x), s) \cdot s v_x(ds) = \int_{\mathbb{R}^n} F'_s(x, u(x), s) v_x(ds) \cdot \int_{\mathbb{R}^n} s v_x(ds), \quad \text{a.e. in} \ \Omega. \quad (20)
\]

The equivalence of the latter two sets of optimality conditions is formulated below.

Theorem 8. Let assumptions (2) and (5) be satisfied. Let \( F(x, u, \cdot) \in C^1(\mathbb{R}^n) \) for a.a. \( x \in \Omega \), and for all \( u \in \mathbb{R} \). Then the set of optimality conditions (15) and (16) is equivalent to the set of optimality conditions (18)–(20).

Proof. Let the conditions (15) and (16) be satisfied. For \( F \) smooth, the condition (16) implies that

\[
\lambda(x) = \int_{\mathbb{R}^n} F'_s(x, u(x), s) v_x(ds).
\]

The condition (18) then follows from the condition (15). It holds that

\[
\max_{s \in \mathbb{R}^n} \{ \lambda(x) \cdot s - F(x, s) \} = \max_{s \in \mathbb{R}^n} \{ \lambda(x) \cdot s - F^{**}(x, s) \}. \quad (21)
\]

As \( F^{**}(x, s) \leq F(x, s) \), the condition (19) then follows from the condition (16). As \( F \) is smooth, the condition (16) implies that

\[
F'_s(x, u(x), s) = \lambda(x) \quad \text{for a.a} \ x \in \Omega \ \text{and for all} \ s \in \text{supp}(v_x).
\]

Thus \( F'_s(x, u(x), s) \) is constant in \( s \) on \( \text{supp}(v_x) \), and the condition (20) is satisfied.
Conversely, let the conditions (19) and (20) be satisfied. Let
\[ y(x) := \int_{\mathbb{R}^n} s \nu_x(ds). \]
Equation (20) can be rewritten as follows:
\[ \int_{\mathbb{R}^n} F'_s(x, u(x), s) \cdot s \nu_x(ds) = \int_{\mathbb{R}^n} F'_s(x, u(x), s) \cdot y(x) \nu_x(ds). \]
The sum of this equality with the identity
\[ \int_{\mathbb{R}^n} [F^{**}']_s(x, u(x), y(x)) \cdot (s - y(x)) \nu_x(ds) = 0 \]
leads to
\[ \int_{\mathbb{R}^n} (F'_s(x, u(x), s) - [F^{**}']_s(x, u(x), y(x))) \cdot (s - y(x)) \nu_x(ds) = 0 \quad \text{a.e. in } \Omega. \]
As \( F(x, u, \cdot) \in C^1(\mathbb{R}^n) \) for a.a. \( x \in \Omega \), and for all \( u \in \mathbb{R} \), also \( F^{**}(x, u, \cdot) \in C^1(\mathbb{R}^n) \) for a.a. \( x \in \Omega \), and for all \( u \in \mathbb{R} \) (cf. Ref. [22]). Moreover \( F'_s(x, u(x), s) = [F^{**}']_s(x, u(x), s) \) for \( s \) such that \( F(x, u(x), s) = F^{**}(x, u(x), s) \), e.g., for \( s \in \text{supp}(\nu_x) \). Therefore the condition (19) together with the equality above gives
\[ \int_{\mathbb{R}^n} ([F^{**}']_s(x, u(x), y(x))) \cdot (s - y(x)) \nu_x(ds) = 0 \quad \text{a.e. in } \Omega. \quad (22) \]
The convexity of \( F^{**} \) implies
\[ ([F^{**}']_s(x, u(x), s) - [F^{**}']_s(x, u(x), y(x))) \cdot (s - y(x)) \geq 0 \]
for all \( s \in \text{supp}(\nu_x) \), a.e. in \( \Omega \).
Then Eq. (22) implies that
\[ [F^{**}']_s(x, u(x), s) = [F^{**}']_s(x, u(x), y(x)) \quad \text{for all } s \in \text{supp}(\nu_x), \text{ a.e. in } \Omega. \]
Let
\[ \lambda(x) := \int_{\mathbb{R}^n} F'_s(x, u(x), s) \nu_s(ds). \]
Then the condition (15) follows from Eq. (18). As
\[ F'_s(x, u(x), s) = [F^{**}']_s(x, u(x), s) = [F^{**}']_s(x, u(x), y(x)) \quad \text{for all } s \in \text{supp}(\nu_x), \text{ a.e. in } \Omega, \]
it holds that
\[ \lambda(x) = \int_{\mathbb{R}^n} \left[ F^{**}(x,u(x),s) v(x)(ds) \right] = [F^{**}]'(x,u(x),s) \nu_x(ds) = [F^{**}]'(x,u(x),s) \nu_x(ds) \]

for all \( s \in \text{supp}(\nu_x) \), a.e. in \( \Omega \).

This is a sufficient condition for the concave function \( s \mapsto \lambda(x) \cdot s - F^{**}(x,s) \) to reach its maximum on \( \text{supp}(\nu_x) \). Then Eq. (21) together with the condition (19) implies the validity of the condition (16). \( \square \)

If the relaxed problem (4) is non-convex, it is suitable to use the optimality conditions of the second order. Let \( U := \{ u \in W^{1,p}(\Omega); u|_{\partial\Omega} = u_D \} \), and let \( K := \{(u,v) \in U \times \mathcal{Y}_p(\Omega;\mathbb{R}^n); \int_{\mathbb{R}^n} \nu_x(ds) = \nabla u(x) \text{ a.e. in } \Omega \} \).

Let for \( (u,v) \in K \),

\[
C(u,v) := \left\{ (\hat{u},\hat{v}) \in U \times L^\infty(\Omega;\text{rca}(\mathbb{R}^n)); \text{ there exist } \mu > 0, \left( \hat{u},\hat{v} \right) \in K, \text{ such that } \left( \hat{u},\hat{v} \right) = \mu \left( (\hat{u},\hat{v}) - (u,v) \right), \text{ and } \right\}
\]

\[
\int_{\Omega} \left[ \int_{\mathbb{R}^n} \frac{\partial F}{\partial u}(x,u(x),s) \hat{u}(x) v_x(ds) + \int_{\mathbb{R}^n} F(x,u(x),s) \hat{v}_x(ds) \right] dx \leq 0
\]

be the so-called critical cone at \( (u,v) \). The necessary optimality condition of the second order can be formulated as follows.

**Theorem 9** (Bonnans and Shapiro [19, Section 3.2.2], here modified). Let assumptions (2), (5)–(7) be satisfied, and let \( (u,v) \) be a solution of the problem (4). Let \( C(u,v) \) be the critical cone at \( (u,v) \) as defined above. Then

\[
\int_{\Omega} \left[ \int_{\mathbb{R}^n} \frac{\partial^2 F}{\partial u^2}(x,u(x),s) \hat{u}(x)^2 v_x(ds) + 2 \int_{\mathbb{R}^n} \frac{\partial F}{\partial u}(x,u(x),s) \hat{u}(x) \hat{v}_x(ds) \right] dx \geq 0
\]

for all \( (\hat{u},\hat{v}) \in C(u,v) \). \( (23) \)

**Proof.** Fix \( (\hat{u},\hat{v}) \in C(u,v) \). Let \( (\tilde{u}(t),\tilde{v}(t)) := (u,v) + t(\hat{u},\hat{v}), \ t \geq 0 \). The definition of \( C(u,v) \) and the convexity of \( K \) assure that there exists \( t^* = 0 \) such that for all \( t \in (0,t^*) \) it holds that \( (\tilde{u}(t),\tilde{v}(t)) \in K \), thus \( \Phi(\tilde{u}(t),\tilde{v}(t)) \geq \Phi(u,v) \). Due to the regularity assured by Lemma 2, the following approximation holds:

\[
\Phi(\tilde{u}(t),\tilde{v}(t)) = \Phi(u,v) + t \int_{\mathbb{R}^n} \left[ \frac{\partial F}{\partial u}(x,u(x),s) \hat{u}(x) v_x(ds) + \int_{\mathbb{R}^n} F(x,u(x),s) \hat{v}_x(ds) \right] dx
\]
\[ + t^2 \int_{\Omega} \left[ \int_{\mathbb{R}^n} \frac{\partial^2 F}{\partial u^2}(x, u(x), s) \hat{u}(x) v_x(ds) \right] dx + o(t^2). \]

As \( \tilde{\Phi}(\tilde{u}(t), \tilde{\nu}(t)) \geq \Phi(u, \nu) \) on \((0, t^*)\), the equality above allows to claim that
\[ \int_{\Omega} \left[ \int_{\mathbb{R}^n} \frac{\partial^2 F}{\partial u^2}(x, u(x), s) \hat{u}(x) v_x(ds) + \int_{\mathbb{R}^n} F(x, u(x), s) \hat{\nu}_x(ds) \right] dx \geq 0. \]

As \((\hat{u}, \hat{\nu}) \in C(u, \nu)\), the inequality above turns to equality, and the approximation above leads to the condition (23).

Let for \((u, \nu) \in K\),
\[ B(u, r) := \{ \tilde{u} \in U; \| \tilde{u} - u \|_{W^{1, p}(\Omega)} \leq r \} \]
and
\[ Y_F(v, r) := \{ \tilde{\nu} \in Y^p(\Omega; \mathbb{R}^n); \text{ess sup}_{x \in \Omega} \left\| \frac{\partial^2 F}{\partial u^2}(x, u(x), s) [\tilde{\nu}_x - v_x](ds) \right\| \leq r \}. \]

The sufficient local optimality condition of the second order can be formulated in the following way.

**Theorem 10.** Let assumptions (2), (5)–(7) be satisfied, and let a couple \((u, \nu) \in W^{1, p}(\Omega) \times Y^p(\Omega; \mathbb{R}^n)\) satisfy the conditions (15)–(17) with some function \( \lambda \in L^{p/(p-1)}(\Omega; \mathbb{R}^n)\), and let it also satisfy the condition
\[ \int \frac{\partial^2 F}{\partial u^2}(x, u(x), s) v_x(ds) \geq \epsilon > 0 \text{ for a.a. } x \in \Omega. \] (24)

Then the couple \((u, \nu)\) is a local minimum of the problem (4), in the sense that there exists \( r > 0 \) such that for all \((\tilde{u}, \tilde{\nu}) \in B(u, r) \times Y_F(v, r)\) it holds that \( \tilde{\Phi}(\tilde{u}, \tilde{\nu}) \geq \Phi(u, \nu) \).

**Proof.** The condition (24) implies that there exists \( \tilde{r} > 0 \) such that for all \( \tilde{\nu} \in Y_F(v, \tilde{r})\) it holds that
\[ \int \frac{\partial^2 F}{\partial u^2}(x, u(x), s) \tilde{\nu}_x(ds) > 0 \text{ for a.a. } x \in \Omega. \] (25)

The regularity of \( \mathcal{N}_F\) assured by Lemma 2 together with the generalized Taylor’s theorem imply that there exists \( r > 0 \) such that \( r < \tilde{r} \) and for each couple \((\tilde{u}, \tilde{\nu}) \in B(u, r) \times Y_F(v, r) \cap K\) there exists a function \( u^* \in B(u, r) \) such that
\[
\tilde{\Phi}(\tilde{u}, \tilde{v}) = \Phi(u, v) + \int_{\Omega} \int_{\mathbb{R}^n} \frac{\partial F}{\partial u}(x, u(x), s)(\tilde{u}(x) - u(x)) \tilde{v}_s(ds)dx \\
+ \int_{\Omega} \int_{\mathbb{R}^n} \frac{\partial^2 F}{\partial u^2}(x, u(x), s)(u^*(x) - u(x))^2 \tilde{v}_s(ds)dx.
\]

As the condition (17) is satisfied and \((\tilde{u}, \tilde{v}) \in K\), it holds that
\[
\tilde{\Phi}(\tilde{u}, \tilde{v}) - \Phi(u, v) = \tilde{\Phi}(u, \tilde{v}) - \Phi(u, v) - \int_{\Omega} \int_{\mathbb{R}^n} \lambda(x) \cdot s[\tilde{v}_s - v_s](ds) \\
+ \int_{\Omega} \int_{\mathbb{R}^n} \frac{\partial F}{\partial u}(x, u(x), s)(\tilde{u}(x) - u(x)) \tilde{v}_s(ds)dx \\
+ \int_{\Omega} \int_{\mathbb{R}^n} \frac{\partial^2 F}{\partial u^2}(x, u(x), s)(u^*(x) - u(x))^2 \tilde{v}_s(ds)dx.
\]

The Green’s theorem with the conditions (15) and (16) then lead to
\[
\tilde{\Phi}(\tilde{u}, \tilde{v}) - \Phi(u, v) \geq \int_{\Omega} \int_{\mathbb{R}^n} \frac{\partial^2 F}{\partial u^2}(x, u(x), s)(u^*(x) - u(x))^2 \tilde{v}_s(ds)dx.
\]

The inequality (25) then assures that the couple \((u, v)\) is the minimum of \(\tilde{\Phi}\) on \(B(u, r) \times Y_F(v, r)\). □

As the condition (24) is rather strong, the neighborhood \(Y_F(v, r)\) is a superset of some neighborhood of \(v\) in the norm topology of \(\mathcal{Y}^p(\Omega; \mathbb{R}^n)\). Still the set \(Y_F(v, r)\) is not a neighborhood of \(v\) in the weak* topology of \(\mathcal{Y}^p(\Omega; \mathbb{R}^n)\).

3. Examples

The following examples illustrate the above studied optimality conditions.

Example 11. The data of the example are \(\Omega = (0, 1)\),
\[
F(x, u, s) = (s^2 - 1)^2 + \frac{1}{4}(s + 1)^2.
\]

This example shows the difference between the set of optimality conditions (13), (14), and the set (15), (16). The feasible Young measure \(\tilde{v} = \delta_{-1}\) together with the Lagrange multiplier \(\lambda(x) = 0\) satisfies the optimality conditions (13), (14), as well as the optimality conditions (15), (16). The optimality condition (14) gives
\[
\text{supp}(v_s) \subset \left\{-1, \frac{2 + \sqrt{2}}{4}\right\}.
\]
While the optimality condition (16) leads to
\[ \text{supp}(\nu_x) = \{-1\}. \]
It shows that the conditions (13), (14) are less selective than the conditions (15), (16).

**Example 12** (Roubíček, not published, here modified). The data of the example are \( \Omega = (0, 1) \),
\[ F(x, u, s) = (s^2 - 1)^2 - u^2, \quad u_D = 0. \]

This example shows the use of the second order necessary condition (23). The feasible couple \( \bar{u} = 0, \bar{\nu} = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1 \) together with \( \lambda = 0 \) satisfies the necessary conditions (15), (16). Thus the couple satisfies the conditions (13), (14) according to Theorem 6, and it also satisfies the conditions (18)–(20) according to Theorem 8. It is shown below that the second order necessary condition (23) is not satisfied at \( (\bar{u}, \bar{\nu}) \), while the second order condition (14) is satisfied, as mentioned above. It shows again that the condition (14) is little selective.

The critical cone at \((\bar{u}, \bar{v})\) (cf. Theorem 9) is
\[ C(\bar{u}, \bar{v}) = \left\{ (\hat{u}, \hat{v}) \in U \times L_w^\infty(\Omega; \text{rca}(\mathbb{R}^n)); \text{ there exist } \mu > 0, (\tilde{u}, \tilde{v}) \in K, \text{ such that } (\hat{u}, \hat{v}) = \mu ((\tilde{u}, \tilde{v}) - (\bar{u}, \bar{v})), \text{ and } \right. \]
\[ \int \int (s^2 - 1)^2 \tilde{v}_x(ds) dx \leq 0 \} . \]
As \( \tilde{v}_x \) is a positive measure, there exists \( f(x) \in L^\infty(0, 1) \) such that \( 0 \leq f(x) \leq 1 \) and
\[ \tilde{v}_x = f(x) \delta_{-1} + (1 - f(x)) \delta_1. \]
As \((\bar{u}, \bar{v}) \in K\), it holds that
\[ \bar{u}(\xi) = \int_0^\xi \bar{u}(x) dx = \int_0^\xi \int_0^s \tilde{v}_x(ds) dx = \int_0^\xi 1 - 2 f(x) dx. \]
The boundary condition \( \bar{u}(1) = 0 \) implies
\[ \int_0^1 f(x) dx = \frac{1}{2} . \]
For example, \( f(x) = x \) satisfies the condition above, thus the couple \((\bar{u}, \bar{v})\) satisfying
\[ \hat{u}(x) = \bar{u}(x) = \int_0^x 1 - 2 \xi d\xi = x - x^2, \]
\[ \hat{v}_x = \bar{v}_x - \bar{v}_x = \left( x - \frac{1}{2} \right) (\delta_{-1} - \delta_1) \]
belongs to the critical cone \( C(\bar{u}, \bar{v}) \). Because the function \( \hat{u} \) does not satisfy the inequality
\[
-2 \int_0^1 \hat{u}^2(x) \, dx \geq 0,
\]
the condition (23) is not satisfied at \((\bar{u}, \bar{v})\).

**Example 13.** The data of the example are \( \Omega = (0, 1) \),
\[
F(x, u, s) = (s^2 - 1)^2 + u^3 - 3u, \quad u_D = 1.
\]

This example shows the use of the second order sufficient condition (24). The feasible couple \( \bar{u} = 1, \bar{v} = \frac{1}{\sqrt{2}} \delta_{1} + \frac{1}{\sqrt{2}} \delta_{-1} \) together with \( \lambda = 0 \) satisfies the necessary conditions (15), (16). It satisfies also the condition (24), because
\[
\frac{\partial^2 F}{\partial u^2}(x, \bar{u}(x), s) = 6\bar{u}(x) = 6.
\]
Thus there exists \( r > 0 \) such that the couple \((\bar{u}, \bar{v})\) is optimal in \( B(\bar{u}, r) \times Y_F(\bar{v}, r) = B(\bar{u}, r) \times \mathcal{Y}(\Omega; \mathbb{R}^n) \) (cf. Theorem 10).

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**References**