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Gelfand–Tsetlin polytopes and Feigin–Fourier–Littelmann–Vinberg polytopes as marked poset polytopes

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ABSTRACT

Stanley (1986) showed how a finite partially ordered set gives rise to two polytopes, called the order polytope and chain polytope, which have the same Ehrhart polynomial despite being quite different combinatorially. We generalize his result to a wider family of polytopes constructed from a poset P with integers assigned to some of its elements.

Through this construction, we explain combinatorially the relationship between the Gelfand–Tsetlin polytopes (1950) and the Feigin–Fourier–Littelmann–Vinberg polytopes (2010, 2005), which arise in the representation theory of the special linear Lie algebra. We then use the generalized Gelfand–Tsetlin polytopes of Berenstein and Zelevinsky (1989) to propose conjectural analogues of the Feigin–Fourier–Littelmann–Vinberg polytopes corresponding to the symplectic and odd orthogonal Lie algebras.

1. Introduction

Consider the simple complex Lie algebra $\mathfrak{sl}_n$. The irreducible representations of $\mathfrak{sl}_n$ are parametrized up to isomorphism by dominant integral weights, i.e., weakly decreasing n-tuples of integers determined up to adding multiples of $(1, \dots, 1)$. Given a dominant integral weight λ, let $V(\lambda)$ denote the corresponding irreducible $\mathfrak{sl}_n$-module. The module $V(\lambda)$ has a distinguished basis, the Gelfand–Tsetlin

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[6] basis, parametrized by the points with integral coordinates ("integral points" or "lattice points" for short) in the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda) \subset \mathbf{R}^{n(n-1)/2}$.

Recently, Feigin, Fourier, and Littelmann [3] constructed a different basis of $V(\lambda)$, conjecturally announced by Vinberg [9]. This basis is related to the Poincaré–Birkhoff–Witt basis of the universal enveloping algebra $U(\mathfrak{n}^-)$, where $\mathfrak{n}^-$ is the span of the negative root spaces. Again, the basis elements are parametrized by the integral points in a certain polytope $\mathrm{FFLV}(\lambda) \subset \mathbf{R}^{n(n-1)/2}$.

Feigin, Fourier, and Littelmann used two subtle algebraic arguments to prove that their basis indeed spans $V(\lambda)$ and is linearly independent. When they had only produced the first half of the proof, they asked the second author of this paper the following question, which would imply the second half:

Question 1.1. (See [5].) Is there a combinatorial explanation for the fact that $\mathrm{GT}(\lambda)$ and $\mathrm{FFLV}(\lambda)$ contain the same number of lattice points?

This question provided the motivation for this paper. We answer it by generalizing a result of Stanley [7] on poset polytopes, as we now describe. Let P be a finite poset. Let A be a subset of P which contains all minimal and maximal elements of P. Let $\lambda = (\lambda_a)_{a \in A}$ be a vector in $\mathbf{R}^A$ such that $\lambda_a \leqslant \lambda_b$ whenever $a \leqslant b$. We think of λ as a marking of the elements of A with real numbers. We call such a triple (P, A, λ) a **marked poset**.

Definition 1.2. The **marked order polytope** of (P, A, λ) is

$$\mathcal{O}(P, A)_\lambda = \left\{x \in \mathbf{R}^{P-A} \;\middle|\; x_p \leqslant x_q \text{ for } p < q,\ \lambda_a \leqslant x_p \text{ for } a < p,\ x_p \leqslant \lambda_a \text{ for } p < a\right\},$$

where p and q represent elements of $P - A$, and a represents an element of A. The **marked chain polytope** of (P, A, λ) is

$$\mathcal{C}(P, A)_\lambda = \left\{x \in \mathbf{R}^{P-A}_{\geqslant 0} \;\middle|\; x_{p_1} + \cdots + x_{p_k} \leqslant \lambda_b - \lambda_a \text{ for } a < p_1 < \cdots < p_k < b\right\},$$

where a and b represent elements of A, and $p_1, \ldots, p_k$ represent elements of $P - A$.

For any polytope with integer coordinates Q there exists a polynomial $E_Q(t)$, the **Ehrhart polynomial** of Q, with the following property: for every positive integer n, the n-th dilate nQ of Q contains exactly $E_Q(n)$ lattice points (see [8]). With this notion, our answer to Question 1.1 is given by the following two results.

Theorem 1.3. *For any marked poset (P, A, λ) with $\lambda \in \mathbf{Z}^A$, the marked order polytope $\mathcal{O}(P, A)_\lambda$ and the marked chain polytope $\mathcal{C}(P, A)_\lambda$ have the same Ehrhart polynomial.*

Theorem 1.4. *For every partition λ there exists a marked poset (P, A, λ) such that $\mathrm{GT}(\lambda) = \mathcal{O}(P, A)_\lambda$ and $\mathrm{FFLV}(\lambda) = \mathcal{C}(P, A)_\lambda$.*

We also consider the extension of these constructions to other Lie algebras. Berenstein and Zelevinsky proposed a construction of generalized Gelfand–Tsetlin polytopes [1] for other semisimple Lie algebras. For the symplectic and odd orthogonal Lie algebras, their polytopes are also in the family of marked order polytopes. Therefore Theorem 1.3 yields candidates for the Feigin–Fourier–Littelmann–Vinberg polytopes in types B_n and C_n.

The paper is organized as follows. In Section 2 we discuss the relevant aspects of the representation theory of the simple complex Lie algebras $\mathfrak{sl}_n$. Section 3 treats marked order and chain polytopes, and gives a bijection between their lattice points. Section 4 discusses the application of the combinatorial results of Section 3 to the representation theoretic polytopes that interest us.

We note that the combinatorial Section 3 is self-contained, and may be of independent interest beyond the representation theoretic application. A possible way to read this article is to skip Section 2 and continue there directly.

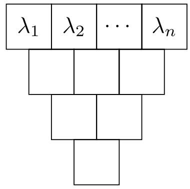


Fig. 1. Board defining Gelfand–Tsetlin patterns.

2. Preliminaries

Consider the simple complex Lie algebra $\mathfrak{sl}_n$. Let $\mathfrak{h}$ be the Cartan subalgebra consisting of its diagonal matrices. For $i = 1, \ldots, n$, let $\varepsilon_i \in \mathfrak{h}^*$ denote the projection onto the i-th diagonal component. As $\varepsilon_1 + \cdots + \varepsilon_n = 0$, the coefficient vector of an integral weight is only determined as an element of $\mathbf{Z}^n/\langle(1, \ldots, 1)\rangle$. We identify an integral weight with the corresponding equivalence class of coefficient vectors. If λ is a weight and we use the symbol λ in a context where it has to be interpreted as an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$, we use the convention that a representative has been chosen implicitly. Fix simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, n-1$. The corresponding fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. Hence dominant integral weights correspond to weakly decreasing n-tuples of integers, or partitions.

Given a dominant integral weight λ, the associated Gelfand–Tsetlin [6] polytope $\mathrm{GT}(\lambda)$ is defined as follows: Consider the board given in Fig. 1.

Each one of the $n(n-1)/2$ empty boxes stands for a real variable. The polytope $\mathrm{GT}(\lambda) \subset \mathbf{R}^{n(n-1)/2}$ is given by the fillings of the board with real numbers with the following property: each number is less than or equal to its upper left neighbor and greater than or equal to its upper right neighbor. Note that the ambiguity in choosing an n-tuple for the weight λ amounts to an integral translation of $\mathrm{GT}(\lambda)$, and hence does not affect its number of integral points. In fact, the integral points in $\mathrm{GT}(\lambda)$ parametrize the Gelfand–Tsetlin basis of $V(\lambda)$, hence $|\mathrm{GT}(\lambda) \cap \mathbf{Z}^{n(n-1)/2}| = \dim V(\lambda)$.

Feigin, Fourier, and Littelmann [3] associate a different polytope with a dominant integral weight λ as follows: The positive roots of $\mathfrak{sl}_n$ are $\Phi_+ = \{\alpha_{i,j} \mid 1 \leqslant i < j \leqslant n\}$, where $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$. In this notation the simple root α_i defined above can be written as $\alpha_i = \alpha_{i,i+1}$. A Dyck path is by definition a sequence $(\beta(0), \ldots, \beta(k))$ in Φ_+ such that $\beta(0)$ and $\beta(k)$ are simple, and if $\beta(l) = \alpha_{i,j}$, then either $\beta(l+1) = \alpha_{i+1,j}$ or $\beta(l+1) = \alpha_{i,j+1}$. Denote the coordinates on $\mathbf{R}^{\Phi_+}$ by s_β for $\beta \in \Phi_+$. Let $\lambda = m_1\omega_1 + \cdots + m_{n-1}\omega_{n-1}$. Then the polytope $\mathrm{FFLV}(\lambda) \subset \mathbf{R}^{\Phi_+}$ is given by the inequalities

$$s_\beta \geqslant 0$$

for all $\beta \in \Phi_+$ and

$$s_{\beta(0)} + \cdots + s_{\beta(k)} \leqslant m_i + \cdots + m_j$$

for all Dyck paths $(\beta(0), \ldots, \beta(k))$ such that $\beta(0) = \alpha_i$ and $\beta(k) = \alpha_j$.

For all $\alpha \in \Phi_+$, let f_α be a nonzero element of the root space $\mathfrak{g}_{-\alpha}$. Let v_λ be a highest weight vector of $V(\lambda)$. Fix any total order on Φ_+. As s ranges over the lattice points of $\mathrm{FFLV}(\lambda)$, the elements $(\prod_{\alpha \in \Phi_+} f_\alpha^{s_\alpha}) v_\lambda$ form a basis of $V(\lambda)$ [3, Theorem 3.11]. Hence $|\mathrm{FFLV}(\lambda) \cap \mathbf{Z}^{\Phi_+}| = \dim V(\lambda)$.

The previous discussion shows that $|\mathrm{FFLV}(\lambda) \cap \mathbf{Z}^{\Phi_+}| = |\mathrm{GT}(\lambda) \cap \mathbf{Z}^{n(n-1)/2}|$. In the sequel, we give a combinatorial explanation and an extension of this fact.

3. Marked poset polytopes

To any finite poset P, Stanley [7] associated two polytopes in $\mathbf{R}^P$: the order polytope and the chain polytope. He showed that there is a continuous, piecewise linear bijection between them, which restricts to a bijection between their sets of integral points. In this section we construct a generalization of the order and chain polytopes, and prove the analogous result. We begin with a review of Stanley's work.

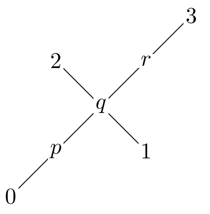


Fig. 2. A marked Hasse diagram defining a partial order on the set $P = \{p, q, r\} \cup A$ with $|A| = 4$ and $\lambda = (3, 2, 1, 0) \in \mathbf{R}^A$.

3.1. Stanley's order and chain polytopes

Let P be a finite poset. For $p, q \in P$ we say that p **covers** q, and write $p \succ q$, when $p > q$ and there is no $r \in P$ with $p > r > q$. We identify P with its **Hasse diagram**: the graph with vertex set P, having an edge going down from p to q whenever p covers q.

The **order polytope** and **chain polytope** of P are,

$$\mathcal{O}(P) = \left\{x \in [0,1]^P \;\middle|\; x_p \leqslant x_q \text{ for all } p < q\right\}, \quad \text{and}$$
$$\mathcal{C}(P) = \left\{x \in [0,1]^P \;\middle|\; x_{p_1} + \cdots + x_{p_k} \leqslant 1 \text{ for all chains } p_1 < \cdots < p_k\right\}$$

respectively.

Stanley proved that, even though $\mathcal{O}(P)$ and $\mathcal{C}(P)$ can have quite different combinatorial structures, they have the same Ehrhart polynomial. He did this as follows. Define the **transfer map** $\varphi : \mathbf{R}^P \to \mathbf{R}^P$ by

$$\varphi(x)_p = \begin{cases} x_p & \text{if } p \text{ is minimal,} \\ \min\{x_p - x_q \mid p \succ q\} & \text{otherwise} \end{cases} \tag{1}$$

for $x \in \mathbf{R}^P$, $p \in P$. Then:

Theorem 3.1. *(See [7, Theorem 3.2].) The transfer map φ restricts to a continuous, piecewise linear bijection from $\mathcal{O}(P)$ onto $\mathcal{C}(P)$. For any $m \in \mathbf{N}$, φ restricts to a bijection from $\mathcal{O}(P) \cap \frac{1}{m}\mathbf{Z}^P$ onto $\mathcal{C}(P) \cap \frac{1}{m}\mathbf{Z}^P$.*

3.2. Marked poset polytopes

We now recall the definition of marked order and chain polytopes, and prove that they satisfy a generalization of Theorem 3.1.

An element of a poset is called **extremal** if it is maximal or minimal.

Definition 3.2. A **marked poset** (P, A, λ) consists of a finite poset P, a subset $A \subseteq P$ containing all its extremal elements, and a vector $\lambda \in \mathbf{R}^A$ such that $\lambda_a \leqslant \lambda_b$ whenever $a \leqslant b$. We identify it with the **marked Hasse diagram**, where we label the elements $a \in A$ with λ_a in the Hasse diagram of P.

Definition 3.3. The **marked order polytope** of (P, A, λ) is

$$\mathcal{O}(P, A)_\lambda = \left\{x \in \mathbf{R}^{P-A} \;\middle|\; x_p \leqslant x_q \text{ for } p < q,\ \lambda_a \leqslant x_p \text{ for } a < p,\ x_p \leqslant \lambda_a \text{ for } p < a\right\},$$

where p and q represent elements of $P - A$, and a represents an element of A. The **marked chain polytope** of (P, A, λ) is

$$\mathcal{C}(P, A)_\lambda = \left\{x \in \mathbf{R}_{\geqslant 0}^{P-A} \;\middle|\; x_{p_1} + \cdots + x_{p_k} \leqslant \lambda_b - \lambda_a \text{ for } a < p_1 < \cdots < p_k < b\right\},$$

where a and b represent elements of A, and $p_1, \ldots, p_k$ represent elements of $P - A$.

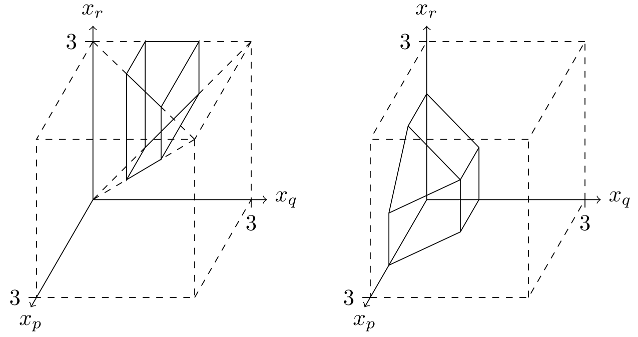


Fig. 3. The marked order polytope of the marked poset in Fig. 2 is given by the inequalities $0 \leqslant x_p \leqslant x_q \leqslant x_r \leqslant 3$ and $1 \leqslant x_q \leqslant 2$. The marked chain polytope is given by the inequalities $x_p, x_q, x_r \geqslant 0$, $x_p + x_q + x_r \leqslant 3$, $x_p + x_q \leqslant 2$, $x_q + x_r \leqslant 2$, and $x_q \leqslant 1$. Note that they are not combinatorially isomorphic.

Stanley's construction is a special case of ours as follows: Given any finite poset P, add a new smallest and largest element to obtain $\tilde{P} = P \cup \{\hat{0}, \hat{1}\}$ for $\hat{0}, \hat{1} \notin P$. Let $A = \{\hat{0}, \hat{1}\}$ and $\lambda = (0, 1)$. Then

$$\mathcal{O}(P) = \mathcal{O}(\tilde{P}, A)_\lambda \quad \text{and} \quad \mathcal{C}(P) = \mathcal{C}(\tilde{P}, A)_\lambda.$$

The following definitions will be needed in the proof of Theorem 3.4: The **length** of a chain $C = \{p_1 < \cdots < p_k\} \subseteq P$ is $\ell(C) = k - 1$. The **height** of $p \in P$ is the length of the longest chain ending at p. If P is graded, the height of an element is just its rank.

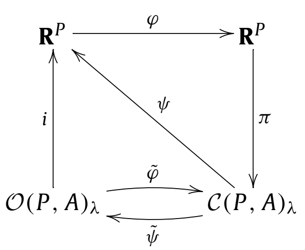
Theorem 3.4. *Let (P, A, λ) be a marked poset. The map $\tilde{\varphi} : \mathbf{R}^{P-A} \to \mathbf{R}^{P-A}$ defined by*

$$\tilde{\varphi}(x)_p = \min\bigl(\{x_p - x_q \mid p \succ q,\ q \notin A\} \cup \{x_p - \lambda_q \mid p \succ q,\ q \in A\}\bigr)$$

for each $p \in P - A$ restricts to a continuous, piecewise affine bijection from $\mathcal{O}(P, A)_\lambda$ onto $\mathcal{C}(P, A)_\lambda$.

The following alternative description of $\tilde{\varphi}$ may be useful. Let $\varphi : \mathbf{R}^P \to \mathbf{R}^P$ be Stanley's transfer map as defined in (1). Let $\pi : \mathbf{R}^P \to \mathbf{R}^{P-A}$ be the canonical projection which forgets the coordinates in A, and let $i : \mathbf{R}^{P-A} \to \mathbf{R}^P$ be the canonical inclusion into the fiber over $\lambda \in \mathbf{R}^A$, which adds a coordinate λ_a to each $a \in A$. Then $\tilde{\varphi} = \pi \circ \varphi \circ i$.

These maps (and some more to be defined in the proof) are illustrated in the following diagram.



Proof. We start by showing that $\tilde{\varphi}(\mathcal{O}(P, A)_\lambda) \subseteq \mathcal{C}(P, A)_\lambda$. Let $x \in \mathcal{O}(P, A)_\lambda$ and $y = \tilde{\varphi}(x)$. Let $a, b \in A$, and $p_1, \ldots, p_k \in P - A$ be such that $a < p_1 < \cdots < p_k < b$. The definition of φ implies that $y_{p_i} \leqslant x_{p_i} - x_{p_{i-1}}$ for all $i = 2, \ldots, k$ and $y_{p_1} \leqslant x_{p_1} - \lambda_a$. Thus,

$$\begin{aligned} y_{p_1} + \cdots + y_{p_k} &\leqslant (x_{p_1} - \lambda_a) + (x_{p_2} - x_{p_1}) + \cdots + (x_{p_k} - x_{p_{k-1}}) \\ &= x_{p_k} - \lambda_a \leqslant \lambda_b - \lambda_a. \end{aligned}$$

Hence, $y \in \mathcal{C}(P, A)_\lambda$.

To show that $\tilde{\varphi}$ is bijective, we construct its inverse $\tilde{\psi} : \mathcal{C}(P, A)_\lambda \to \mathcal{O}(P, A)_\lambda$. We first define a map $\psi : \mathbf{R}^{P-A} \to \mathbf{R}^P$, where we define $\psi(y)_p$ recursively by going up the poset according to the rule:

$$\psi(y)_p = \begin{cases} \lambda_p & \text{if } p \in A, \\ y_p + \max\{\psi(y)_q \mid p \succ q\} & \text{if } p \notin A. \end{cases}$$

Since all the elements of height 0 are in A, $\psi(y)$ is well-defined. We then define $\tilde{\psi} = \pi \circ \psi$ by applying ψ and then forgetting the A-coordinates. We will prove that, when restricted to $\mathcal{C}(P, A)_\lambda$, the map $\tilde{\psi}$ is the inverse of $\tilde{\varphi}$.

First we show that $\tilde{\psi} \circ \tilde{\varphi}$ is the identity on $\mathcal{O}(P, A)_\lambda$. We begin by showing that $\psi \circ \tilde{\varphi} = i$; i.e., that if $x \in \mathcal{O}(P, A)_\lambda$ and $y = \tilde{\varphi}(x)$ then $i(x) = \psi(y)$. We prove $i(x)_p = \psi(y)_p$ by induction on $\mathrm{ht}(p)$. The claim certainly holds for $\mathrm{ht}(p) = 0$. Suppose that we have proved it for all elements of height at most n, and let p have height $n + 1$. If $p \in A$, then

$$\psi(y)_p = \lambda_p = i(x)_p$$

by definition. Otherwise, if $p \notin A$, we have

$$\begin{aligned} \psi(y)_p &= y_p + \max\bigl\{\psi(y)_q \bigm| p \succ q\bigr\} \\ &= y_p + \max\bigl\{i(x)_q \bigm| p \succ q\bigr\} \end{aligned}$$

by the inductive hypothesis. As

$$\begin{aligned} y_p = \tilde{\varphi}(x)_p = \pi\bigl(\varphi\bigl(i(x)\bigr)\bigr)_p &= \varphi\bigl(i(x)\bigr)_p \\ &= \min\bigl\{i(x)_p - i(x)_q \bigm| p \succ q\bigr\} \\ &= i(x)_p - \max\bigl\{i(x)_q \bigm| p \succ q\bigr\}, \end{aligned}$$

we conclude that $\psi(y)_p = i(x)_p$, as desired.

We have shown that $\psi \circ \tilde{\varphi} = i$. By composing with the projection which forgets the A coordinates, we obtain that $\tilde{\psi} \circ \tilde{\varphi}$ is the identity on $\mathcal{O}(P, A)_\lambda$. Hence $\tilde{\varphi}$ is injective.

To prove surjectivity, let $y \in \mathcal{C}(P, A)_\lambda$ and define $x = \tilde{\psi}(y) \in \mathbf{R}^{P-A}$. We start by showing that $x \in \mathcal{O}(P, A)_\lambda$. Let $p \in P - A$. By definition,

$$x_p = \psi(y)_p = y_p + \max\bigl\{\psi(y)_q \bigm| p \succ q\bigr\}.$$

As $y_p \geqslant 0$, this implies $x_p \geqslant \psi(y)_q$ for all q such that $p \succ q$. If $q \in A$, this says that $x_p \geqslant \lambda_q$. If $q \notin A$, this says that $x_p \geqslant x_q$. To show that $x_p \leqslant \lambda_b$ for any $b \in A$ exceeding p, choose

$$p =: p_{k+1} \succ p_k \succ \cdots \succ p_1 \succ p_0$$

such that $p_k, \ldots, p_1 \in P - A$ and $p_0 \in A$, and that the maximum in the definition of $\varphi(y)_{p_i}$ is attained at $q = p_{i-1}$ for $i = k+1, \ldots, 1$. For any $b \in A$ with $b > p$, from the definition of $\mathcal{C}(P, A)_\lambda$ is follows that $y_{p_1} + \cdots + y_{p_k} \leqslant \lambda_b - \lambda_{p_0}$. By the choice of the p_i, we have

$$x_p = \varphi(y)_p = \lambda_{p_0} + y_{p_1} + \cdots + y_{p_k}.$$

Hence $x_p \leqslant \lambda_b$. As p is arbitrary, it follows that $x \in \mathcal{O}(P, A)_\lambda$.

Finally, we claim that $\tilde{\varphi}(x) = y$. Once again, we prove that $\tilde{\varphi}(x)_p = y_p$ for all $p \in P - A$ by induction on the height of p. For height 0 this statement is vacuous. Suppose that it holds for all elements of height at most n, and consider $p \in P - A$ with $\mathrm{ht}(p) = n + 1$. Then

$$\begin{aligned} \tilde{\varphi}(x)_p &= \min\bigl\{i(x)_p - i(x)_q \bigm| p \succ q\bigr\} \\ &= \min\bigl\{\psi(y)_p - \psi(y)_q \bigm| p \succ q\bigr\} \\ &= \psi(y)_p - \max\bigl\{\psi(y)_q \bigm| p \succ q\bigr\} \\ &= y_p + \max\bigl\{\psi(y)_q \bigm| p \succ q\bigr\} - \max\bigl\{\psi(y)_q \bigm| p \succ q\bigr\} \\ &= y_p, \end{aligned}$$

as desired. We have shown that $\tilde{\varphi} \circ \tilde{\psi}$ is the identity on $\mathcal{C}(P, A)_\lambda$, hence $\tilde{\varphi}$ is surjective.

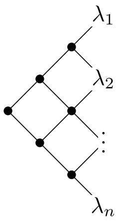


Fig. 4. Marked Hasse diagram for $\mathfrak{sl}_n$.

We conclude that $\tilde{\psi} : \mathcal{C}(P, A)_\lambda \to \mathcal{O}(P, A)_\lambda$ and $\tilde{\varphi} : \mathcal{O}(P, A)_\lambda \to \mathcal{C}(P, A)_\lambda$ are inverse functions, and therefore bijective, as we wished to show. The fact that they are continuous and piecewise affine follows directly from the definitions. □

We conclude this section with the generalization of the second part of Theorem 3.1, the compatibility of the transfer map with the integral lattice.

Lemma 3.5. *If (P, A, λ) is a marked poset with λ integral, then the polytopes $\mathcal{O}(P, A)_\lambda$ and $\mathcal{C}(P, A)_\lambda$ are integral.*

Proof. It is immediate from its defining inequalities that $\mathcal{O}(P, A)_\lambda$ is integral. We now "transfer" this property to $\mathcal{C}(P, A)_\lambda$.

Consider the subdivision of $\mathcal{O}(P, A)_\lambda$ induced by the braid arrangement, which consists of the hyperplanes $x_p = x_q$ for p and q in $P - A$. Each polytope C in this subdivision is integral. Notice that $\tilde{\varphi}$ is linear on C and, since $\tilde{\varphi}$ maps lattice points to lattice points, $\tilde{\varphi}(C)$ is also integral. Therefore $\mathcal{C}(P, A)_\lambda$ has a subdivision into integral polytopes, and hence must be integral. □

It is worth remarking that the integrality of the polytope $\mathcal{C}(P, A)_\lambda$ is quite subtle, in the sense that the inequalities defining it do not necessarily form a totally unimodular matrix.

Since $\mathcal{O}(P, A)_\lambda$ and $\mathcal{C}(P, A)_\lambda$ are integral, they have an Ehrhart polynomial. In fact:

Theorem 3.6. *Let (P, A, λ) be a marked poset with $\lambda \in \mathbf{Z}^A$. Then $\mathcal{O}(P, A)_\lambda$ and $\mathcal{C}(P, A)_\lambda$ have the same Ehrhart polynomial.*

Proof. This follows immediately from the proof of Theorem 3.4, which shows that $\tilde{\varphi}$ restricts to a bijection between $\mathcal{O}(P, A)_\lambda \cap \frac{1}{m}\mathbf{Z}^{P-A}$ and $\mathcal{C}(P, A)_\lambda \cap \frac{1}{m}\mathbf{Z}^{P-A}$. □

Theorem 3.6 does not hold for general $\lambda \in \mathbf{R}^A$.

4. Applications

We now show how marked poset polytopes occur "in nature" in the representation theory of semisimple Lie algebras. More concretely, marked order polytopes occur as Gelfand–Tsetlin polytopes in types A, B, and C, and marked chain polytopes occur as Feigin–Fourier–Littelmann–Vinberg polytopes in type A.

4.1. Type A

Let λ be a dominant integral weight for $\mathfrak{sl}_n$. Let $\mathcal{O}(P, A)_\lambda$ and $\mathcal{C}(P, A)_\lambda$ be the marked order and chain polytopes determined by the marked Hasse diagram given in Fig. 4. Note that Fig. 4 is obtained from Fig. 1 by a clockwise rotation by 90°. Hence from the definitions it is immediate that $\mathrm{GT}(\lambda) = \mathcal{O}(P, A)_\lambda$. Similarly, it follows immediately from the definitions that $\mathrm{FFLV}(\lambda) = \mathcal{C}(P, A)_\lambda$. Hence the equation

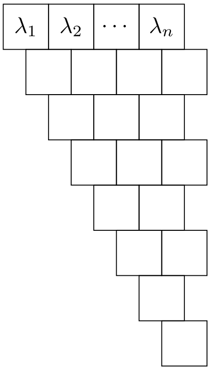


Fig. 5. Board defining generalized Gelfand–Tsetlin patterns for $\mathfrak{sp}_{2n}$ and $\mathfrak{o}_{2n+1}$.

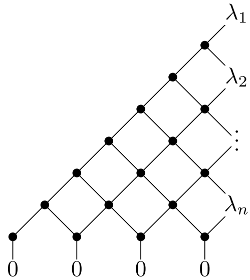


Fig. 6. Marked Hasse diagram for $\mathfrak{sp}_{2n}$ and $\mathfrak{o}_{2n+1}$.

$$\left|\,\mathrm{FFLV}(\lambda)\cap \mathbf{Z}^{\Phi_+}\right| = \left|\mathrm{GT}(\lambda)\cap \mathbf{Z}^{n(n-1)/2}\right|$$

is implied by Theorem 3.6.

It would be interesting to see whether the explicit bijection of Theorem 3.6 gives interesting information about the transition matrix between the Gelfand–Tsetlin basis and the Feigin–Fourier–Littelmann–Vinberg basis of $V(\lambda)$.

4.2. Type C

Now consider the symplectic Lie algebra $\mathfrak{sp}_{2n}$. Here the role of Gelfand–Tsetlin patterns is played by the generalized Gelfand–Tsetlin patterns defined by Berenstein and Zelevinsky [1]. Fix a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{sp}_{2n}$. Choose simple roots $\alpha_1,\ldots,\alpha_n\in\mathfrak{h}^*$ such that $\alpha_i\not\perp\alpha_{i+1}$ for $i<n$ and α_n is the long root. Let $\varepsilon_1,\ldots,\varepsilon_n$ be the basis of $\mathfrak{h}^*$ such that $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$ for $i<n$ and $\alpha_n=2\varepsilon_n$. The corresponding fundamental weights are $\omega_i=\varepsilon_1+\cdots+\varepsilon_i$. This is the setting as used by Bourbaki [2]. We identify a weight λ with the n-tuple $(\lambda_1,\ldots,\lambda_n)$ of its coefficients with respect to the basis $\varepsilon_1,\ldots,\varepsilon_n$. Then dominant integral weights correspond to weakly decreasing n-tuples of nonnegative integers. Given a dominant integral weight λ, Berenstein and Zelevinsky define an $\mathfrak{sp}_{2n}$-**pattern** of highest weight λ to be a filling of the board in Fig. 5 with nonnegative integers, such that every number is bounded from above by its upper left neighbor and bounded from below by its upper right neighbor (if any). They show that $\dim V(\lambda)$ is the number of such patterns [1, Theorem 4.2].

Let $\mathcal{O}(P,A)_{(\lambda,0)}$ and $\mathcal{C}(P,A)_{(\lambda,0)}$ be the marked order and chain polytopes determined by the marked Hasse diagram given in Fig. 6. Note that Fig. 6 is obtained from Fig. 5 by a clockwise rotation by 90° and apposition of the zeroes. From the definitions it is immediate that the $\mathfrak{sp}_{2n}$-patterns of highest weight λ are the integral points in $\mathcal{O}(P,A)_{(\lambda,0)}$. This suggests the following:

Conjecture 1. *The lattice points in $\mathcal{C}(P,A)_{(\lambda,0)}$ parametrize a PBW basis of $V(\lambda)$ for the symplectic Lie algebras, as described in Section* 2 *and in* [3, Theorem 3.11].

Indeed, this conjecture is proved in a forthcoming article by Feigin, Fourier, and Littelmann [4].

4.3. Type B

For the odd orthogonal Lie algebra $\mathfrak{o}_{2n+1}$, the situation is a bit more complicated. Fix a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{o}_{2n+1}$. Choose simple roots $\alpha_1,\ldots,\alpha_n\in\mathfrak{h}^*$ such that $\alpha_i\not\perp\alpha_{i+1}$ for $i<n$ and α_n is the short root. Let $\varepsilon_1,\ldots,\varepsilon_n$ be the basis of $\mathfrak{h}^*$ such that $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$ for $i<n$ and $\alpha_n=\varepsilon_n$. The corresponding fundamental weights are $\omega_i=\varepsilon_1+\cdots+\varepsilon_i$ for $i<n$ and $\omega_n=\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_n)$. This is the setting as used by Bourbaki [2]. We identify a weight λ with the n-tuple $(\lambda_1,\ldots,\lambda_n)$ of its

coefficients with respect to the basis $\varepsilon_1, \ldots, \varepsilon_n$. Then dominant integral weights correspond to weakly decreasing n-tuples in $\frac{1}{2}\mathbf{Z}_{\geqslant 0}$ such that either all or none of the components are integers. Given a dominant integral weight λ, Berenstein and Zelevinsky [1] define an $\mathfrak{o}_{2n+1}$**-pattern** of highest weight λ to be a filling of the board in Fig. 5 with elements of $\frac{1}{2}\mathbf{Z}_{\geqslant 0}$ such that every number is bounded from above by its upper left neighbor and bounded from below by its upper right neighbor (if any), and such that all numbers which possess an upper right neighbor are congruent to λ_1 modulo $\mathbf{Z}$. Let $R(\lambda)$ be the set of $\mathfrak{o}_{2n+1}$-patterns of highest weight λ.

As in type C, let $\mathcal{O}(P, A)_{(\lambda,0)}$ be the marked order polytope defined by the marked Hasse diagram in Fig. 6. Then $R(\lambda) \subset \mathcal{O}(P, A)_{(\lambda,0)}$, but $R(\lambda)$ does not consist of the integral points, but of the points determined by more complicated congruence conditions. Namely, decompose

$$P - A = P' \cup P'' \cup P''',$$

where P', P'', and P''' consist of all elements in P of height 1, 2, and $\geqslant 3$, respectively, that are not contained in A. Then $R(\lambda)$ consists of all $x \in \mathcal{O}(P, A)_{(\lambda,0)} \cap (\frac{1}{2}\mathbf{Z})^{P-A}$ such that $x_p + \lambda_1 \in \mathbf{Z}$ for all $p \in P'' \cup P'''$. Hence $S(\lambda) = \tilde{\varphi}(R(\lambda))$ consists of all

$$y \in \mathcal{C}(P, A)_{(\lambda,0)} \cap \left(\left(\frac{1}{2}\mathbf{Z}\right)^{P' \cup P''} \times \mathbf{Z}^{P'''}\right)$$

such that

$$\max\{y_q\colon\ p \succ q\} + y_p + \lambda_1 \in \mathbf{Z}$$

for all $p \in P''$. From the point of view taken in this article, $S(\lambda)$ appears to be the most natural candidate to parametrize a PBW basis of [3] in type C. Note that the elements of $S(\lambda)$ cannot appear directly as exponent vectors of a PBW basis, as their components are not necessarily integral, so we are missing at least a change of coordinates in this case.

Question 4.1. Is there a way to modify $S(\lambda)$ so that it parametrizes a PBW basis of $V(\lambda)$ for the odd orthogonal Lie algebras, as described in Section 2 and in [3, Theorem 3.11]?

4.4. Type D

Some of the inequalities defining the generalized Gelfand–Tsetlin polytopes [1] for the even orthogonal Lie algebras $\mathfrak{o}_{2n}$ involve several summands, so these polytopes are not marked order polytopes. Also, as for type B, the lattice used to define the integral points is not the canonical lattice. Hence our methods do not apply here directly. It would be interesting to find either a suitable modification of our results to this case, or a suitable change of coordinates to represent the polytopes as marked order polytopes.

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