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Stein's method, Jack measure, and the Metropolis algorithm

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Abstract

The one parameter family of Jack_α measures on partitions is an important discrete analog of Dyson's β ensembles of random matrix theory. Except for special values of $\alpha = \frac{1}{2}, 1, 2$ which have group theoretic interpretations, the Jack_α measure has been difficult if not intractable to analyze. This paper proves a central limit theorem (with an error term) for Jack_α measure which works for arbitrary values of α. For $\alpha = 1$ we recover a known central limit theorem on the distribution of character ratios of random representations of the symmetric group on transpositions. The case $\alpha = 2$ gives a new central limit theorem for random spherical functions of a Gelfand pair (or equivalently for the spectrum of a natural random walk on perfect matchings in the complete graph). The proof uses Stein's method and has interesting combinatorial ingredients: an intruiging construction of an exchangeable pair, properties of Jack polynomials, and work of Hanlon relating Jack polynomials to the Metropolis algorithm.

MSC: primary, 05E10; secondary, 60C05

Keywords: Plancherel measure; Stein's method; Spherical function; Jack polynomial; Central limit theorem

1. Introduction

The purpose of this paper is to give a new approach to studying a certain probability measure on the set of all partitions of size n, known as Jack_α measure. Here $\alpha > 0$, and this

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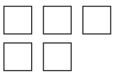
measure chooses a partition λ of size n with probability

$$\frac{\alpha^n n!}{\prod_{s\in\lambda}(\alpha a(s)+l(s)+1)(\alpha a(s)+l(s)+\alpha)},$$

where the product is over all boxes in the partition. Here $a(s)$ denotes the number of boxes in the same row of s and to the right of s (the "arm" of s) and $l(s)$ denotes the number of boxes in the same column of s and below s (the "leg" of s). For example the partition of 5 below



would have Jack_α measure

$$\frac{60\alpha^2}{(2\alpha+2)(3\alpha+1)(\alpha+2)(2\alpha+1)(\alpha+1)}.$$

Before proceeding, it should be mentioned that there is significant interest in the study of statistical properties of Jack_α measure when α is fixed. The case $\alpha=1$ corresponds to the Plancherel measure of the symmetric group, which is now well understood due to numerous results in the past few years. The surveys [AlD,De,O2] and the seminal papers [BOO,J,O1] indicate how the Plancherel measure of the symmetric group is a discrete analog of random matrix theory, and describe its importance in representation theory and geometry. The case $\alpha=2$ corresponds to the Gelfand pair (S_{2n}, H_{2n}) where S_{2n} is a symmetric group and H_{2n} is the hyperoctahedral group of size $2^n n!$. When $\alpha=\frac{1}{2}$, Jack polynomials arise in the study of the Gelfand pair $(GL(n,H), U(n,H))$ where H denotes the division ring of quaternions and GL, U denote general linear and unitary group. Okounkov [O2] emphasizes that the study of Jack_α measure is an important open problem, about which relatively little is known [BO1]. It is a discrete analog of Dyson's β ensembles, which are tractable for the three values $\beta=1,2,4$. In particular, the correlation functions of Jack_α measure are not known, so the traditional techniques for studying discrete analogs of random matrix theory are not obviously applicable.

In the current paper we study Jack_α measure using a remarkable probability technique known as Stein's method. Although Stein's method can be quite hard to work with, there are some problems where it seems to be the only option available (see [RR] for such an example involving the antivoter model). Good surveys of Stein's method (two of them books) are [ArGG,BHJ,Stn1,Stn2].

The current paper is a continuation of [F1], which applied Stein's method to the study of Plancherel measure of the symmetric group S_n. Let $\chi^\lambda_{(2,1^{n-2})}$ denote the character of the irreducible representation of S_n parameterized by λ on the conjugacy class of transpositions. Let $dim(\lambda)$ denote the dimension of the irreducible representation parameterized by λ. Letting P_α denote the probability of an event under Jack_α measure (so that P_1 corresponds to Plancherel measure), the following central limit theorem was proved:

Theorem 1.1 (*Fulman [F1]*). *For $n \geqslant 2$ and all real x_0,*

$$|P_1\left(\frac{n-1}{\sqrt{2}}\frac{\chi^{\lambda}_{(2,1^{n-2})}}{dim(\lambda)} \leqslant x_0\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0} e^{-\frac{x^2}{2}}\,dx| \leqslant 40.1n^{-1/4}.$$

This result sharpened earlier work of Kerov [K1] (see [IO] for a detailed exposition of Kerov's argument) and Hora [Ho], who both obtained a central limit theorem by the method of moments, but with no error bound. We remark that statistical properties of the quantity $\frac{\chi^{\lambda}_{(2,1^{n-2})}}{dim(\lambda)}$ (also called a character ratio) have important applications to random walk [DSh] and to the moduli space of curves [EO].

The main result of the current paper is the following deformation of Theorem 1.1. To state it one needs some notation about partitions. Let λ be a partition of some non-negative integer $|\lambda|$ into integer parts $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$. The symbol $m_i(\lambda)$ will denote the number of parts of λ of size i. Let $l(\lambda)$ denote $\sum_{i \geqslant 1} m_i(\lambda)$, the number of parts of λ. Let $n(\lambda)$ be the quantity $\sum_{i \geqslant 1}(i-1)\lambda_i$. One defines λ' to be the partition dual to λ in the sense that $\lambda'_i = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. Geometrically this corresponds to flipping the diagram of λ.

Theorem 1.2. *Suppose that $\alpha \geqslant 1$. Let $W_{\alpha}(\lambda) = \frac{\alpha n(\lambda') - n(\lambda)}{\sqrt{\alpha \binom{n}{2}}}$. For $n \geqslant 2$ and all real x_0,*

$$|P_{\alpha}(W_{\alpha} \leqslant x_0) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0} e^{-\frac{x^2}{2}}\,dx| \leqslant A_{\alpha} n^{-1/4},$$

where A_{α} depends on α but not on n.

Note that the assumption that $\alpha \geqslant 1$ is merely for convenience. Indeed, from the definition of Jack measure it is clear that the Jack$_{\alpha}$ probability of λ is equal to the Jack$_{1/\alpha}$ probability of λ'. From this one concludes that the Jack$_{\alpha}$ probability that $W_{\alpha} = w$ is equal to the Jack$_{1/\alpha}$ probability that $W_{1/\alpha} = -w$, so that a central limit theorem holds for α if and only if it holds for $\frac{1}{\alpha}$.

We conjecture that the convergence rate upper bound in Theorem 1.2 can be improved to a universal constant multiplied by the maximum of $\frac{1}{\sqrt{n}}$ and $\frac{\sqrt{\alpha}}{n}$. In fact the third moment of W_{α} is $\frac{\alpha - 1}{\sqrt{\alpha \binom{n}{2}}}$ (see Corollary 5.3), so certainly $\frac{\sqrt{\alpha}}{n} \to 0$ is necessary for W_{α} to be asymptotically normal. Of course typically one is interested in α fixed, as α is a parameter which represents the symmetries of the system. In this case the conjecture has recently been proved [CF].

A result of Frobenius [Fr] is that

$$\frac{\chi^{\lambda}_{(2,1^{n-2})}}{dim(\lambda)} = \frac{n(\lambda') - n(\lambda)}{\binom{n}{2}}.$$

Hence Theorem 1.2 is a generalization of Theorem 1.1 in the case $\alpha = 1$. It is also of group theoretic interest in the case $\alpha = 2$. By p. 410 of Macdonald [M] one sees for the $\alpha = 2$ case that $\frac{2n(\lambda') - n(\lambda)}{2\binom{n}{2}}$ is the value of a spherical function corresponding to the Gelfand pair

(S_{2n}, H_{2n}), where H_{2n} is the hyperoctahedral group of size $2^n n!$. Moreover when $\alpha = 2$, Theorem 1.2 gives a central limit theorem for the spectrum of a natural random walk on perfect matchings of the complete graph. For a definition and analysis of the convergence rate of this random walk on matchings, see [DHol], where it was studied in connection with phylogenetic trees. Note that their Corollary 1 shows that the eigenvalues of that random walk are indexed by partitions λ of n, and are $\frac{W_2(\lambda)}{\sqrt{n(n-1)}}$, occurring with multiplicity proportional to the Jack_2 measure on λ.

Next, we make some remarks about the proof of Theorem 1.2. The argument is not a straightforward modification of arguments used in [F1], and requires new ideas. The reason for this is that for general α the Jack_α measure does not have a known interpretation in terms of representation theory of finite groups. Hence, the proof of [F1], which used concepts such as induction and restriction of characters, can not be applied. There is another fundamental difference between the case of Plancherel measure and Jack_α measure. In the Plancherel case the argument of [F1] can be pushed through to conjugacy classes other than transpositions, but the same is not clearly so for the Jack_α case. This is because the Jack_α case uses connections between Jack polynomials and the Metropolis algorithm (due to Hanlon [Ha] and to be reviewed in Section 5) and it is not clear that these connections work for classes other than transpositions.

Theorem 1.2 will be a consequence of the following bound of Stein. Recall that if W, W^* are random variables, they are called exchangeable if for all w_1, w_2, $P(W = w_1, W^* = w_2)$ is equal to $P(W = w_2, W^* = w_1)$. The notation $E^W(\cdot)$ means the expected value given W. Note from [Stn1] that there are minor variations on Theorem 1.3 (and thus for Theorem 1.2) for $h(W)$ where h is a bounded continuous function with bounded piecewise continuous derivative. For simplicity we only state the result when h is the indicator function of an interval.

Theorem 1.3 (*Stein [Stn1]*). *Let (W, W^*) be an exchangeable pair of real random variables such that $E^W(W^*) = (1-\tau)W$ with $0 < \tau < 1$. Then for all real x_0,*

$$\begin{aligned}|P(W \leqslant x_0) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0} e^{-\frac{x^2}{2}}\,dx| \\ \leqslant 2\sqrt{E[1-\frac{1}{2\tau}E^W(W^*-W)^2]^2} + (2\pi)^{-\frac{1}{4}}\sqrt{\frac{1}{\tau}E|W^*-W|^3}.\end{aligned}$$

In order to apply Theorem 1.3 to study a statistic W, one clearly needs an exchangeable pair (W, W^*) such that $E^W(W^*) = (1-\tau)W$. A Markov chain K (with chance of going from x to y denoted by $K(x, y)$) on a finite set X is called reversible with respect to a probability distribution π if $\pi(x)K(x, y) = \pi(y)K(y, x)$. This condition implies that π is a stationary distribution for K. The idea is to use a reversible Markov chain on the set of partitions of size n whose stationary distribution is Jack_α measure, to let λ^* be obtained from λ by one step in the chain where λ is sampled from π, and then set $(W, W^*) = (W(\lambda), W(\lambda^*))$. A main contribution of this paper is the construction and analysis of an exchangeable pair which is useful for Stein's method.

Section 2 revisits and generalizes the construction of an exchangeable pair for Plancherel measure of the symmetric group. We give a connection between harmonic functions on

Bratelli diagrams and decomposition of tensor products and extend some results in [F2]. Section 3 reviews necessary facts about Jack polynomials. Motivated by the discussion in Section 2, Section 4 constructs an exchangeable pair (W_α, W_α^*) to be used in the proof of Theorem 1.2. The combinatorics in this section is quite interesting. Section 5 recalls needed work of Hanlon [Ha] relating Jack polynomials to the Metropolis algorithm. Section 6 combines the ingredients of the previous sections to prove Theorem 1.2.

To close the introduction, we mention some follow up work to this paper. Paper [F3] sharpens the bound in Theorem 1.2 using martingale theory. The forthcoming paper [CF] extends the approach of this paper to other Gelfand pairs (where the limit need not be a Gaussian law). It also further sharpens the bound of Theorem 1.2.

2. Plancherel measure revisited

To begin, we revisit the construction of an exchangeable pair (W, W') for the special case $\alpha = 1$, corresponding to Plancherel measure, which was studied in [F1]. In doing so we clarify and generalize some of the results there and in [F2]. This will be very helpful for treating the case of general α.

As mentioned in the introduction, to construct an exchangeable pair (W, W^*) with respect to a probability measure π on a finite set X, it is enough to construct a Markov chain on X which is reversible with respect to π. Indeed, choosing x from π and letting x^* be obtained from x by one step of the chain, it follows that $(W, W^*) := (W(x), W(x^*))$ is an exchangeable pair. Of course one wants to construct the Markov chain in such a way that the exchangeable pair is useful for Stein's method, and more precisely useful for Theorem 1.3.

2.1. Known constructions

To start we consider the situation for an arbitrary finite group G. Let $Irr(G)$ denote the set of irreducible representations of G. Then the Plancherel measure on $Irr(G)$ chooses a representation λ with probability $\frac{dim(\lambda)^2}{|G|}$, where $dim(\lambda)$ denotes the dimension of λ. In [F2] we constructed a Markov chain M_H on $Irr(G)$ which is reversible with respect to Plancherel measure. To define this Markov chain, one first fixes a subgroup H of G. For $\tau \in Irr(H)$ and $\rho \in Irr(G)$, we let $\kappa(\tau, \rho)$ denote the multiplicity of ρ in the representation of G obtained by inducing τ from H (by Frobenius reciprocity, this is also equal to the multiplicity of τ in the representation of H obtaining by restricting ρ). Then [F2] defined the transition probability $M_H(\lambda, \rho)$ of moving from a representation λ to a representation ρ by

$$\frac{|H|}{|G|}\frac{dim(\rho)}{dim(\lambda)}\sum_{\tau\in Irr(H)}\kappa(\tau, \lambda)\kappa(\tau, \rho).$$

It was proved there that these transition probabilities sum to one, and that the Markov chain with transition mechanism M_H is indeed reversible with respect to the Plancherel measure of G.

For arbitrary groups, this construction can be recast in terms of harmonic functions on Bratelli diagrams. We recommend [K2] or [BO2] for an introduction to this subject. One starts with a Bratteli diagram; that is an oriented graded graph $\Gamma = \bigcup_{n \geqslant 0} \Gamma_n$ such that

(1) Γ_0 is a single vertex $\emptyset$.
(2) If the starting vertex of an edge is in Γ_i, then its end vertex is in Γ_{i+1}.
(3) Every vertex has at least one outgoing edge.
(4) All Γ_i are finite.

For two vertices $\lambda, \Lambda \in \Gamma$, one writes $\lambda \nearrow \Lambda$ if there is an edge from λ to Λ. Part of the underlying data is a multiplicity function $\kappa(\lambda, \Lambda)$. Letting the weight of a path in Γ be the product of the multiplicities of its edges, one defines the dimension $dim(\Lambda)$ of a vertex Λ to be the sum of the weights over all minimal length paths from $\emptyset$ to Λ. Given a Bratteli diagram with a multiplicity function, one calls a function ϕ *harmonic* if $\phi(0) = 1$, $\phi(\lambda) \geqslant 0$ for all $\lambda \in \Gamma$, and

$$\phi(\lambda) = \sum_{\Lambda:\lambda \nearrow \Lambda} \kappa(\lambda, \Lambda)\phi(\Lambda).$$

An equivalent concept is that of coherent probability distributions. Namely a set $\{M_n\}$ of probability distributions M_n on Γ_n is called *coherent* if

$$M_{n-1}(\lambda) = \sum_{\Lambda:\lambda \nearrow \Lambda} \frac{dim(\lambda)\kappa(\lambda, \Lambda)}{dim(\Lambda)} M_n(\Lambda).$$

The formula showing the concepts to be equivalent is $\phi(\lambda) = \frac{M_n(\lambda)}{dim(\lambda)}$. Note that in this setting there is a natural transition mechanism for moving up or down a step in the Bratelli diagram. Namely the chance of moving from λ to Λ is $\frac{\kappa(\lambda,\Lambda)M_n(\Lambda)dim(\lambda)}{M_{n-1}(\lambda)dim(\Lambda)}$, and the chance of moving from Λ to λ is $\frac{dim(\lambda)\kappa(\lambda,\Lambda)}{dim(\Lambda)}$.

Let $H_0 = \{id\} \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ be a tower of subgroups of G. Consider the Bratelli diagram whose jth level consists of irreducible representations of H_j, with edge multiplicity given by $\kappa(\tau, \lambda)$ as in the first paragraph of this subsection. It is proved in [F2] that the Plancherel measures of the groups form a coherent family of probability distributions (this was known for the symmetric group [K1]). Moreover it was shown that if one transitions from level n to level $n-1$, and then from level $n-1$ to level n, that the resulting Markov chain on irreducible representations of H_n is exactly the chain $M_{H_{n-1}}$.

2.2. New construction

Next, we give a new Markov chain L_η on the set of irreducible representations of G which is reversible with respect to Plancherel measure, and which generalizes the chain M_H. First fix η, any representation (not necessarily irreducible) of G whose character is real valued. Let $< \phi, \psi >$ be the usual inner product on class functions of G defined as $\frac{1}{|G|}\sum_{g\in G} \phi(g)\overline{\psi(g)}$. Then the probability that the chain L_η transitions from λ to ρ is

$$\frac{dim(\rho)}{dim(\eta)dim(\lambda)} < \chi^\rho, \chi^\eta\chi^\lambda > .$$

Note that this is nonnegative because $< \chi^\rho, \chi^\eta\chi^\lambda >$ is the multiplicity of ρ in the tensor product of η and λ.

Lemma 2.1. *Let η be a representation of a finite group G whose character is real valued. Then the transition probabilities of L_η sum to 1, and the Markov chain L_η is reversible with respect to the Plancherel measure of G.*

Proof. To see that the transition probabilities do as claimed sum to 1, observe that $\sum_\rho dim(\rho)\chi^\rho$ is the character of the regular representation of G, so takes value $|G|$ at the identity element and 0 elsewhere. The reversibility assertion uses the fact that $< \chi^\rho, \chi^\eta\chi^\lambda >$ is equal to $< \chi^\eta\chi^\rho, \chi^\lambda >$, which is true since χ^η is real valued. □

We remark that the second part of Lemma 2.1 needs χ^η to be real valued. An instructive counterexample when χ^η is not real valued is obtained by letting G be a cyclic group of order n and taking η to be the representation whose value on a fixed generator is $e^{\frac{2\pi i}{n}}$.

One can also define a chain with transition probability

$$\frac{dim(\rho)}{dim(\eta)dim(\lambda)} < \chi^\rho\chi^\lambda, \chi^\eta >$$

which would not require η real valued in Lemma 2.1 but this is less useful for the applications at hand, since then Proposition 2.3 would fail as any reversible Markov chain has real eigenvalues.

Proposition 2.2 shows that M_H is in fact a special case of L_η.

Proposition 2.2. *Let M_H be the Markov chain on irreducible representations of G corresponding to the choice of subgroup H. Let L_η be the Markov chain on the irreducible representations of G corresponding to the choice that η is the representation of G on cosets of H (i.e. the induction of the trivial representation of H to G). Then $M_H = L_\eta$.*

Proof. Throughout the proof we let Res, Ind denote restriction and induction of characters.

$$\begin{aligned}
L_\eta(\lambda,\rho) &= \frac{|H|}{|G|}\frac{dim(\rho)}{dim(\lambda)} < \chi^\rho, \chi^\lambda Ind_H^G[1] >_G \\
&= \frac{|H|}{|G|}\frac{dim(\rho)}{dim(\lambda)} < \chi^\rho\overline{\chi^\lambda}, Ind_H^G[1] >_G \\
&= \frac{|H|}{|G|}\frac{dim(\rho)}{dim(\lambda)} < Res_H(\chi^\rho\overline{\chi^\lambda}), \quad 1 >_H \\
&= \frac{|H|}{|G|}\frac{dim(\rho)}{dim(\lambda)} < Res_H(\chi^\rho), \quad Res_H(\chi^\lambda) >_H \\
&= \frac{|H|}{|G|}\frac{dim(\rho)}{dim(\lambda)} \sum_{\tau\in Irr(H)} \kappa(\tau,\lambda)\kappa(\tau,\rho) \\
&= M_H(\lambda,\rho).
\end{aligned}$$

Note that the third equality is Frobenius reciprocity. □

Next, we note that the chain L_η can be explicitly diagonalized, a fact which has implications for the decomposition of tensor products. As this directly generalizes results from [F2] (which explains their importance) and can be proved by a similar technique, we omit the proofs.

Proposition 2.3. *Let G be a finite group and η any representation of G whose character is real valued. Let π denote the Plancherel measure of G. Then the eigenvalues and eigenfunctions of the Markov chain L_η are indexed by conjugacy classes C of G.*

(1) *The eigenvalue parameterized by C is* $\frac{\chi^\eta(C)}{dim(\eta)}$.
(2) *An orthonormal basis of eigenfunctions* ψ_C in $L^2(\pi)$ *is defined by* $\psi_C(\rho) = \frac{|C|^{\frac{1}{2}}\chi^\rho(C)}{dim(\rho)}$.

Proposition 2.4. *Let η be a representation of a finite group G whose character χ^η is real valued. Suppose that $|G| > 1$. Let $\beta = \max_{g \neq 1} \left| \frac{\chi^\eta(g)}{dim(\eta)} \right|$ and let π denote the Plancherel measure of G. Then for integer $r \geqslant 1$,*

$$\sum_{\rho \in Irr(G)} \left| \frac{dim(\rho)}{dim(\eta)^r} < \chi^\rho, (\chi^\eta)^r > -\pi(\rho) \right| \leqslant |G|^{1/2} \beta^r.$$

3. Properties of Jack polynomials

The purpose of this section is to collect properties of Jack polynomials which will be crucial in the proof of Theorem 1.2. A thorough introduction to Jack polynomials is in Chapter 6 of Macdonald [M]. We conform to Macdonald's notation and let $J_\lambda^{(\alpha)}$ denote the Jack polynomial with parameter α associated to the partition λ. When $\alpha = 1$, the Jack polynomials are Schur functions, and when $\alpha = 2$ or $\alpha = \frac{1}{2}$, they are zonal polynomials corresponding to spherical functions of a Gelfand pair.

As in the introduction, given a box s in the diagram of λ, let $a(s)$ and $l(s)$ denote the arm and leg of s, respectively. One defines quantities

$$c_\lambda(\alpha) = \prod_{s \in \lambda} (\alpha a(s) + l(s) + 1)$$

$$c'_\lambda(\alpha) = \prod_{s \in \lambda} (\alpha a(s) + l(s) + \alpha).$$

Recall that $m_i(\lambda)$ denotes the number of parts of λ of size i and that $l(\lambda)$ denotes the total number of parts of λ. We let $z_\lambda = \prod_{i \geqslant 1} i^{m_i(\lambda)} m_i(\lambda)!$, the size of the centralizer of a permutation of cycle type λ in the symmetric group.

Let $\theta_\mu^\lambda(\alpha)$ denote the coefficient of the power sum symmetric function p_μ in $J_\lambda^{(\alpha)}$. Lemma 3.1 gives orthogonality relations for these coefficients. We remark that when $\alpha = 1$, $\theta_\mu^\lambda(1)$ is equal to $\frac{n!}{z_\mu} \frac{\chi_\mu^\lambda}{dim(\lambda)}$ where χ_μ^λ is the character value of the representation of S_n

parameterized by λ on elements of cycle type μ. Thus when $\alpha = 1$, Lemma 3.1 specializes to the orthogonality relations for characters of the symmetric group.

Lemma 3.1 (*Macdonald [M, p. 382]*).

(1)

$$\sum_{|\mu|=n} z_\mu \alpha^{l(\mu)} \theta_\mu^\rho(\alpha)\theta_\mu^\lambda(\alpha) = \delta_{\rho,\lambda} c_\rho(\alpha) c'_\rho(\alpha).$$

(2)

$$\sum_{|\rho|=n} \frac{\theta_\mu^\rho(\alpha)\theta_\nu^\rho(\alpha)}{c_\rho(\alpha)c'_\rho(\alpha)} = \delta_{\mu,\nu}\frac{1}{z_\mu \alpha^{l(\mu)}}.$$

The following special values of $\theta_\mu^\lambda(\alpha)$ will be needed.

Lemma 3.2. (1) (Macdonald [M, p.382])

$$\theta_{(1^n)}^\lambda(\alpha) = 1.$$

(2) (Macdonald [M, p.383])

$$\theta_\mu^{(n)}(\alpha) = \frac{n!}{z_\mu}\alpha^{n-l(\mu)}.$$

(3) (Macdonald [M, p.384])

$$\theta_{(2,1^{n-2})}^\lambda(\alpha) = n(\lambda')\alpha - n(\lambda).$$

(4) (Stanley [St, p.107])

$$\theta_\mu^{(n-1,1)}(\alpha) = \frac{\alpha^{n-l(\mu)} n!}{z_\mu}\frac{(\alpha(n-1)+1)m_1(\mu) - n}{\alpha n(n-1)}.$$

Next, we consider the ring of symmetric functions, with inner product defined by the orthogonality condition $< p_\nu, p_\mu >_\alpha = \delta_{\nu,\mu} z_\mu \alpha^{l(\mu)}$. By Lemma 3.1, this is equivalent to the condition that $< J_\eta^{(\alpha)}, J_\lambda^{(\alpha)} >_\alpha = \delta_{\eta,\lambda} c_\lambda(\alpha) c'_\lambda(\alpha)$. For a symmetric function *f*, its adjoint $f^\perp$ is defined by the condition $< fg, h >_\alpha = < g, f^\perp h >_\alpha$ for all g, h in the ring of symmetric functions. It is straightforward to check that $p_1^\perp = \alpha \frac{\partial}{\partial p_1}$ (for the case $\alpha = 1$ see p. 76 of Macdonald [M]).

Let

$$\psi'_{\lambda/\tau}(\alpha) = \prod_{s \in C_{\lambda/\tau} - R_{\lambda/\tau}} \frac{(\alpha a_\lambda(s) + l_\lambda(s) + 1)}{(\alpha a_\lambda(s) + l_\lambda(s) + \alpha)} \frac{(\alpha a_\tau(s) + l_\tau(s) + \alpha)}{(\alpha a_\tau(s) + l_\tau(s) + 1)},$$

where $C_{\lambda/\tau}$ is the union of columns of λ that intersect $\lambda - \tau$ and $R_{\lambda/\tau}$ is the union of rows of λ that intersect $\lambda - \tau$.

Lemma 3.3.

$$p_1^{\perp} J_{\lambda}^{(\alpha)} = \sum_{|\tau|=n-1} \frac{c'_{\lambda}(\alpha)\psi'_{\lambda/\tau}(\alpha)}{c'_{\tau}(\alpha)} J_{\tau}^{(\alpha)}.$$

Proof. Take the inner product of both sides with $J_{\tau}^{(\alpha)}$. The left-hand side becomes

$$< p_1^{\perp} J_{\lambda}^{(\alpha)}, J_{\tau}^{(\alpha)} >_{\alpha} = < J_{\lambda}^{(\alpha)}, p_1 J_{\tau}^{(\alpha)} >_{\alpha} .$$

Using the Pieri rule for Jack symmetric functions ([M, p.340]), this becomes

$$\frac{c_{\tau}(\alpha)}{c_{\lambda}(\alpha)} \psi'_{\lambda/\tau}(\alpha) < J_{\lambda}^{(\alpha)}, J_{\lambda}^{(\alpha)} >_{\alpha} = c_{\tau}(\alpha) c'_{\lambda}(\alpha) \psi'_{\lambda/\tau}(\alpha).$$

By the orthogonality relations for the $J's$, this is equal to the inner product of the right-hand side with $J_{\tau}^{(\alpha)}$. $\square$

4. Construction of an exchangeable pair

The purpose of this section is twofold. First, we use the theory of harmonic functions on Bratelli diagrams to construct an exchangeable pair $(W_{\alpha}, W_{\alpha}^{*})$ with respect to Jack$_{\alpha}$ measure on the set of partitions of size n (and as usual, we suppose without loss of generality that $\alpha \geqslant 1$). We give a Markov chain M_{α} which is a deformation of the chain M_H from Section 2 (when $\alpha = 1$ it corresponds to the case that $G = S_n$ and $H = S_{n-1}$). The second and more subtle part of this section is to show that this construction is closely related to a chain L_{α} which is a deformation of the chain L_{η} from Section 2 (when $\alpha = 1$ it corresponds to the case that $G = S_n$ and η is the irreducible representation of the symmetric group of shape $(n-1, 1)$). In fact much of this paper can be pushed through for generalizations of M_{α} and L_{α} corresponding to more vigorous walks on the set of partitions, but for Stein's method it is preferable to use local walks.

The use of both M_{α} and L_{α} will be crucial to this paper. An interesting result in this section will be that (except for holding probabilities), L_{α} is a rescaling of M_{α}, so that one can work with whichever is more convenient. For instance it will be clear from the definition that the transition probabilities of M_{α} are always non-negative. But except for cases such as $\alpha = 1, 2$ where there is a group theoretic reason, it will not be clear that the transition probabilities of L_{α} are always non-negative. But to prove that W_{α} is an eigenvector of M_{α}, it will be convenient to use connections with L_{α}.

In order to define M_{α}, we first recall results on the theory of harmonic functions on Bratelli diagrams. The basic language was reviewed in Section 2. The level Γ_n consists of all partitions of size n. The multiplicity function $\kappa_{\alpha}(\tau, \lambda)$ is defined as $\psi'_{\lambda/\tau}(\alpha)$ where $\psi'_{\lambda/\tau}(\alpha)$ was defined in Section 3. A result of Stanley [St] is that $dim_{\alpha}(\lambda) = \frac{n!\alpha^n}{c'_{\lambda}(\alpha)}$. Then [K3] shows that the Jack$_{\alpha}$ measure

$$\pi_{\alpha}(\lambda) = \frac{\alpha^n n!}{c_{\lambda}(\alpha) c'_{\lambda}(\alpha)}$$

forms a coherent set of probability distributions for this Bratelli diagram.

Motivated by the discussion in Section 2, for $\lambda, \rho \in \Gamma_n$, we define (for $\alpha \geqslant 1$) the transition probability $M_\alpha(\lambda, \rho)$ to be

$$\frac{\pi_\alpha(\rho)}{dim_\alpha(\lambda) dim_\alpha(\rho)} \sum_{|\tau|=n-1} \frac{dim_\alpha(\tau)^2 \kappa_\alpha(\tau, \rho) \kappa_\alpha(\tau, \lambda)}{\pi_\alpha(\tau)}$$
$$= \frac{c'_\lambda(\alpha)}{\alpha n c_\rho(\alpha)} \sum_{|\tau|=n-1} \frac{\psi'_{\lambda/\tau}(\alpha) \psi'_{\rho/\tau}(\alpha) c_\tau(\alpha)}{c'_\tau(\alpha)}.$$

Note that this corresponds to transitioning down a level and then up a level in the Bratelli diagram. The expression for $M_\alpha(\lambda, \rho)$ is a mess, but three useful observations can be made. First, being a sum of non-negative terms, it is non-negative. Second, it is clear that the transition mechanism M_α proceeds by local moves, in the sense that if $M_\alpha(\lambda, \rho) \neq 0$, then λ and ρ have a common descendant. Third, M_α is reversible with respect to Jack$_\alpha$ measure.

As an example, when $n = 3$ the reader can verify that the M_α transition probabilities (rows add to 1) are

	(3)	$(2,1)$	(1^3)
(3)	$\frac{1}{2\alpha+1}$	$\frac{2\alpha}{2\alpha+1}$	0
$(2,1)$	$\frac{\alpha+2}{3(\alpha+1)(2\alpha+1)}$	$\frac{2(\alpha^2+7\alpha+1)}{3(\alpha+2)(2\alpha+1)}$	$\frac{\alpha(2\alpha+1)}{3(\alpha+1)(\alpha+2)}$
(1^3)	0	$\frac{2}{\alpha+2}$	$\frac{\alpha}{\alpha+2}$

Next, we define (for $\alpha \geqslant 1$) a chain L_α to have transition "probability"

$$L_\alpha(\lambda, \rho) = \frac{1}{c_\rho(\alpha) c'_\rho(\alpha) \alpha^n n!} \sum_{|\mu|=n} (z_\mu)^2 \alpha^{2l(\mu)} \theta^\lambda_\mu(\alpha) \theta^\rho_\mu(\alpha) \theta^{(n-1,1)}_\mu(\alpha).$$

As an example, when $n = 3$ using the special values of the θ's given in Lemma 3.2 (and also the value $\theta^\lambda_{(3)}(\alpha)$ which is determined from the other values by the orthogonality relations Lemma 3.1), the reader can verify that the L_α transition probabilities (rows add to 1) are

	(3)	$(2,1)$	(1^3)
(3)	0	1	0
$(2,1)$	$\frac{\alpha+2}{6\alpha(\alpha+1)}$	$\frac{2\alpha^2+11\alpha-4}{6\alpha(\alpha+2)}$	$\frac{(2\alpha+1)^2}{6(\alpha+1)(\alpha+2)}$
(1^3)	0	$\frac{2\alpha+1}{\alpha(\alpha+2)}$	$\frac{\alpha^2-1}{\alpha(\alpha+2)}$

Since the θ's can be negative it is not clear (see more discussion below) that these transition "probabilities" are non-negative. However L_α is clearly "reversible" with respect to Jack$_\alpha$ measure. Proposition 4.1 shows that the transition probabilities sum to one.

Proposition 4.1.

$$\sum_{|\rho|=n} L_\alpha(\lambda, \rho) = 1.$$

Proof. By definition $\sum_{|\rho|=n} L_\alpha(\lambda,\rho)$ is equal to

$$\sum_{|\rho|=n} \frac{1}{c_\rho(\alpha)c'_\rho(\alpha)\alpha^n n!} \sum_{|\mu|=n} (z_\mu)^2 \alpha^{2l(\mu)} \theta^\lambda_\mu(\alpha)\theta^\rho_\mu(\alpha)\theta^{(n-1,1)}_\mu(\alpha).$$

Using the fact from part 1 of Lemma 3.2 that $\theta^\rho_{(1^n)} = 1$, this can be rewritten as

$$\sum_{|\mu|=n} (z_\mu)^2 \alpha^{2l(\mu)} \theta^\lambda_\mu(\alpha)\theta^{(n-1,1)}_\mu(\alpha) \sum_{|\rho|=n} \frac{\theta^\rho_\mu(\alpha)\theta^\rho_{(1^n)}}{c_\rho(\alpha)c'_\rho(\alpha)\alpha^n n!}.$$

The result now follows from part 2 of Lemma 3.1 and part 1 of Lemma 3.2. □

Theorem 4.2 establishes a fundamental relationship between the chains M_α and L_α.

Theorem 4.2. *If $\lambda \neq \rho$, then*

$$L_\alpha(\lambda,\rho) = \frac{\alpha(n-1)+1}{\alpha(n-1)} M_\alpha(\lambda,\rho).$$

Proof. By part 4 of Lemma 3.2, $L_\alpha(\lambda,\rho)$ is equal to

$$\frac{1}{\alpha n(n-1)c_\rho(\alpha)c'_\rho(\alpha)} \sum_{|\mu|=n} \theta^\lambda_\mu(\alpha)\theta^\rho_\mu(\alpha)\alpha^{l(\mu)} z_\mu((\alpha(n-1)+1)m_1(\mu)-n).$$

Since $\lambda \neq \rho$, part 1 of Lemma 3.1 shows that this is equal to

$$\frac{(\alpha(n-1)+1)}{\alpha n(n-1)c_\rho(\alpha)c'_\rho(\alpha)} \sum_{|\mu|=n} \theta^\lambda_\mu(\alpha)\theta^\rho_\mu(\alpha)\alpha^{l(\mu)} z_\mu m_1(\mu).$$

Bearing in mind the results from Section 3, this can be rewritten as

$$\begin{aligned}
&\frac{(\alpha(n-1)+1)}{\alpha n(n-1)c_\rho(\alpha)c'_\rho(\alpha)} \sum_{|\mu|=n} < p_1 \frac{\partial}{\partial p_1} \sum_{|\mu|=n} \theta^\lambda_\mu(\alpha) p_\mu, \sum_{|\mu|=n} \theta^\rho_\mu(\alpha) p_\mu >_\alpha \\
&= \frac{(\alpha(n-1)+1)}{\alpha^2 n(n-1)c_\rho(\alpha)c'_\rho(\alpha)} < p_1^\perp J^{(\alpha)}_\lambda, p_1^\perp J^{(\alpha)}_\rho >_\alpha \\
&= \frac{(\alpha(n-1)+1)}{\alpha^2 n(n-1)c_\rho(\alpha)c'_\rho(\alpha)} \\
&\quad < \sum_{|\tau|=n-1} \frac{\psi'_{\lambda/\tau}(\alpha)c'_\lambda(\alpha)}{c'_\tau(\alpha)} J^{(\alpha)}_\tau, \sum_{|\tau|=n-1} \frac{\psi'_{\rho/\tau}(\alpha)c'_\rho(\alpha)}{c'_\tau(\alpha)} J^{(\alpha)}_\tau >_\alpha \\
&= \frac{(\alpha(n-1)+1)}{\alpha(n-1)} \sum_{|\tau|=n-1} \frac{c'_\lambda(\alpha)c_\tau(\alpha)\psi'_{\lambda/\tau}(\alpha)\psi'_{\rho/\tau}(\alpha)}{\alpha n c'_\tau(\alpha)c_\rho(\alpha)} \\
&= \frac{\alpha(n-1)+1}{\alpha(n-1)} M_\alpha(\lambda,\rho). \qquad \square
\end{aligned}$$

Note that Theorem 4.2 implies that $L_\alpha(\lambda,\rho)\geqslant 0$ for $\lambda\neq\rho$. We conjecture that $L_\alpha(\lambda,\lambda)\geqslant 0$ for all λ and $\alpha\geqslant 1$. Using Theorem 4.2, this is equivalent to the assertion that $M_\alpha(\lambda,\lambda)\geqslant \frac{1}{\alpha(n-1)+1}$ for all λ. However as this paper only uses non-negativity of M_α, this conjecture is somewhat of a distraction and we do not pursue it here. The proof should not be too difficult.

In fact since $L_1(\lambda,\rho)$ is simply the chain L_η of Section 2 with η the irreducible representation of shape $(n-1,1)$, non-negativity of L_1 is clear. To conclude this section we give a similar group theoretic argument that $L_2(\lambda,\rho)\geqslant 0$ for all λ,ρ.

Proposition 4.3. $L_2(\lambda,\rho)\geqslant 0$ for all λ,ρ.

Proof. Let H_{2n} be the hyperoctahedral group of order $2^n n!$. Using the notation of Section 7.2 of Macdonald [M] for the Gelfand pair (S_{2n},H_{2n}), given λ,μ partitions of n, let ω^λ_μ be the value of the spherical function ω^λ on a double coset of type μ. It follows that

$$L_2(\lambda,\rho)=\frac{(2^n n!)^2}{c_\rho(2)c'_\rho(2)}\sum_{|\mu|=n}\frac{1}{2^{l(\mu)}z_\mu}\omega^\lambda_\mu\omega^\rho_\mu\omega^{(n-1,1)}_\mu .$$

It is a general fact ([M, p.396]) that if $\omega_1,\dots,\omega_t$ are spherical functions for a Gelfand pair (G,K) and a^k_{ij} are defined by

$$\omega_i\omega_j=\sum_k a^k_{ij}\omega_k$$

(where the multiplication $\omega_i\omega_j$ denotes the pointwise product) then a^k_{ij} are real and $\geqslant 0$. The proposition now follows from the orthogonality relation

$$\sum_\mu\frac{1}{2^{l(\mu)}z_\mu}\omega^\lambda_\mu\omega^\nu_\mu=\delta_{\lambda,\nu}\frac{c_\lambda(2)c'_\lambda(2)}{(2^n n!)^2}$$

on p. 406 of Macdonald [M]. $\square$

5. Jack polynomials and the Metropolis algorithm

To begin we recall the Metropolis algorithm [MRRTT] for sampling from a positive probability $\pi(x)$ on a finite set X. A marvelous survey of the Metropolis algorithm, containing references and many examples is [DSa]. The Metropolis algorithm is especially useful when one can understand the ratios $r_{y,x}=\frac{\pi(y)}{\pi(x)}$, but cannot easily compute $\pi(x)$ (for instance in Ising-type models). Let $S(x,y)$ (the base chain) be the transition matrix of a symmetric irreducible Markov chain on X. Define the Metropolis chain T by letting $T(x,y)$, the probability of moving from x to y be defined by

$$\begin{cases} S(x,y)r_{y,x} & \text{if } r_{y,x}<1\\ S(x,y) & \text{if } y\neq x \text{ and } r_{y,x}\geqslant 1\\ S(x,x)+\sum_{\substack{z\neq x\\ r_{z,x}<1}} S(x,z)(1-r_{z,x}) & \text{if } y=x\end{cases}$$

This chain has desirable properties. First, is easy to implement. From x, pick y with probability $S(x, y)$. If $y \neq x$ and $r_{y,x} \geqslant 1$, the chain moves to y. If $y \neq x$ and $r_{y,x} < 1$, flip a coin with success probability $r_{y,x}$. If the coin toss succeeds, the chain moves to y. Otherwise the chain stays at x. Second, the chain $T(x, y)$ is irreducible and aperiodic with stationary distribution π. Thus taking sufficiently many steps according to the chain T one obtains an arbitrarily good approximate sample of π.

A remarkable result of Hanlon [Ha] relates the Metropolis algorithm to Jack symmetric functions. Fix $\alpha \geqslant 1$. Hanlon defines a Markov chain T_α on the symmetric group S_n as follows. Let $\pi(x)$ be the probability measure on S_n which chooses x with probability proportional to $\alpha^{-c(x)}$ where $c(x)$ is the number of cycles of x (ironically for sampling purposes one does not need to use the Metropolis algorithm as the constant of proportionality can be exactly computed in this case). Let $S(x, y) = \frac{1}{\binom{n}{2}}$ if $x^{-1}y$ is a transposition, and 0 otherwise. Then Hanlon defines $T_\alpha(x, y)$ to be the resulting Metropolis chain. To be explicit, if λ_x is the partition whose rows are the cycle lengths of x, then the chance $T_\alpha(x, y)$ of moving from x to y is

$$\begin{cases} \frac{(\alpha-1)n(\lambda_x')}{\alpha\binom{n}{2}} & \text{if } y = x \\ \frac{1}{\binom{n}{2}} & \text{if } y = x(i,j) \text{ and } c(y) = c(x) - 1 \\ \frac{1}{\alpha\binom{n}{2}} & \text{if } y = x(i,j) \text{ and } c(y) = c(x) + 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus for $n = 3$ the transition matrix is (rows sum to 1)

$$\begin{array}{ccccccc} & id & (12) & (13) & (23) & (123) & (132) \\ id & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ (12) & \frac{1}{3\alpha} & \frac{\alpha-1}{3\alpha} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ (13) & \frac{1}{3\alpha} & 0 & \frac{\alpha-1}{3\alpha} & 0 & \frac{1}{3} & \frac{1}{3} \\ (23) & \frac{1}{3\alpha} & 0 & 0 & \frac{\alpha-1}{3\alpha} & \frac{1}{3} & \frac{1}{3} \\ (123) & 0 & \frac{1}{3\alpha} & \frac{1}{3\alpha} & \frac{1}{3\alpha} & 1-\frac{1}{\alpha} & 0 \\ (132) & 0 & \frac{1}{3\alpha} & \frac{1}{3\alpha} & \frac{1}{3\alpha} & 0 & 1-\frac{1}{\alpha} \end{array}.$$

It is clear that the transition matrix for T_α commutes with the action of S_n on itself by conjugation. Thus lumping the chain T_α to conjugacy classes gives a Markov chain on conjugacy classes of S_n. We denote this lumped Metropolis chain by K_α. The transition probability $K_\alpha(\mu, \nu)$ is defined as $\sum T_\alpha(x, y)$ where x is any permutation in the class μ and y ranges over all permutations in the class ν. For instance when $n = 3$ the transition matrix (rows sum to 1) is

$$\begin{array}{cccc} & (1^3) & (2,1) & (3) \\ (1^3) & 0 & 1 & 0 \\ (2,1) & \frac{1}{3\alpha} & \frac{\alpha-1}{3\alpha} & \frac{2}{3} \\ (3) & 0 & \frac{1}{\alpha} & 1-\frac{1}{\alpha} \end{array}$$

Theorem 5.1 is due to Hanlon and is quite deep. In [DHa] it is applied to analyze the convergence rate of the Metropolis chain T_α. The case $\alpha = 1$ of Theorem 5.1 is the usual Fourier analysis on the symmetric group (see [DSh] for details and an application to analyzing the convergence rate of random walk generated by random transpositions).

Theorem 5.1 (*Hanlon [Ha]*). *Suppose that* $\alpha \geqslant 1$. *Then the chance that the lumped Metropolis chain* K_α *on partitions moves from* (1^n) *to the partition* μ *after* r *steps is equal to*

$$\alpha^n n! \sum_{|\rho|=n} \frac{\theta_\mu^\rho(\alpha)}{c_\rho(\alpha)c'_\rho(\alpha)} \left(\frac{\alpha n(\rho') - n(\rho)}{\alpha\binom{n}{2}} \right)^r .$$

The following consequence is worth recording.

Corollary 5.2. *Suppose that* $\alpha \geqslant 1$. *Then the chance that the lumped Metropolis chain* K_α *on partitions of size n moves from the partition* (1^n) *to itself after r steps is the rth moment of the statistic* $\frac{W_\alpha}{\sqrt{\alpha\binom{n}{2}}}$ *under* Jack$_\alpha$ *measure*.

Proof. By part 1 of Lemma 3.2, $\theta_{(1^n)}^\rho(\alpha) = 1$. The result is now clear from Theorem 5.1. $\square$

Corollary 5.2 allows one to compute the rth moment of W_α in terms of return probabilities of the Metropolis chain K_α. This opens the door to the method of moments approach to proving a central limit theorem for W_α, as in [Ho] for the special case $\alpha = 1$. However, we prefer the Stein's method approach, as it comes with an error term. But in passing we note a consequence which indicates that the scaling of W_α has been chosen correctly.

Corollary 5.3. *Suppose that* $\alpha \geqslant 1$. *Then* $E(W_\alpha) = 0$, $E(W_\alpha^2) = 1$, *and* $E(W_\alpha^3) = \frac{\alpha - 1}{\sqrt{\alpha\binom{n}{2}}}$.

Proof. The chance that K_α goes from (1^n) to itself in one step is 0. Hence $E(W_\alpha) = 0$. The chance that K_α goes from (1^n) to itself in two steps is computed to be $\frac{1}{\alpha\binom{n}{2}}$. Hence $E(W_\alpha^2) = 1$. The chance that K_α goes from (1^n) to itself in three steps is equal to the chance of going from (1^n) to $(2, 1^{n-2})$ in two steps, and then back to (1^n). This chance is $\frac{\alpha-1}{\alpha^2\binom{n}{2}^2}$. Hence $E(W_\alpha^3) = \frac{\alpha-1}{\sqrt{\alpha\binom{n}{2}}}$. $\square$

6. Central limit theorem for Jack measure

In this section, we prove Theorem 1.2. Thus $\alpha \geqslant 1$ is fixed and we aim to show that $W_\alpha(\lambda) = \frac{\alpha n(\lambda') - n(\lambda)}{\sqrt{\alpha\binom{n}{2}}}$ satisfies a central limit theorem when λ is chosen from Jack$_\alpha$ measure.

Let (W_α, W_α^*) be the exchangeable pair constructed in Section 4 using the Markov chain M_α. Abusing notation due to possible negativity issues, it is also convenient to let (W_α, W'_α) be the exchangeable pair constructed in Section 4 using L_α. To apply Stein's method it is necessary to work with the genuine exchangeable pair (W_α, W_α^*), but Theorem 4.2 will reduce computations involving it to the more tractable pair (W_α, W_α').

Proposition 6.1 shows that the hypothesis needed to apply the Stein method bound (Theorem 1.3) is satisfied. It also tells us that W_α is an eigenvector for the Markov chain M_α, with eigenvalue $1-\frac{2}{n}$. It is perhaps unexpected that this eigenvalue is independent of α.

Proposition 6.1. $E^{W_\alpha}(W_\alpha^*) = (1-\frac{2}{n})W_\alpha$.

Proof. Theorem 4.2 implies that

$$E^\lambda(W_\alpha^* - W_\alpha) = \frac{\alpha(n-1)}{\alpha(n-1)+1} E^\lambda(W_\alpha' - W_\alpha).$$

Using the definition of the chain L_α and part 3 of Lemma 3.2, it follows that

$$\begin{aligned}
&E^\lambda(W_\alpha') \\
&= \frac{1}{\sqrt{\alpha\binom{n}{2}}} \sum_{|\rho|=n} L_\alpha(\lambda,\rho)\theta^\rho_{(2,1^{n-2})}(\alpha) \\
&= \frac{1}{\sqrt{\alpha\binom{n}{2}}} \sum_{|\rho|=n} \frac{\theta^\rho_{(2,1^{n-2})}(\alpha)}{c_\rho(\alpha)c_\rho'(\alpha)\alpha^n n!} \sum_{|\mu|=n} (z_\mu)^2\alpha^{2l(\mu)}\theta^\lambda_\mu(\alpha)\theta^\rho_\mu(\alpha)\theta^{(n-1,1)}_\mu(\alpha) \\
&= \frac{1}{\sqrt{\alpha\binom{n}{2}}} \sum_{|\mu|=n} (z_\mu)^2\alpha^{2l(\mu)}\theta^\lambda_\mu(\alpha)\theta^{(n-1,1)}_\mu(\alpha) \sum_{|\rho|=n} \frac{\theta^\rho_\mu(\alpha)\theta^\rho_{(2,1^{n-2})}(\alpha)}{c_\rho(\alpha)c_\rho'(\alpha)\alpha^n n!}.
\end{aligned}$$

Using part 2 of Lemma 3.1, one sees that only the term $\mu = (2, 1^{n-2})$ makes a non-zero contribution. Thus

$$\begin{aligned}
E^\lambda(W_\alpha') &= \frac{2}{\alpha n(n-1)}\theta^{(n-1,1)}_{(2,1^{n-2})}(\alpha)\frac{\theta^\lambda_{(2,1^{n-2})}(\alpha)}{\sqrt{\alpha\binom{n}{2}}} \\
&= \frac{2}{\alpha n(n-1)}\theta^{(n-1,1)}_{(2,1^{n-2})}(\alpha)W_\alpha \\
&= \left(1 - \frac{2(\alpha n-\alpha+1)}{n(n-1)\alpha}\right)W_\alpha.
\end{aligned}$$

The last two equations used Lemma 3.2. Consequently

$$E^\lambda(W_\alpha' - W_\alpha) = -\left(\frac{2(\alpha n-\alpha+1)}{n(n-1)\alpha}\right)W_\alpha.$$

Thus $E^\lambda(W_\alpha^* - W_\alpha) = -\frac{2}{n}W_\alpha$, and since this depends on λ only through W_α, the result follows. □

More generally, the following proposition (proved using the same method as for Proposition 6.1) holds.

Proposition 6.2. *Fix ν a partition of n. Then $\theta_\nu^\lambda(\alpha)$ is an eigenvector of L_α with eigenvalue $\frac{z_\nu}{\alpha^{n-l(\nu)}n!}\theta_\nu^{(n-1,1)}(\alpha)$ and an eigenvector of M_α with eigenvalue*

$$1+\frac{\alpha(n-1)}{\alpha(n-1)+1}\left(\frac{z_\nu}{\alpha^{n-l(\nu)}n!}\theta_\nu^{(n-1,1)}(\alpha)-1\right).$$

As a consequence of Proposition 6.1, we see that the mean $E(W_\alpha)$ is equal to 0.

Corollary 6.3. $E(W_\alpha)=0$.

Proof. Since the pair (W_α, W_α^*) is exchangeable, $E(W_\alpha^*-W_\alpha)=0$. Using Proposition 6.1, we see that

$$E(W_\alpha^*-W_\alpha)=E(E^{W_\alpha}(W_\alpha^*-W_\alpha))=-\frac{2}{n}E(W_\alpha).$$

Hence $E(W_\alpha)=0$. □

Next, we compute $E^\lambda(W'_\alpha)^2$. Recall that this notation means the expected value of $(W'_\alpha)^2$ given λ. This will be useful for analyzing the error term in Theorem 1.3.

Proposition 6.4.

$$\begin{aligned}E^\lambda((W'_\alpha)^2)=1&+\theta^\lambda_{(2,1^{n-2})}(\alpha)\frac{4(\alpha-1)(\alpha\binom{n-1}{2}-1)}{\alpha^2n^2(n-1)^2}\\&+\theta^\lambda_{(3,1^{n-3})}(\alpha)\frac{6\,(\alpha(n-1)(n-3)-3)}{\alpha^2n^2(n-1)^2}\\&+\theta^\lambda_{(2^2,1^{n-4})}(\alpha)\frac{4\,(\alpha(n-1)(n-4)-4)}{\alpha^2n^2(n-1)^2}.\end{aligned}$$

Proof.

$$\begin{aligned}E^\lambda((W'_\alpha)^2)&=\alpha\binom{n}{2}\sum_{|\rho|=n}L_\alpha(\lambda,\rho)\left(\frac{\alpha n(\rho')-n(\rho)}{\alpha\binom{n}{2}}\right)^2\\&=\alpha\binom{n}{2}\sum_{|\rho|=n}\frac{1}{c_\rho(\alpha)c'_\rho(\alpha)\alpha^n n!}\\&\quad\times\sum_{|\mu|=n}(z_\mu)^2\alpha^{2l(\mu)}\theta^\lambda_\mu(\alpha)\theta^\rho_\mu(\alpha)\theta^{(n-1,1)}_\mu(\alpha)\left(\frac{\alpha n(\rho')-n(\rho)}{\alpha\binom{n}{2}}\right)^2\\&=\alpha\binom{n}{2}\sum_{|\mu|=n}\theta^\lambda_\mu(\alpha)\theta^{(n-1,1)}_\mu(\alpha)\frac{(z_\mu)^2\alpha^{2l(\mu)}}{\alpha^n n!}\\&\quad\times\sum_{|\rho|=n}\frac{\theta^\rho_\mu(\alpha)}{c_\rho(\alpha)c'_\rho(\alpha)}\left(\frac{\alpha n(\rho')-n(\rho)}{\alpha\binom{n}{2}}\right)^2.\end{aligned}$$

Next observe that using Theorem 5.1, one can compute the sum

$$\alpha^n n! \sum_{|\rho|=n} \frac{\theta_\mu^\rho(\alpha)}{c_\rho(\alpha)c'_\rho(\alpha)} \left(\frac{\alpha n(\rho') - n(\rho)}{\alpha\binom{n}{2}}\right)^2$$

for any partition μ. Indeed, it is simply the probability that the lumped Metropolis chain K_α moves from (1^n) to μ in two steps. From the explicit description of the transition rule of K_α, it is straightforward to calculate that this probability is $\frac{1}{\alpha\binom{n}{2}}$ when $\mu = (1^n)$, is $\frac{\alpha-1}{\alpha\binom{n}{2}}$ when $\mu = (2, 1^{n-2})$, is $\frac{4(n-2)}{n(n-1)}$ when $\mu = (3, 1^{n-3})$, and is $\frac{(n-2)(n-3)}{n(n-1)}$ when $\mu = (2^2, 1^{n-4})$. Together with part 4 of Lemma 3.2, this completes the proof of the proposition. □

One can use Proposition 6.4 to give a Stein's method proof of the fact that $Var(W_\alpha) = 1$, but in light of Corollary 5.3 there is no need to do so.

In order to prove Theorem 1.2, we have to analyze the error terms in Theorem 1.3. To begin we study

$$E\left(-1 + \frac{n}{4}E^\lambda(W_\alpha^* - W_\alpha)^2\right)^2,$$

obtaining an exact formula. From Jensen's inequality for conditional expectations, (see Lemma 5 of [F4] for details) the fact that W_α is determined by λ implies that

$$E[E^{W_\alpha}(W_\alpha^* - W_\alpha)^2]^2 \leqslant E[E^\lambda(W_\alpha^* - W_\alpha)^2]^2.$$

Hence Proposition 6.5 gives an upper bound on

$$E\left(-1 + \frac{n}{4}E^{W_\alpha}(W_\alpha^* - W_\alpha)^2\right)^2.$$

Proposition 6.5.

$$E\left(-1 + \frac{n}{4}E^\lambda(W_\alpha^* - W_\alpha)^2\right)^2 = \frac{3\alpha n + 2\alpha^2 - 10\alpha + 2}{4\alpha n(n-1)}.$$

Proof. By Theorem 4.2 and Proposition 6.1,

$$\begin{aligned}
E^\lambda(W_\alpha^* - W_\alpha)^2 &= \frac{\alpha(n-1)}{\alpha(n-1)+1}E^\lambda(W'_\alpha - W_\alpha)^2\\
&= \frac{\alpha(n-1)}{\alpha(n-1)+1}(W_\alpha^2 - 2W_\alpha E^\lambda(W'_\alpha) + E^\lambda(W'_\alpha)^2)\\
&= \frac{\alpha(n-1)}{\alpha(n-1)+1}\left((\frac{4(\alpha n - \alpha + 1)}{\alpha n(n-1)} - 1)W_\alpha^2 + E^\lambda(W'_\alpha)^2\right).
\end{aligned}$$

Combining this with Proposition 6.4, it follows that $-1 + \frac{n}{4}E^\lambda(W_\alpha^* - W_\alpha)^2$ is equal to $A + B + C + D + E$ where

(1) $A = -1 + \frac{n}{4}\frac{\alpha(n-1)}{\alpha(n-1)+1}$

(2) $B = \frac{(\alpha-1)(\alpha\binom{n-1}{2}-1)}{\alpha n(n-1)(\alpha n-\alpha+1)}\theta^{\lambda}_{(2,1^{n-2})}(\alpha)$
(3) $C = \frac{3(\alpha(n-1)(n-3)-3)}{2\alpha n(n-1)(\alpha n-\alpha+1)}\theta^{\lambda}_{(3,1^{n-3})}(\alpha)$
(4) $D = \frac{\alpha(n-1)(n-4)-4}{\alpha n(n-1)(\alpha n-\alpha+1)}\theta^{\lambda}_{(2^2,1^{n-4})}(\alpha)$
(5)

$$E = \frac{n}{4}\frac{\alpha(n-1)}{\alpha n-\alpha+1}(\frac{4(\alpha n-\alpha+1)}{\alpha n(n-1)}-1)\alpha\binom{n}{2}\left(\frac{\alpha n(\lambda')-n(\lambda)}{\alpha\binom{n}{2}}\right)^2$$
$$= \frac{n}{4}\frac{\alpha(n-1)}{\alpha n-\alpha+1}(\frac{4(\alpha n-\alpha+1)}{\alpha n(n-1)}-1)\frac{1}{\alpha\binom{n}{2}}(\theta^{\lambda}_{(2,1^{n-2})}(\alpha))^2.$$

We need to compute the Jack$_\alpha$ average of $(A+B+C+D+E)^2$. Since A^2 is constant, the average of A^2 is $\left(-1+\frac{n}{4}\frac{\alpha(n-1)}{\alpha(n-1)+1}\right)^2$. The Jack$_\alpha$ averages of B^2,C^2,D^2 can all be computed using part 2 of Lemma 3.1. To compute the Jack$_\alpha$ average of E^2 one uses Theorem 5.1 to reduce to computing the probability that after three steps taken by the chain K_α started from the partition (1^n), that one is at the partition $(2,1^{n-2})$. From the description of the entries of the transition matrix of K_α, one computes this probability to be $\frac{2(3\alpha n^2+\alpha n+2\alpha^2-16\alpha+2)}{\alpha^2n^2(n-1)^2}$. The Jack$_\alpha$ averages of $2AB, 2AC, 2AD, 2BC, 2BD, 2CD$ are all 0 by part 2 of Lemma 3.1. The Jack$_\alpha$ average of $2AE$ is computed using the second expression for E and part 2 of Lemma 3.1. Finally, Theorem 5.1 reduces computation of the Jack$_\alpha$ average of $2BE$ (respectively $2CE$ and $2DE$) to the probability that after two steps taken by the chain K_α started at (1^n), that one is at the partition $(2,1^{n-2})$ (respectively $(3,1^{n-3})$ and $(2^2,1^{n-4})$). Thus all of the enumerations are elementary and adding up the terms yields the proposition. □

The final ingredient needed to prove Theorem 1.2 is an upper bound on $E|W^*-W|^3$. Typically this is the crudest term in applications of Stein's method.

Lemma 6.6 bounds the tail probabilities for λ_1, λ_1' under Jack$_\alpha$ measure.

Lemma 6.6. *Suppose that* $\alpha > 0$.

(1) *The Jack*$_\alpha$ *probability that* $\lambda_1 \geqslant 2e\sqrt{\frac{n}{\alpha}}$ *is at most* $\frac{\alpha n^2}{4^{2e\sqrt{\frac{n}{\alpha}}}}$.
(2) *The Jack*$_\alpha$ *probability that* $\lambda_1' \geqslant 2e\sqrt{\alpha n}$ *is at most* $\frac{n^2}{\alpha 4^{2e\sqrt{n\alpha}}}$.

Proof. Given a partition λ, let τ be the partition of $n-\lambda_1$ given by removing the first row of λ. Then by the definition of Jack$_\alpha$ measure, it follows that the Jack$_\alpha$ measure of λ is at most $\frac{n!}{(n-\lambda_1)!\lambda_1!(\alpha(\lambda_1-1)+1)\cdots(\alpha+1)}$ multiplied by the Jack$_\alpha$ measure of τ. It follows that the Jack$_\alpha$ probability that $\lambda_1 = l$ is at most

$$\frac{n!}{(n-l)!l!}\frac{1}{\alpha^{l-1}(l-1)!} \leqslant (\frac{n}{\alpha})^l\frac{\alpha l}{l!^2}.$$

Using the inequality $y! \geqslant (y/e)^y$ and assuming that $l \geqslant 2e\sqrt{\frac{n}{\alpha}}$ this is at most

$$\left(\frac{ne^2}{\alpha l^2}\right)^l \alpha l \leqslant \frac{\alpha n}{4^{2e\sqrt{\frac{n}{\alpha}}}}.$$

The first assertion follows by summing over l with $n \geqslant l \geqslant 2e\sqrt{\frac{n}{\alpha}}$.

The second assertion follows from the first assertion by symmetry. Indeed, since the Jack$_\alpha$ measure of λ' is the Jack$_{\frac{1}{\alpha}}$ measure of λ, the Jack$_\alpha$ probability that $\lambda_1' \geqslant 2e\sqrt{\alpha n}$ is equal to the Jack$_{\frac{1}{\alpha}}$ probability that $\lambda_1 \geqslant 2e\sqrt{\alpha n}$. Now apply part 1 of the lemma with α replaced by $\frac{1}{\alpha}$. □

Proposition 6.7. *Suppose that $\alpha \geqslant 1$. Then there is a constant C_α depending on α such that*

$$E|W^* - W|^3 \leqslant C_\alpha n^{-3/2}$$

for all n.

Proof. Recall that

$$W = \frac{1}{\sqrt{\alpha\binom{n}{2}}}(\alpha n(\lambda') - n(\lambda)).$$

From the definition of M_α, it is clear that λ^* is obtained from λ by removing a box from the diagram of λ and reattaching it somewhere. It follows that

$$|W^* - W| \leqslant \frac{1}{\sqrt{\alpha\binom{n}{2}}}(\alpha(\lambda_1 + 1) + \lambda_1' + 1).$$

Indeed, suppose that λ^* is obtained from λ by moving a box from row a and column b to a different row c and column d. Then

$$W^* - W = \frac{1}{\sqrt{\alpha\binom{n}{2}}}(\alpha(\lambda_c - \lambda_a + 1) + (\lambda_b' - \lambda_d' - 1)).$$

Suppose that $\lambda_1 \leqslant 2e\sqrt{\frac{n}{\alpha}}$ and that $\lambda_1' \leqslant 2e\sqrt{\alpha n}$. Then by the previous paragraph

$$|W^* - W| \leqslant \frac{C_0}{\sqrt{n}}$$

for a universal constant C_0 (not even depending on α). Note by the first paragraph, that even if $\lambda_1 > 2e\sqrt{\frac{n}{\alpha}}$ or $\lambda_1' > 2e\sqrt{\alpha n}$ occurs, then $|W^* - W| \leqslant C_1\sqrt{\alpha}$ for a universal constant C_1. The result now follows by Lemma 6.6, which shows that these events occur with very low probability for α fixed. □

Summarizing, now we prove Theorem 1.2 (the main result).

Proof of Theorem 1.2. We use Theorem 1.3 with the exchangeable pair (W, W^*) constructed in Section 4. Proposition 6.1 shows this to be possible with $\tau = \frac{2}{n}$. The result now follows from Proposition 6.5 (together with the paragraph before it) and Proposition 6.7. □

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