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A least squares approach to reduce stable discrete linear systems preserving their stability

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Abstract

A new stability preserving model reduction algorithm for discrete linear SISO-systems based on a least squares approach is proposed. Similar to the Padé approximation, an equation system for the Markov parameters involving a high dimensional Hankel matrix is considered. It is proved that approximate solutions, computed via the Moore–Penrose pseudo-inverse, give rise to a stability preserving reduction scheme. Furthermore, the proposed algorithm is compared to the balanced truncation method, showing comparable performance of the reduced systems.

Keywords: Discrete linear systems; Model reduction; Stability; Hankel matrix; Stein equation

1. Introduction

Generally the problem of model reduction lies in the replacement of a given mathematical description Σ of a natural process by a description $\widehat{\Sigma}$, which is much smaller than the ν-dimensional system Σ, but still, guarantees the main properties of the original process. For discrete linear systems

$$\Sigma : \begin{matrix} x_{k+1} = \mathscr{A}x_k + \mathscr{B}u_k \\ y_k = \mathscr{C}x_k \end{matrix}, \quad x_0 = o, \quad \mathscr{A} \in \mathbb{C}^{\nu\times\nu}, \quad \mathscr{B}, \mathscr{C}^{\mathrm{T}} \in \mathbb{C}^{\nu} \tag{1}$$

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it makes sense to call an n-dimensional system $\widehat{\Sigma}$ *smaller* than Σ, if the inequality $n < \nu$ is satisfied. A typical property, which should be preserved by changing over to the reduced system, is stability. A discrete system Σ is said to be *stable*, if Σ has all its eigenvalues inside the open unit disc $\mathscr{D}$. Clearly, the reduced system $\widehat{\Sigma}$ should moreover fulfil certain approximation requirements. For discrete systems it is natural to demand the similarity of their impulse responses. More precisely, we are interested in the solution of the following problem: let a ν-dimensional stable discrete linear single-input-single-output-system (SISO) (1) be given, we then are looking for an n-dimensional stable system $\widehat{\Sigma}$, such that $n < \nu$, and the quantity

$$\varepsilon := \frac{\|G - \widehat{G}\|_2}{\|G\|_2}, \quad \|G\|_2 := \left(\tfrac{1}{2\pi}\int_0^{2\pi} |G(\mathrm{e}^{\mathbf{i}\omega})|^2 \,\mathrm{d}\omega\right)^{1/2}, \tag{2}$$

is smaller than a prescribed error bound ε_0. Here, G and $\widehat{G}$ denote the transfer functions of Σ and $\widehat{\Sigma}$. Similar impulse responses are reflected by a small ε. A huge number of papers exists which deal with approximation problems of this nature. Two keywords are *Padé approximation* and *balanced truncation*. A good source to obtain an overview of the first topic is [9], and concerning balanced model reduction we refer the reader to [33,35]. It would be desirable to estimate the appropriate size n of $\widehat{\Sigma}$ a priori, such that the resulting approximation error ε, defined by (2), turns out to be smaller than a prescribed ε_0. We are aware, that this problem will not be solved here. In order to determine the appropriate dimension for the reduced system, we are only able to compute a sequence of stable systems $(\widehat{\Sigma}_n)_{n=1}^{\nu-1}$ and to observe the corresponding decreasing number sequence $(\|G - \widehat{G}_n\|_2)_{n=1}^{\nu-1}$.

For the discrete time case the Padé approximation problem reads as follows: let a strictly proper rational function G of degree ν be given. Then the nth Padé approximation problem of order N is to find a rational function $\widehat{G}$ of degree $n < \nu$, such that in a neighbourhood of ∞ the difference $G - \widehat{G}$ admits the representation

$$G(s) - \widehat{G}(s) = \sum_{k=N}^{\infty} d_k s^{-(k+1)}.$$

The context to the model reduction problem is based on the following two facts:

- the impulse response of (1) is given by the Laurent coefficient sequence $(h_k)_{k=0}^{\infty}$ of its transfer function $G(s) := \sum_{k=0}^{\infty} h_k s^{-(k+1)} := \mathscr{C}(sI - \mathscr{A})^{-1}\mathscr{B}$, and
- the dimension of a minimal realization and the degree of a rational function $\widehat{G}$ coincide.

Consequently, $\widehat{G}$ admits an n-dimensional realization by a system with the same impulse response as Σ up to the Nth time step. In particular, the numbers h_k are called the *Markov parameters* of Σ. The nth Padé approximation problem of order N is equivalent to the *partial realization problem*: let a finite complex number sequence $h := (h_k)_{k=0}^{N-1}$ be given. The problem is to find a matrix triple $\widehat{\Sigma} := (A, B, C)$, $A \in \mathbb{C}^{n\times n}$, $B, C^{\mathrm{T}} \in \mathbb{C}^n$, such that for $k = 0, \ldots, N-1$ the equation $h_k = CA^kB$ holds.

One deals with the *minimal partial realization* (MPR) *problem* if one is interested in solutions with minimal state space dimension n. For details we refer the reader to [1,5,8,11–13,16,17,21,23,24,27,30,31,32,34].

In connection with model reduction problems the sequence h is defined by the first N Markov parameters of Σ, abbreviated by $h(\Sigma, N)$. For non-stable minimal systems, i.e. $\mathscr{A}$ is not an iteration matrix, for large N the calculation of $h(\Sigma, N)$ turns out to be impossible, because the sequence $(\mathscr{C}\mathscr{A}^k\mathscr{B})_{k=0}^{\infty}$ is non-convergent. Even for small N, if the associated Hankel matrix

$$H_N^n := \begin{bmatrix} h_0 & \cdots & h_{n-1} \\ \vdots & & \vdots \\ h_{\ell-1} & \cdots & h_{N-2} \end{bmatrix} \in \mathbb{C}^{\ell\times n}, \quad \ell + n = N, \tag{3}$$

exists, it then is frequently ill conditioned. These numerical difficulties can be avoided by exploiting the connection between Padé approximants and the Lanczos process (Padé approximation via Lanczos process, PVL). For details confer [7,18,19], and the references therein.

An important drawback of Padé approximation methods is that they generally do not preserve stability. That means one cannot ensure that all poles of $\widehat{G}$ are again in the open left half plane or in the open unit disc even if this holds for G. For the continuous time case an illustrating example can be found in [6], and for the discrete time case, we demonstrate this disadvantage with Example 1. One possibility to overcome this problem consists in the reduction of the order N of the Padé approximation problem to obtain a solution family of non-minimal partial realizations for $h(\Sigma, N)$ containing more than one element. The gained freedom then can be used to try to find a stable one. For example, one starts with an arbitrary stable denominator polynomial of degree n, and computes the corresponding numerator polynomial such that at least the first n members of $h(\Sigma, N)$ are matched. For details see [9], Section 12, and the references therein. Our reduction method is inspired by Padé approximation as well, however we overcome the stability problem by allowing small deviations between $\hat{h}_k$ and h_k for all indices. For compensational reasons, we however try to approximate a large number N of Markov parameters h_k.

The nth Padé approximation problem of order N for G is solved, if one is able to compute a vector $\xi \in \mathbb{C}^n$, such that

$$H_N^n \xi = h_N^n, \quad h_N^n := [h_n, \ldots, h_{N-1}]^{\mathrm{T}}. \tag{4}$$

Then the denominator polynomial of a fraction representation of $\widehat{G}$ is given by

$$Q_n(s) := s^n - [s^0, \ldots, s^{n-1}]\xi. \tag{5}$$

However for $n \ll N$ it cannot be expected, that the vector h_N^n belongs to the subspace span $H_N^n \subset \mathbb{C}^\ell$. In that case, it seems natural for us to set

$$\xi := {H_N^n}^{\dagger} h_N^n, \tag{6}$$

where $F^{\dagger}$ denotes the Moore–Penrose pseudo-inverse of the matrix F. Our main result (Theorem 2) concerns the stability of the polynomial (5) for

$$\xi = \lim_{N\to\infty} {H_N^n}^{\dagger} h_N^n.$$

To measure the "similarity" of the impulse responses, the quantity $\|G - \widehat{G}\|_2$ is appropriate. For its estimation, the identification of the residual vector

$$\mathscr{E}^n := H_\infty^n \xi - h_\infty^n$$

as ℓ_2-sequence is useful, since there holds:

$$\|G - \widehat{G}\|_2 \leqslant \|\mathscr{E}^n\|_2 (1 - r_{\widehat{\Sigma}})^{-n}.$$

Here, $r_{\widehat{\Sigma}}$ denotes the maximum of the moduli of the zeros of Q_n.

The above mentioned stability problems do not arise if one reduces Σ via the *balanced truncation* method. The crucial step in this approach is the transformation of Σ into a balanced form, which requires the solution of two ν-dimensional Stein equations. This step however is computationally quite expensive for large ν and additionally often turns out to be ill conditioned. For details confer [29], and [35, Chapter 21.8]. The advantage of our method is the avoidance of those ν-dimensional matrix equation problems. We therefore are able to solve high dimensional problems. For example for original state space dimension $\nu := 1000$, reduced state space dimension $n := 30$, and $N := 500$ used Markov parameters one obtains Fig. 1.

In the upper part the Nyquist plot of G (solid line) is compared to the Nyquist plot of $\widehat{G}$ (dotted), and in the lower part the eigenvalues of Σ (dots) and $\widehat{\Sigma}$ (stars) are

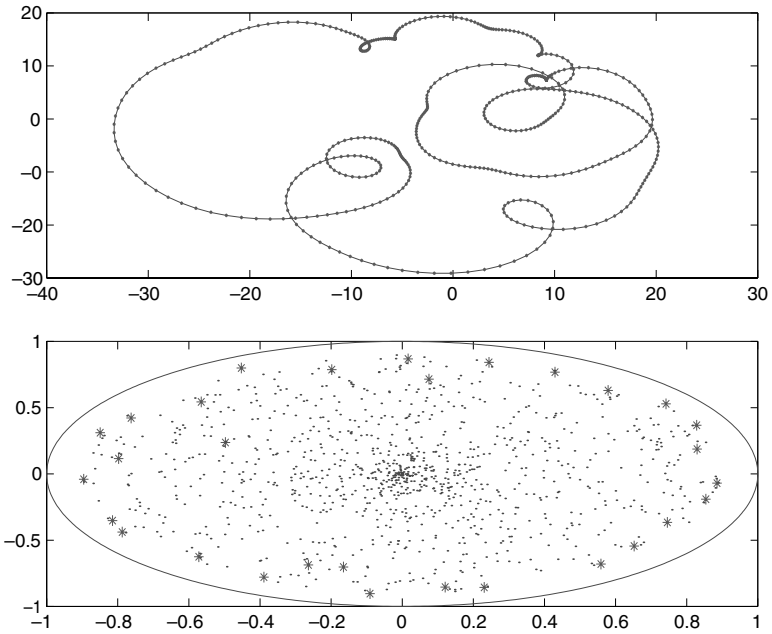

Fig. 1. $[r_\Sigma, \nu, n, N] = [0.9, 1000, 30, 500]$.

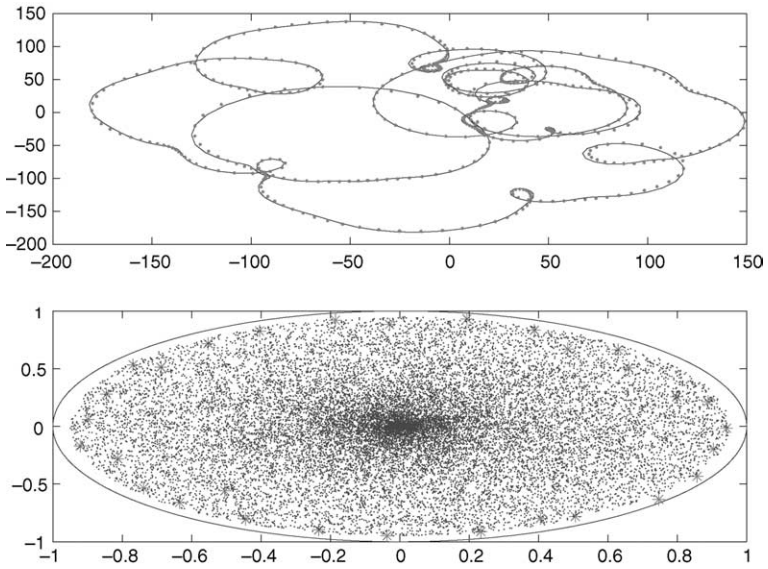

Fig. 2. $[r_\Sigma, \nu, n, N] = [0.95, 20000, 30, 500]$.

shown. To avoid numerical problems, the moduli of the eigenvalues of the system Σ are bounded by $r_\Sigma := 0.9$. For $[r_\Sigma, \nu, n, N] = [0.95, 20\,000, 30, 500]$ one obtains for example Fig. 2.

In both examples, the Moore–Penrose pseudo-inverse ${H_N^n}^\dagger$ has been computed via

$$(H_N^{n\,*} H_N^n)^{-1} H_N^{n\,*}.$$

For the case where r_Σ is almost 1, this approach does not lead to satisfying numerical results. Thus it is natural to ask for numerical algorithms, which compute ${H_N^n}^\dagger$ by exploitation of its Hankel structure. In the literature the Toeplitz structure is exploited in two different directions: one approach leads to algorithms of low complexity (fast and superfast algorithms), and a second approach to algorithms, which take care for accuracy and stability (high performance algorithms). For fast and superfast algorithms confer among others [2–4,10,14,15,28], and for high performance algorithms confer [20,22,25]. The paper [26] of Gu goes in both directions.

The paper is organised as follows. First we repeat the solution procedure for the MPR-problem. Then, the difference of the Markov parameter sequences of Σ and its reduction $\widehat{\Sigma}$ is studied, where $\widehat{\Sigma}$ is obtained via the polynomial (5) with coefficient vector (6). In Section 4 the proof of Theorem 2 is prepared. Then we proceed with its formulation and discussion of certain consequences. In Section 6 a reduction example is given and in Section 7 the transfer functions of $\widehat{\Sigma}$ and of the corresponding system Σ_{trunc} obtained by balanced truncation are compared.

2. The MPR-problem

A solution of the MPR-problem can be found by considering the Hankel matrix sequence $\mathfrak{H} := (H_{\ell,n})_{n=1}^{N}$, $\ell + n = N + 1$:

$$H_{\ell,n} = \begin{bmatrix} h_0 & \cdots & h_{n-1} \\ \vdots & & \vdots \\ h_{\ell-1} & \cdots & h_{N-1} \end{bmatrix} \in \mathbb{C}^{\ell\times n}.$$

Here, the polynomial spaces

$$\mathscr{H}_n := \{\psi(s,n)\xi : \xi \in \ker H_{\ell,n}, \ell + n = N + 1\} \subset \mathbb{C}[s],$$
$$\mathscr{H}_{N+1} := \{p(s) \in \mathbb{C}[s] : \deg p \leqslant N\},$$

are of special interest, where $\psi(s,n) := [s^0, \ldots, s^{n-1}]$. The Hankel structure of the members of $\mathfrak{H}$ implies

$$s\mathscr{H}_n \subseteq \mathscr{H}_{n+1}, \quad \mathscr{H}_1 \subset \mathscr{H}_2 \subset \cdots \subset \mathscr{H}_{N+1}.$$

In [27] the following is proved.

Proposition 1. *For any non-zero sequence $h := (h_k)_{k=0}^{N-1}$ there exist two integers d_1, d_2 with $d_1 \leqslant d_2$, $d_1 + d_2 = N + 1$ and two monic polynomials $q_1 \in \mathscr{H}_{d_1+1}$, $q_2 \in \mathscr{H}_{d_2+1}$, such that for all $n = 1, \ldots, N + 1$:*

$$\mathscr{H}_n = \{q_1p_1 + q_2p_2 \mid p_i \in \mathbb{C}[s], \deg p_i < n - d_i, i = 1, 2\}.$$

Here we have to set $p_i \equiv 0$ if $n \leqslant d_i$.

The polynomial system $\{q_1, q_2\}$ is called a *fundamental system* and the two integers d_1, d_2 are called the *characteristic* degrees for h. Theorem 1, taken from [27], relates the characteristic degrees of h to the dimension of its minimal partial realizations (MPRs).

Theorem 1. *Let d_1, d_2 be the characteristic degrees, $\{q_1, q_2\}$ be a fundamental system, and ν be the state space dimension of a MPR of h. Then*

1. *If* $\deg q_1 = d_1$, *then* $\nu = d_1$, *else* $\nu = d_2$.
2. *If $\nu = d_1 < d_2$, then h has a unique MPR up to similarity.*
3. *If $\nu = d_2$, then h has infinitely many MPRs with diagonalisable main operator A.*

Starting with

$$Q(s) := s^{\nu} - \sum_{k=0}^{\nu-1} \xi_k s^k := \begin{cases} q_1(s), & \text{if } \deg q_1 = d_1, \\ q_2(s), & \text{if } \deg q_1 < d_1, \end{cases}$$
$$\xi := [\xi_0, \ldots, \xi_{\nu-1}],$$

a MPR $\widehat{\Sigma}$ of h is constructed as follows: to the polynomial Q a polynomial P is assigned according to $P(s) := s^{\nu-1}p_{\nu-1} + \cdots + p_0$, where

$$\begin{aligned} &[p_{\nu-1}, \ldots, p_0]^{\mathrm{T}} := T(h, \nu)[1, -\xi_{\nu-1}, \ldots, -\xi_1]^{\mathrm{T}}, \\ &T(h, \nu) := \begin{bmatrix} h_0 & & \\ \vdots & \ddots & \\ h_{\nu-1} & \cdots & h_0 \end{bmatrix}. \end{aligned} \tag{7}$$

The polynomial P is said to be the *residual polynomial* of Q with respect to h, and the vector $\mathrm{res}(Q, h)$ is defined by $\mathrm{res}(Q, h) := [p_0, \ldots, p_{\nu-1}]$. Now, the matrix triple

$$\widehat{\Sigma} := (C_Q^{\mathrm{T}}, e_\nu, \mathrm{res}(Q, h)), \quad C_Q := \left[\begin{array}{c|c} 0 \ldots 0 & \\ \hline I_{\nu-1} & \xi \end{array}\right], \quad e_\nu := [0, \ldots, 0, 1]^{\mathrm{T}} \tag{8}$$

represents a MPR of h, and is known as the *controller form realization* of the rational function P/Q. The matrix C_Q is referred to as the *companion matrix* of Q. As already mentioned above, the Padé approximation may lead to non-stable model reduction results, even if the original system is stable. This disadvantage is shown in Example 1.

Example 1. For all $\alpha \in (0, 1)$ the three-dimensional system

$$\Sigma(\alpha) : \mathscr{A} = \mathrm{diag}(\alpha, 0, -\alpha), \quad \mathscr{B} = [2, -1, -1]^{\mathrm{T}}, \quad \mathscr{C} = [1, 1, 1]$$

is stable. With respect to the above described solution procedure, for certain $\alpha \in (0, 1)$ we compute a two-dimensional MPR $\widehat{\Sigma}$ of $h(\Sigma(\alpha), 4)$, which turns out to be non-stable and unique up to similarity. Hence, for the transfer function G_α of $\Sigma(\alpha)$ no stable Padé approximant $\widehat{G}$ exists, which fulfils $G(s) - \widehat{G}(s) = \sum_{i=4}^{\infty} \tilde{h}_i s^{-i-1}$, and can be realized by a system with state space dimension less than 3. Obviously, $h_0 = 0$, $h_1 = 3\alpha$, $h_2 = \alpha^2$, $h_3 = 3\alpha^3$, and the solution of the equation system

$$H_4^2 \xi = h_4^2, \quad H_4^2 = \begin{bmatrix} h_0 & h_1 \\ h_1 & h_2 \end{bmatrix}, \quad h_4^2 = \begin{bmatrix} h_2 \\ h_3 \end{bmatrix},$$

is given by $\xi := [8\alpha^2/9, \alpha/3]^{\mathrm{T}}$, where in particular for $\alpha \neq 0$ the Hankel matrix H_4^2 is regular. Consequently, the first characteristic degree d_1 of h is equal to 2, and the first characteristic polynomial q_1 is given by

$$q_1(s) := s^2 - \psi(s, 2)\xi = s^2 - (\alpha/3)s - (8/9)\alpha^2$$

with the zeros $\frac{(\sqrt{33}\pm 1)}{6}\alpha$. Due to $1 < (\sqrt{33}+1)/6$, one is able to choose $\alpha \in (0, 1)$, such that $1 < \frac{\sqrt{33}+1}{6}\alpha$ is fulfilled, namely $\alpha \in (6/(\sqrt{33}+1), 1)$. For these α the matrix C_{q_1} is non-stable, which is equivalent to the non-stability of the system

$\widehat{\Sigma} := (C_{q_1}^{\mathrm{T}}, e_2, \mathrm{res}(q_1, h))$. Finally, we have $2 = \nu = d_1 < d_2 = 3$. Hence, Theorem 1 states the uniqueness of $\widehat{\Sigma}$ up to similarity.

To overcome the stability problem in all that follows we loosen the demand for exact coincidence between h_k and $\hat{h}_k$ for $k = 0, \ldots, N-1$. For compensational reasons, we however try to approximate a large number N of Markov parameters.

3. Projection of h_N^n on span H_N^n

The condition $Q_n \in \mathscr{H}_{n+1}$, $\deg Q_n = n$ is equivalent to the existence of a solution for

$$H_N^n x = h_N^n,$$

where H_N^n and h_N^n are defined as in (3) and (4). In the case where $h_N^n \notin \operatorname{span} H_N^n$, and one is satisfied with a vector ξ, such that

$$\mathscr{E}_N^n := H_N^n \xi - h_N^n, \quad \|\mathscr{E}_N^n\|_2 \approx 0,$$

we will show that the system $\widehat{\Sigma} := (C_{Q_n}^{\mathrm{T}}, e_n, \mathrm{res}(Q_n, h))$, $Q_n(s) := s^n - \psi(s, n)\xi$, becomes stable, if ξ is chosen to be

$$\xi := \lim_{N\to\infty} {H_N^n}^{\dagger} h_N^n.$$

Moreover, due to the properties of the Moore–Penrose pseudo-inverse, supposed that the columns of H_N^n are linear independent, the residual vector $\mathscr{E}_N^n$ satisfies the relation

$$\|\mathscr{E}_N^n\|_2 = \min_{x\in\mathbb{C}^n} \|H_N^n x - h_N^n\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{C}^\ell$. Theorem 1 implies $\|\mathscr{E}_N^n\|_2 = 0$, if $\nu \leqslant n$, and $0 < \|\mathscr{E}_N^n\|_2$, if $n < \nu$, where ν denotes the dimension of the MPRs of $h(\Sigma, N)$. The following Proposition relates the chosen state space dimension $n < \nu$ to the differences of the Markov parameters from $\widehat{\Sigma}$ and Σ.

Proposition 2. *Let h be the Markov parameter sequence of a ν-dimensional system Σ. Consider for $n < \nu$ the monic polynomial Q and the system $\widehat{\Sigma}$, defined by $Q(s) := s^n - \psi(s, n){H_N^n}^{\dagger} h_N^n$ and $\widehat{\Sigma} := (C_Q^{\mathrm{T}}, e_n, \mathrm{res}(Q, h))$. Then the differences $d_k = \hat{h}_k - h_k$, $k = 0, \ldots, N-1$, can be represented via the residual vector $\mathscr{E}_N^n$ of the projection of h_N^n on* span H_N^n. *More precisely they are given by the output of the system*

$$\begin{aligned} x_{k+1} &= C_Q^{\mathrm{T}} x_k + e_n \varepsilon_k \\ d_k &= e_n^{\mathrm{T}} x_k \end{aligned}, \quad x_0 = o, \tag{9}$$

with respect to the input $\mathscr{E} = [\varepsilon_0, \ldots, \varepsilon_{n-1}, (\mathscr{E}_N^n)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{C}^N$, defined by

$$\varepsilon_0 = \cdots = \varepsilon_{n-1} = 0, \quad \mathscr{E}_N^n = H_N^n {H_N^n}^{\dagger} h_N^n - h_N^n.$$

Proof. Because the third component of $\widehat{\Sigma}$ is defined by $\mathrm{res}(Q, h)$, for $k < n$ we have $h_k = \hat{h}_k$, which means $d_0 = \cdots = d_{n-1} = 0$. To prove for $n \leqslant k$ the validity of (9), we consider the equation system

$$\begin{bmatrix} \hat{h}_n \\ \vdots \\ \hat{h}_{N-1} \end{bmatrix} = \begin{bmatrix} \hat{h}_0 & \cdots & \hat{h}_{n-1} \\ \vdots & & \vdots \\ \hat{h}_{\ell-1} & \cdots & \hat{h}_{N-2} \end{bmatrix} \begin{bmatrix} \xi_0 \\ \vdots \\ \xi_{n-1} \end{bmatrix},$$

$$\begin{bmatrix} \xi_0 \\ \vdots \\ \xi_{n-1} \end{bmatrix} := H_N^{n\dagger} h_N^n, \quad \ell + n = N,$$

which follows immediately by the definition of $\hat{h}_k$. Replacement of $\hat{h}_k$ by $h_k + d_k$ yields

$$\begin{aligned} \hat{h}_{n+k} &= \sum_{i=0}^{n-1} \hat{h}_{k+i}\xi_i = \sum_{i=0}^{n-1} (h_{k+i} + d_{k+i})\xi_i = \sum_{i=0}^{n-1} h_{k+i}\xi_i + \sum_{i=0}^{n-1} d_{k+i}\xi_i \\ &= (h_{n+k} + \varepsilon_k) + \sum_{i=0}^{n-1} d_{k+i}\xi_i, \end{aligned}$$

which means that the differences d_{n+k} fulfil the recurrence relation

$$d_{n+k} = \sum_{i=0}^{n-1} d_{k+i}\xi_i + \varepsilon_k, \quad d_0 = \cdots = d_{n-1} = 0,$$

which is equivalent to (9). □

To prove the stability of $\widehat{\Sigma}$, for $n < \nu$ we associate the vector ξ with the solution of a certain symmetric Stein-type displacement equation.

4. The associated symmetric Stein-type displacement equation

Let $\Sigma = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a ν-dimensional stable system, $h := h(\Sigma, \infty)$ be its Markov parameter sequence, and let the n-dimensional system $\widehat{\Sigma}$ be defined by

$$\begin{aligned} &\widehat{\Sigma} := (C_{Q_n}^{\mathrm{T}}, e_n, \mathrm{res}(Q_n, h)), \quad Q_n(s) := s^n - \psi(s, n)\xi, \\ &\xi := \lim_{N\to\infty} H_N^{n\,\dagger} h_N^n. \end{aligned} \tag{10}$$

The polynomial Q_n exists, because Σ is supposed to be stable. In the following for $n = \nu - 1$ we use the term *single-step-reduction*, and for $n < \nu - 1$ the term *multi-step-reduction*. It turns out, that for a convenient vector $W \in \mathbb{C}^\nu$, the

coefficient vector ξ of Q_n can be represented as a linear combination of the columns of a matrix M, which solves the symmetric Stein-type displacement equation

$$X - C_u X C_u^* = WW^*, \quad C_u \in \mathbb{C}^{\nu\times\nu}, \quad u(s) = \det(sI_\nu - \mathscr{A}). \tag{11}$$

Here C_u denotes the companion matrix of $u(s)$ as defined in (8).

Proposition 3. *For the ν-dimensional, stable, minimal system $\Sigma = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ and $n < \nu$, let the polynomial Q_n and the matrix M be defined by*

$$Q_n(s) = s^n - \psi(s,n) H_\infty^{n\,\dagger} h_\infty^n \in \mathbb{C}_n[s], \quad M = (H_\infty^{\nu *} H_\infty^\nu)^{-1} \in \mathbb{C}^{\nu\times\nu}.$$

If M is partitioned according to $M = [M_{ij}]_{ij=1}^2$, $M_{11} \in \mathbb{C}^{n\times n}$, then Q_n and M are related by

$$Q_n(s) = \psi(s,\nu) M \begin{bmatrix} o_n \\ \mu \end{bmatrix}, \quad \mu = M_{22}^{-1} e_1. \tag{12}$$

Here, e_1 denotes the vector $[1,0,\ldots,0]^{\mathrm{T}} \in \mathbb{R}^{\nu-n}$.

Proof. Using the abbreviations $M_n := (H_\infty^{n*} H_\infty^n)^{-1} \in \mathbb{C}^{n\times n}$, and $c_n := (e_n^{\mathrm{T}} M_n e_n)^{-1}$ one obtains $M = M_\nu$ and the recurrence relation

$$M_{n+1} = \left[\begin{array}{c|c} M_n^{-1} & H_\infty^{n*} h_\infty^n \\ \hline h_\infty^{n*} H_\infty^n & h_\infty^{n*} h_\infty^n \end{array}\right]^{-1}.$$

Consequently,

$$M_{n+1} e_{n+1} c_{n+1} = \begin{bmatrix} -M_n H_\infty^{n*} h_\infty^n \\ 1 \end{bmatrix} = \begin{bmatrix} -H_\infty^{n\,\dagger} h_\infty^n \\ 1 \end{bmatrix},$$
$$Q_n(s) = \psi(s, n+1) M_{n+1} e_{n+1} c_{n+1}.$$

Let the vector τ, the matrices N_{ij}, and the constant c be defined by

$$\tau := M \begin{bmatrix} o_n \\ \mu \end{bmatrix}, \quad \left[\begin{array}{c|c} M_{n+1}^{-1} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right] := M^{-1}, \quad c := e_1^{\mathrm{T}} \mu.$$

Then $[M_{n+1}^{-1}, N_{12}]\tau = e_{n+1} c$, and because τ is of the form

$$\tau = \begin{bmatrix} \vartheta \\ o \end{bmatrix} \in \mathbb{C}^\nu, \quad \vartheta := \begin{bmatrix} \mu \\ 1 \end{bmatrix} \in \mathbb{C}^{n+1},$$

one obtains $M_{n+1}^{-1} \vartheta = e_{n+1} c$. Hence, $M_{n+1} e_{n+1} = \vartheta c^{-1}$, and

$$\begin{aligned} Q_n(s) &= \psi(s, n+1) M_{n+1} e_{n+1} c_{n+1} \\ &= \psi(s, n+1) \vartheta c^{-1} c_{n+1} \\ &= \psi(s, \nu) \tau c^{-1} c_{n+1}. \end{aligned}$$

By definition Q_n is monic, and the last component of ϑ is equal to 1. Hence $c_{n+1} = c$. $\square$

Assuming M solves an equation of the form (11), then Proposition 4 ensures the stability of polynomials of the form (12).

Proposition 4. *Let u be a stable monic polynomial, and the solution M of*

$$X - C_u X C_u^* = WW^* \tag{13}$$

be positive definite. Let M be partitioned according to $M = [M_{ij}]_{ij=1}^2$, $M_{11} \in \mathbb{C}^{n\times n}$, and the polynomial Q_n be defined by

$$Q_n(s) := \psi(s,\nu)\tau, \quad \tau := M\varrho, \quad \varrho := \begin{bmatrix} o_n \\ \mu \end{bmatrix}, \quad \mu := M_{22}^{-1} e_1.$$

Then $\sigma(Q_n) \subset \mathscr{D}$.

Proof. Assume that $Q_n(\delta) = 0$, $u(\delta) \neq 0$, and $\eta \in \mathbb{C}^\nu$ is defined by $\eta := \left(\overline{\delta} I - C_u^*\right)^{-1} \varrho$. We show that η and τ are orthogonal. With the abbreviations

$$S := C_{s^\nu}^{\mathrm{T}}, \quad \tilde{u}(s) := u(s^{-1})s^\nu, \quad e(s) := s^0 + \cdots + s^{\nu-2},$$

the equation

$$(\delta I - C_u)^{-1} = \tilde{u}(S)\psi(\delta^{-1},\nu)^{\mathrm{T}}\delta^{\nu-1}u(\delta)^{-1}\psi(\delta,\nu) - Se(\delta S)$$

holds, and hence together with $\varrho^* = [o_n^*, \mu^*]$ one obtains

$$\begin{aligned}\eta^* &= \varrho^*(\delta I - C_u)^{-1} \\ &= \underbrace{\varrho^*\tilde{u}(S)\psi(\delta^{-1},\nu)^{\mathrm{T}}\delta^{\nu-1}u(\delta)^{-1}}_{=:\varphi\in\mathbb{C}}\psi(\delta,\nu) - \underbrace{[o_n^*,\mu^*]Se(\delta S)}_{=:[o_{n+1}^*,\omega^*]} \\ &= \varphi\psi(\delta,\nu) - [o_{n+1}^*,\omega^*].\end{aligned}$$

Because τ is of the form $\tau = \begin{bmatrix} \vartheta \\ o \end{bmatrix}$, $\vartheta \in \mathbb{C}^{n+1}$, the orthogonality of η and τ follows:

$$\eta^*\tau = (\varphi\psi(\delta,\nu) - [o_{n+1}^*,\omega^*])\tau = \varphi\psi(\delta,\nu)\tau = \varphi Q_n(\delta) = 0.$$

The equations $WW^* = M - C_u M C_u^*$, $\eta^* C_u = \delta\eta^* - \varrho^*$ and the definition of the positive real number $\kappa := \eta^* M \eta$ now imply

$$\begin{aligned}0 \leqslant \kappa - \eta^* C_u M C_u^* \eta &= \kappa - (\delta\eta^* - \varrho^*)M(\overline{\delta}\eta - \varrho) \\ &= \kappa(1-|\delta|^2) + 2\,\Re e(\delta \underbrace{\eta^*\tau}_{=0}) - \varrho^* M\varrho = \kappa(1-|\delta|^2) - \underbrace{\varrho^* M\varrho}_{>0} < \kappa(1-|\delta|^2).\end{aligned}$$

Consequently, $\delta \in \mathscr{D}$. □

In the following corollary we use Matlab notation. When $M = [m_{ij}]_{i,j=1}^{k,\ell}$, then

$$\begin{aligned}M(n_1 : n_2, m_1 : m_2) &:= [m_{ij}]_{i=n_1,\, j=m_1}^{n_2,\, m_2}, \\ M(n_1 : n_2, m) &:= M(n_1 : n_2, m : m).\end{aligned}$$

Corollary 1. *Let $h = (h_k)_{k=1}^N$, $h_N \neq 0$, be a finite complex number sequence, and let for $n \in \{1, \ldots, N-1\}$ the polynomial Q_n be defined by $Q_n(s) = s^n - \psi(s, n)\xi$, where*

$$\xi = H_\Delta(1:N, 1:n)^\dagger H_\Delta(1:N, n+1), \quad H_\Delta = \begin{bmatrix} h_1 & \cdots & h_N \\ \vdots & \iddots & \\ h_N & & \end{bmatrix}.$$

Then $\sigma(Q_n) \subset \mathscr{D}$.

Proof. It is sufficient to show that $M := (H_\Delta^* H_\Delta)^{-1}$ satisfies for appropriate stable polynomial u and vector W the symmetric Stein-type displacement equation (13). Then by Propositions 3 and 4 the statement follows. We set $u(s) = s^N$, $e_N = [0, \ldots, 0, 1]^{\mathrm{T}} \in \mathbb{R}^N$, and $W = H_\Delta^{-1} e_N$. Then $H_\Delta C_u = C_u^* H_\Delta$ and

$$\begin{aligned} M - C_u M C_u^* &= (H_\Delta^* H_\Delta)^{-1} - C_u (H_\Delta^* H_\Delta)^{-1} C_u^* \\ &= H_\Delta^{-1}(I - H_\Delta C_u H_\Delta^{-1} H_\Delta^{-*} C_u^* H_\Delta^*) H_\Delta^{-*} \\ &= H_\Delta^{-1}(I - C_u^* C_u) H_\Delta^{-*} \\ &= H_\Delta^{-1} e_N e_N^* H_\Delta^{-*} \\ &= W W^*. \qquad \square \end{aligned}$$

If one defines $\mathscr{Z}(h) := \bigcup_{n=1}^{N-1} \{\sigma(Q_n)\}$, then Corollary 1 implies $\mathscr{Z}(h) \subset \mathscr{D}$.

Example 2. Fig. 3 visualises $\mathscr{Z}(h)$ for $h := \left(\cos\left(\frac{2k\pi\mathbf{i}}{N}\right)\right)_{k=1}^N$, and $N = 70$.

For the proof our main result it remains to construct a vector W, such that the matrix M as defined in Proposition 3 solves Eq. (13). For the sake of simplicity we assume that the main operator of Σ possesses ν pairwise distinct eigenvalues $\{\alpha_1, \ldots, \alpha_\nu\} \subset \mathscr{D}$. We define

$$\begin{aligned} &u(s) := \prod_{i=1}^{\nu}(s - \alpha_i), \quad \hat{u}(s) := \prod_{i=1}^{\nu}(1 - \overline{\alpha_i} s), \quad u'(s) := \frac{\mathrm{d}u(s)}{\mathrm{d}s}, \\ &r(s) := \frac{\hat{u}(s)}{u'(s)}, \quad \alpha := [\alpha_1, \ldots, \alpha_\nu]^{\mathrm{T}}, \quad r(\alpha) := [r(\alpha_1), \ldots, r(\alpha_\nu)]^{\mathrm{T}}, \\ &D(r(\alpha)) := \mathrm{diag}(r(\alpha)), \quad V := [\alpha_i^{j-1}]_{i,j=1}^{\nu}, \quad e := [1, \ldots, 1]. \end{aligned}$$

Proposition 5. *Let for the minimal system $\Sigma = (\mathscr{A}, \mathscr{B}, \mathscr{C})$, $\sigma(\mathscr{A}) = \{\alpha_1, \ldots, \alpha_\nu\} \subset \mathscr{D}$, $\mathscr{A} \in \mathbb{C}^{\nu\times\nu}$, the matrix M be defined by $M = (H_\infty^{\nu *} H_\infty^\nu)^{-1} \in \mathbb{C}^{\nu\times\nu}$. Then M satisfies*

$$M - C_u M C_u^* = W W^*, \quad W := V^{-1} D(g)^{-1} D(r(\alpha)) e. \tag{14}$$

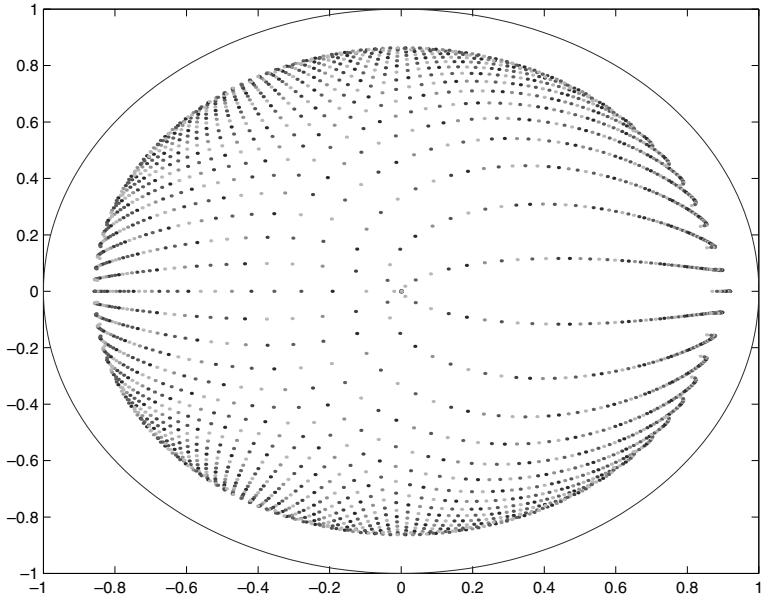

Fig. 3. $\mathscr{Z}(h)$ for $h = \left(\cos\left(\frac{2k\pi\mathbf{i}}{70}\right)\right)_{k=1}^{70}$.

Here, $\sum_{i=1}^{\nu} \frac{g_i}{s-\alpha_i}$ *represents the transfer function of* Σ*, and*

$$D(g) := \operatorname{diag}(g), \quad g := [g_1, \ldots, g_\nu]^{\mathrm{T}}.$$

Proof. In order to prove (14), depending on $\sigma(\mathscr{A}) \subset \mathscr{D}$ the Cauchy matrix

$$\Omega := [(1 - \overline{\alpha_i}\alpha_j)^{-1}]_{i,j=1}^{\nu} \in \mathbb{C}^{\nu\times\nu}$$

is introduced. Since all eigenvalues of $\mathscr{A}$ are pairwise distinct, Ω is regular. The non-negativity of $\Omega^{-1} - D(\alpha)\Omega^{-1}D(\alpha)^*$ is responsible for the definition (14) of W. To make that evident, let $\overline{\Omega} := [(1 - \alpha_i\overline{\alpha_j})^{-1}]_{i,j=1}^{\nu}$. Then $\Omega^{-1} = D(r(\alpha))\overline{\Omega}D(r(\alpha))^*$, and together with the displacement equation

$$\overline{\Omega} - D(\alpha)\overline{\Omega}D(\alpha)^* = ee^{\mathrm{T}},$$

one concludes

$$\begin{aligned}\Omega^{-1} - D(\alpha)\Omega^{-1}D(\alpha)^* &= D(r(\alpha))(\overline{\Omega} - D(\alpha)\overline{\Omega}D(\alpha)^*)D(r(\alpha))^* \\ &= D(r(\alpha))ee^{\mathrm{T}}D(r(\alpha))^*.\end{aligned} \tag{15}$$

Since $G(s) = \mathscr{C}(sI - \mathscr{A})^{-1}\mathscr{B} = \sum_{i=1}^{\nu} \frac{g_i}{s-\alpha_i}$, $g_i \neq 0$, the Markov parameters of Σ take the form $h_k = \sum_{i=1}^{\nu} g_i\alpha_i^k$. Hence, the matrix $H_\infty^{\nu *}H_\infty^{\nu}$ $(= M^{-1})$ admits the factorisation

$$H_\infty^{\nu *}H_\infty^{\nu} = V^*D(g)^*\Omega D(g)V, \tag{16}$$

which makes it easy to check (14), namely by virtue of $VC_u = D(\alpha)V$, and (15) one gets

$$\begin{aligned}
M - C_u M C_u^* &= (V^* D(g)^* \Omega D(g) V)^{-1} - C_u (V^* D(g)^* \Omega D(g) V)^{-1} C_u^* \\
&= V^{-1} D(g)^{-1} \Omega^{-1} D(g)^{-*} V^{-*} - C_u V^{-1} D(g)^{-1} \Omega^{-1} D(g)^{-*} V^{-*} C_u^* \\
&= V^{-1} D(g)^{-1} \Omega^{-1} D(g)^{-*} V^{-*} \\
&\quad - V^{-1} D(\alpha) D(g)^{-1} \Omega^{-1} D(g)^{-*} D(\alpha)^* V^{-*} \\
&= V^{-1} D(g)^{-1} (\Omega^{-1} - D(\alpha) \Omega^{-1} D(\alpha)^*) D(g)^{-*} V^{-*} \\
&= \underbrace{V^{-1} D(g)^{-1} D(r(\alpha)) e}_{=W} \underbrace{e^{\mathrm{T}} D(r(\alpha))^* D(g)^{-*} V^{-*}}_{=W^*} \\
&= W W^*. \qquad \square
\end{aligned}$$

As usual let ℓ_2 be the normed vector space of all square-summable complex number sequences, and $L_2(\mathbb{T})$ be the normed vector space of all Lebesgue square-integrable functions on $\mathbb{T} := \partial\mathscr{D}$, more precisely

$$a := (a_k)_{k=0}^{\infty} \in \ell_2 \Leftrightarrow \|a\|_2 := \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2} < \infty,$$

$$G \in L_2(\mathbb{T}) \Leftrightarrow \|G\|_2 := \left(\frac{1}{2\pi} \int_0^{2\pi} |G(\mathrm{e}^{\mathrm{i}\omega})|^2 d\omega \right)^{1/2} < \infty.$$

For stable systems it is not difficult to show, that the ℓ_2-distance between h_∞^n and span H_∞^n is finite. Therefore the residual vector $\mathscr{E}^n := H_\infty^n \xi - h_\infty^n$ can be used to define an $L_2(\mathbb{T})$-function, which is then appropriate to estimate $\|G - \widehat{G}\|_2$, where G and $\widehat{G}$ denote the transfer functions of Σ and $\widehat{\Sigma}$.

5. The main theorem

The Propositions 2–5 lead us to our main theorem.

Theorem 2. *Let $\Sigma = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a minimal ν-dimensional system, where $\sigma(\mathscr{A}) = \{\alpha_1, \ldots, \alpha_\nu\} \subset \mathscr{D}$. Then the system* (10) *is again stable. In particular,*

$$\|G - \widehat{G}\|_2 \leqslant \frac{\|\mathscr{E}^n\|_2}{\min_{s \in \mathbb{T}} |Q_n(s)|} \leqslant \frac{\|\mathscr{E}^n\|_2}{(1 - r_{\widehat{\Sigma}})^n},$$

where $\mathscr{E}^n := H_\infty^n \xi - h_\infty^n \in \ell_2$ and $r_{\widehat{\Sigma}}$ is defined by the maximum of the moduli of the zeros of Q_n.

Proof. By virtue of Propositions 3–5 we know that Q_n is stable.

To prove the inequalities, we first note that due to (16) the columns of H_∞^n as well as h_∞^n can be interpreted as elements of ℓ_2. Consequently the residual vector $\mathscr{E}^n$, is interpretable as ℓ_2-element, too. Let $d(s) = \widehat{G}(s) - G(s) = \sum_{k=0}^{\infty} d_k s^{-(k+1)}$, and $\mathscr{E}(s) := \sum_{k=n}^{\infty} \varepsilon_k s^{1-k}$, where $\mathscr{E}^n =: (\varepsilon_k)_{k=n}^{\infty}$. Due to $\mathscr{E}^n \in \ell_2$, the relation $\mathscr{E}(s) \in L_2(\mathbb{T})$ can be stated. With respect to Proposition 2 one obtains

$$d(s) = e_n^{\mathrm{T}}(sI - C_{Q_n}^{\mathrm{T}})^{-1} e_n \mathscr{E}(s) = s^{n-1} Q_n(s)^{-1} \mathscr{E}(s).$$

Hence,

$$\|G - \widehat{G}\|_2 = \|Q_n^{-1}\mathscr{E}\|_2 \leqslant \max_{s\in\mathbb{T}} |Q_n(s)^{-1}| \|\mathscr{E}^n\|_2 = \frac{\|\mathscr{E}^n\|_2}{\min_{s\in\mathbb{T}} |Q_n(s)|}.$$

Finally, $\min_{s\in\mathbb{T}} |Q_n(s)| = \min_{s\in\mathbb{T}} |s - \widehat{\alpha}_1| \cdots |s - \widehat{\alpha}_n| \geqslant (1 - r_{\widehat{\Sigma}})^n$. □

Remark 1. It is natural to ask for the minimality of $\widehat{\Sigma}$. The following example shows, that a single-step-reduction does not necessarily lead to a minimal system. As example we consider the degenerated case where the transfer function of Σ is of the form

$$G(s) = \sum_{k=0}^{\nu-1} (s - \alpha_k)^{-1}, \quad \alpha_k = r_\Sigma \mathrm{e}^{\mathbf{i}\frac{2\pi}{\nu}k},\ r_\Sigma < 1.$$

Then the transfer function $\widehat{G}$ of $\widehat{\Sigma}$ is equal to ν/s, which means $\widehat{G}$ can be realized by the one-dimensional system $\widetilde{\Sigma} := (0, 1, \nu)$. Whereby the made assumption leads to

$$h_k = \begin{cases} 0, & \text{if } k \notin \nu\mathbb{N}_0, \\ \nu r_\Sigma^k, & \text{if } k \in \nu\mathbb{N}_0. \end{cases}$$

Consequently, for every non-vanishing component of $h_\infty^{\nu-1}$ the corresponding row of $H_\infty^{\nu-1}$ vanishes and vice versa, every non-vanishing row of $H_\infty^{\nu-1}$ corresponds with a zero component of $h_\infty^{\nu-1}$. Therefore, $h_\infty^{\nu-1} \in (\operatorname{span} H_\infty^{\nu-1})^{\perp}$, which is equivalent to $\xi = {H_\infty^{\nu-1}}^{\dagger} h_\infty^{\nu-1} = o$. Thus, $Q(s) = s^{\nu-1}$, and $\operatorname{res}(Q, h) = \nu e_{\nu-1}^{\mathrm{T}}$. Finally

$$\widehat{G}(s) = \operatorname{res}(Q, h)(sI - C_Q^{\mathrm{T}})^{-1} e_\nu = \nu s^{\nu-2}/s^{\nu-1} = \nu/s.$$

Remark 2. It is natural to ask for the similarity of the systems Σ_{cons} and $\widetilde{\Sigma}$, where the first one is obtained by two consecutive single-step-reductions, whereas the second one is obtained by a two-step-reduction. The following example shows, that, in general, non-similar systems are obtained. Let Σ be given by:

$$\mathscr{A} = r_\Sigma \operatorname{diag}(1, \mathbf{i}, -\mathbf{i}), \quad r_\Sigma = 0.99, \quad \mathscr{B} = \mathscr{C}^{\mathrm{T}}, \quad \mathscr{C} = [1, 1, 1].$$

Then $h_k = r_\Sigma^k(1 + \mathbf{i}^k + (-\mathbf{i})^k)$ and consecutive single-step-reduction leads to the systems

$$\Sigma_2 := (A, B, C), \quad A = \begin{bmatrix} 0 & 1 \\ -0.4852 & 0.4899 \end{bmatrix}, \quad B = [0, 1]^{\mathrm{T}},$$
$$C = [-0.4798, 3],$$

$$\Sigma_{\text{cons}} := \Sigma_1 := (\alpha, 1, h_0), \quad \alpha := \widetilde{H}_\infty^{1\dagger}\widetilde{h}_\infty^1 = \begin{bmatrix} \tilde{h}_0 \\ \tilde{h}_1 \\ \vdots \end{bmatrix}^\dagger \begin{bmatrix} \tilde{h}_1 \\ \tilde{h}_2 \\ \vdots \end{bmatrix} = 0.2501,$$

$$\tilde{h}_k := CA^kB.$$

On the other side, the two-step-reduction leads to

$$\widetilde{\Sigma} = (\widetilde{\alpha}, 1, h_0), \quad \widetilde{\alpha} := H_\infty^{1\dagger}h_\infty^1 = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \end{bmatrix}^\dagger \begin{bmatrix} h_1 \\ h_2 \\ \vdots \end{bmatrix} = 0.3236.$$

Since $\alpha \neq \widetilde{\alpha}$, Σ_{cons} and $\widetilde{\Sigma}$ are non-similar.

6. Comparison of consecutive single-step-reductions to multi-step-reductions

According to Theorem 2 for every stable, diagonalisable system Σ a $(\nu - 1)$-dimensional stable system $\widehat{\Sigma}$ exists with Markov parameters, which are "similar" to the original ones. In particular, $\widehat{\Sigma}$ generically again fulfils the reduction assumptions, such that $\widehat{\Sigma}$ itself can be reduced. Thus one obtains a sequence of stable systems with descending state space dimensions and similar impulse responses. Based on the Nyquist plots, we now compare the approximation quality related to consecutive single-step- versus multi-step-reductions for one concrete example. For stable discrete systems the complex number set

$$G(\mathbb{T}) = \{G(\mathrm{e}^{\mathbf{i}\omega}) : \omega \in [0, 2\pi]\} \subset \mathbb{C}$$

relates the output $(y_k)_{k=1}^\infty$ to an input of the form $(u_k)_{k=0}^\infty$, $u_k = \cos(\omega k)$, according to

$$y_k \approx |G(\mathrm{e}^{\mathbf{i}\omega})| \cos(\omega k + \varphi(\omega)), \quad \varphi(\omega) = \arg G(\mathrm{e}^{\mathbf{i}\omega}).$$

Therefore, the quantity $\|G - \widehat{G}\|_2$ yields a measure for the preservation of the IO-behaviour by the passage from Σ to $\widehat{\Sigma}$.

Example 3. Let $\nu = 40$, $r_\Sigma = 0.93$, $\mathscr{A} = \operatorname{diag}(\alpha_k)_{k=0}^{\nu-1} \in \mathbb{C}^{\nu\times\nu}$, $\alpha_k = r_\Sigma \mathrm{e}^{\frac{2\pi\mathbf{i}}{\nu}k}$ $\mathscr{B} = [1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^\nu$, and

$$\mathscr{C} = [g_1, g_2, \ldots, g_{20}, g_{21}, g_{20}, \ldots, g_2],$$
$$[g_1, \ldots, g_{21}] := [8, 8, 2, 6, 5, 5, 3, 8, 8, 8, 6, 7, 2, 6, 8, 10, 4, 4, 10, 3, -9].$$

Then $G(s) = \sum_{k=1}^\nu g_k(s - \alpha_k)^{-1}$, and Fig. 4 (Nyquist plot) compares $G(\mathbb{T})$ (solid line) to $G_{\text{cons}}(\mathbb{T})$ (dotted line), where G_{cons} represents the transfer function of the

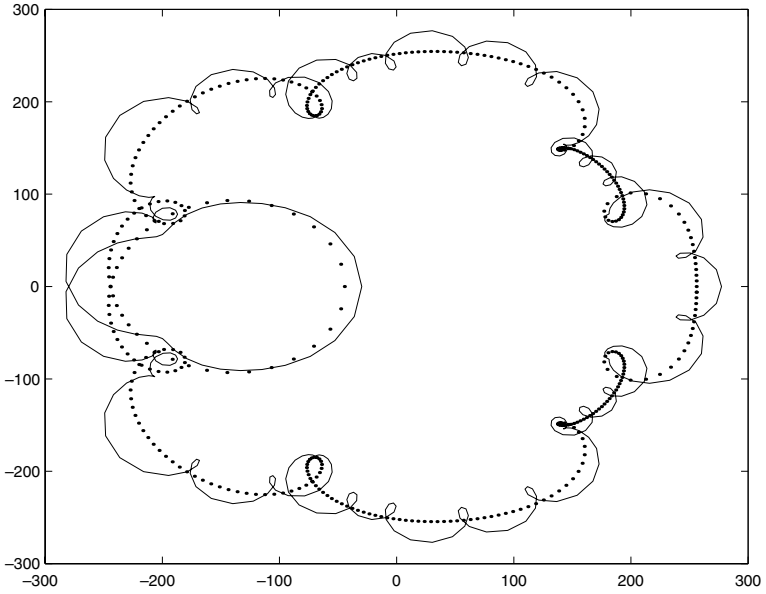

Fig. 4. $G(\mathbb{T}) \mathrel{\widehat{=}} —$, $G_{\mathrm{cons}}(\mathbb{T}) \mathrel{\widehat{=}} \cdots$

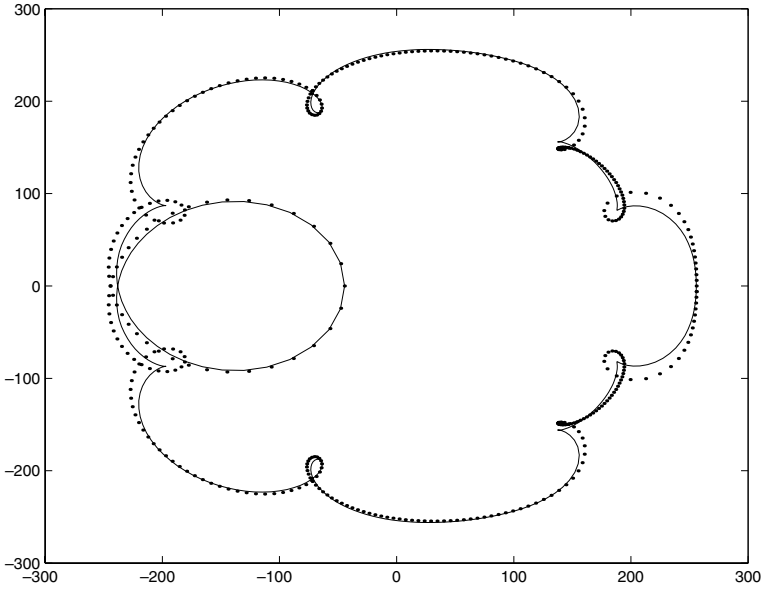

Fig. 5. $\widehat{G}(\mathbb{T}) \mathrel{\widehat{=}} —$, $G_{\mathrm{cons}}(\mathbb{T}) \mathrel{\widehat{=}} \cdots$

10-dimensional system Σ_{cons}, obtained after 30 consecutive single-step-reductions. In every reduction step $N := 500$ Markov parameters have been used.

Fig. 5 compares $G_{\text{cons}}(\mathbb{T})$ (dotted line) to $\widehat{G}(\mathbb{T})$ (solid line), where $\widehat{G}$ represents the transfer function of the system $\widehat{\Sigma}$, obtained by reduction of 30 state space dimensions in one step.

Obviously both reduction procedures lead to a satisfying reproduction of the original Nyquist plot. By virtue of the second figure it seems that some of the small loops are better preserved by the consecutive single-step-reductions. Such phenomena will be investigated by the authors in a future article.

7. Comparison of multi-step-reductions to reductions obtained by balanced truncation

A stable discrete system $\Sigma = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ is said to be *balanced*, if a non-negative diagonal matrix $X =: \operatorname{diag}(\sigma_1, \ldots, \sigma_\nu)$, $\sigma_1 \geqslant \cdots \geqslant \sigma_\nu \geqslant 0$ exists, which simultaneously solves the Stein equations

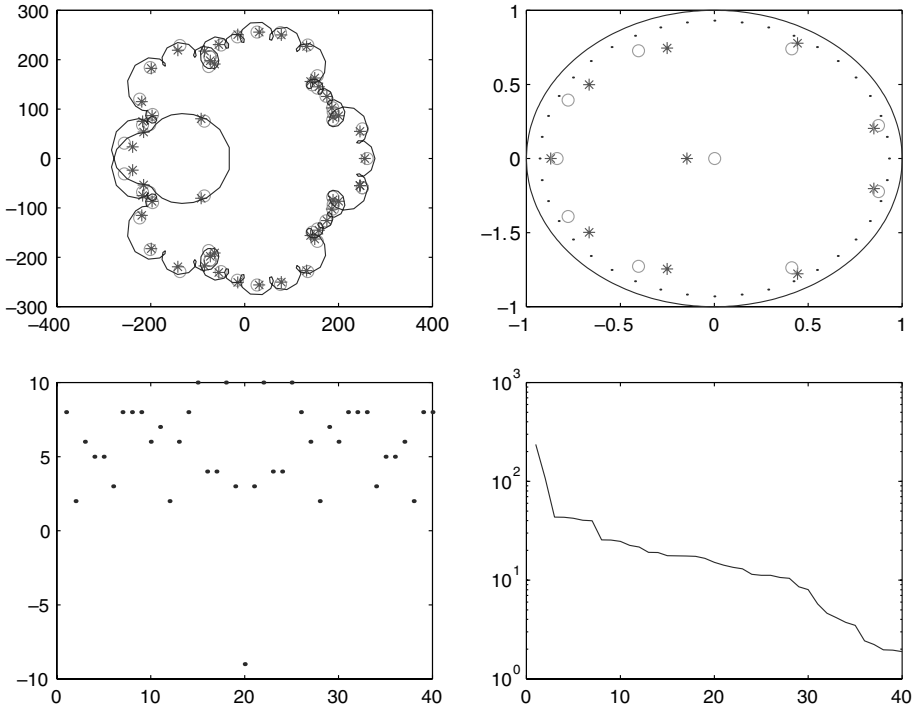

Fig. 6. $G(\mathbb{T})$, $\widehat{G}(\mathbb{T})$, $G_{\text{trunc}}(\mathbb{T})$, $\sigma(\mathscr{A})$, $\sigma(C_{\mathcal{Q}_n})$, $\sigma(A_{11})$, g_k, σ_k.

$$X - \mathscr{A} X \mathscr{A}^* = \mathscr{B}\mathscr{B}^*, \quad X - \mathscr{A}^* X \mathscr{A} = \mathscr{C}^*\mathscr{C}.$$

The decreasingly ordered numbers σ_i are called the *singular values* of Σ. For every stable system a transformation matrix T exists, such that

$$\Sigma_{\text{bal}} := (A_{\text{bal}}, B_{\text{bal}}, C_{\text{bal}}) := (T\mathscr{A}T^{-1}, T\mathscr{B}, \mathscr{C}T^{-1})$$

is balanced. To find T, assume that Σ is minimal, and that $\mathscr{P}$ and $\mathscr{Q}$ satisfy

$$\mathscr{P} - \mathscr{A}\mathscr{P}\mathscr{A}^* = \mathscr{B}\mathscr{B}^*, \quad \mathscr{Q} - \mathscr{A}^*\mathscr{Q}\mathscr{A} = \mathscr{C}^*\mathscr{C}.$$

Then $\mathscr{P} = \sum_{k=0}^{\infty} \mathscr{A}^k \mathscr{B}\mathscr{B}^* (\mathscr{A}^*)^k$, and $\mathscr{Q} = \sum_{k=0}^{\infty} (\mathscr{A}^*)^k \mathscr{C}^*\mathscr{C}\mathscr{A}^k$, which means that $\mathscr{P}$ and $\mathscr{Q}$ are positive definite, and the transformation matrix T is given by $T := \widehat{X}^{1/4} U^* R^{-*}$. Here, R is obtained by the Cholesky decomposition of $\mathscr{P} =: R^*R$. The diagonal matrix $\widehat{X}$ and the unitary matrix U are obtained by the diagonalisation of the product $R\mathscr{Q}R^* =: U\widehat{X}U^*$. In particular, the singular values of Σ are the square roots of the eigenvalues of $R\mathscr{Q}R^*$: $X = \widehat{X}^{1/2}$. Now, truncation of Σ_{bal} yields a stable system $\Sigma_{\text{trunc}} := (A_{11}, B_1, C_1)$, where

$$[A_{ij}]_{i,j=1}^{2} := A_{\text{bal}}, \quad [B_1^{\mathrm{T}}, B_2^{\mathrm{T}}]^{\mathrm{T}} := B_{\text{bal}}, \quad [C_1, C_2] := C_{\text{bal}},$$
$$A_{11} \in \mathbb{C}^{n\times n}, \quad C_1^{\mathrm{T}}, B_1 \in \mathbb{C}^n.$$

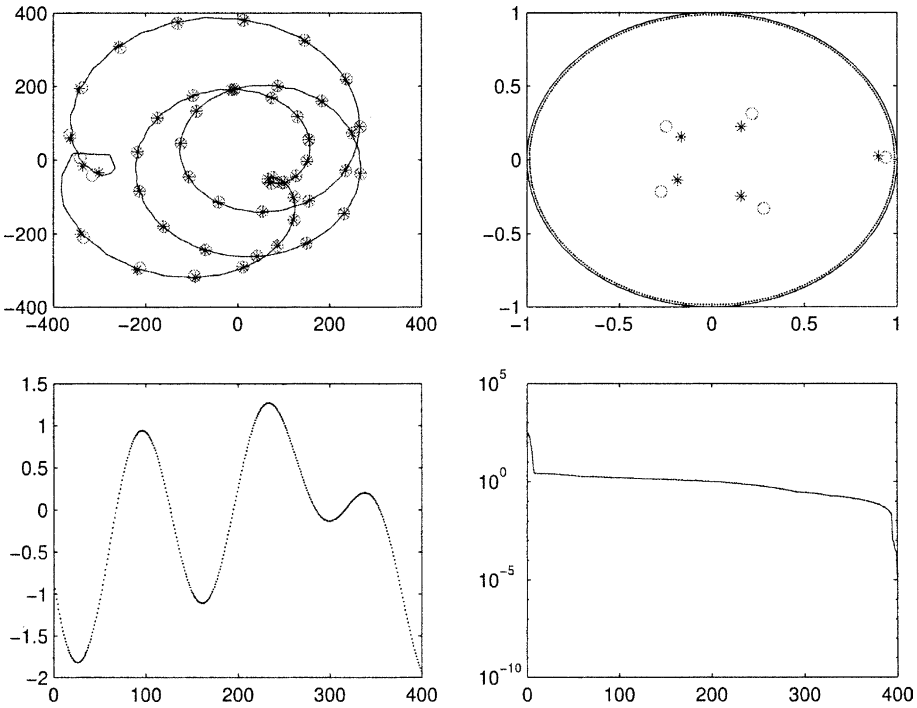

Fig. 7. $[r_\Sigma, \nu, n, N] = [0.986, 400, 5, 1000]$.

Estimates for $\max_{\omega\in[0,2\pi]} |G(e^{i\omega}) - G_{\text{trunc}}(e^{i\omega})|$ can be found in [29], and in [35, Chapter 21.8].

To transform Σ into balanced form one has to solve two ν-dimensional matrix equations and to realize two decompositions. On the other hand for our multi-step-reduction procedure one has to determine the Markov parameter sequence $(\mathscr{C}\mathscr{A}^k\mathscr{B})_{k=0}^{N-1}$, and to invert the n-dimensional Hankel matrix product $H_N^{n\,*}H_N^n$ for sufficient large N. Hence, for $n \ll \nu$ our approach avoids the numerical solution of ν-dimensional matrix problems. The following example tries to illustrate, that the obtained reduction results are very similar despite their very different computational efforts.

Example 4. For the 40-dimensional system of Example 3 one obtains Fig. 6.

Fig. 6.1.1 displays the Nyquist plots of the original system Σ (solid) together with the plots of the reduced systems $\widehat{\Sigma}$ (stars) and Σ_{trunc} (circles). Obviously, both reduced model curves are in good accordance with the reference curve. In Fig. 6.1.2 the distribution of the points $\alpha_k \in \mathscr{D}$ (dots), the poles of $\widehat{G}$ (stars), and the poles of G_{trunc} (circles) are depicted, showing similar distributions for both reduced models. In Fig. 6.2.1 the distribution of the partial fraction coefficients g_k of $G(s)$ is shown,

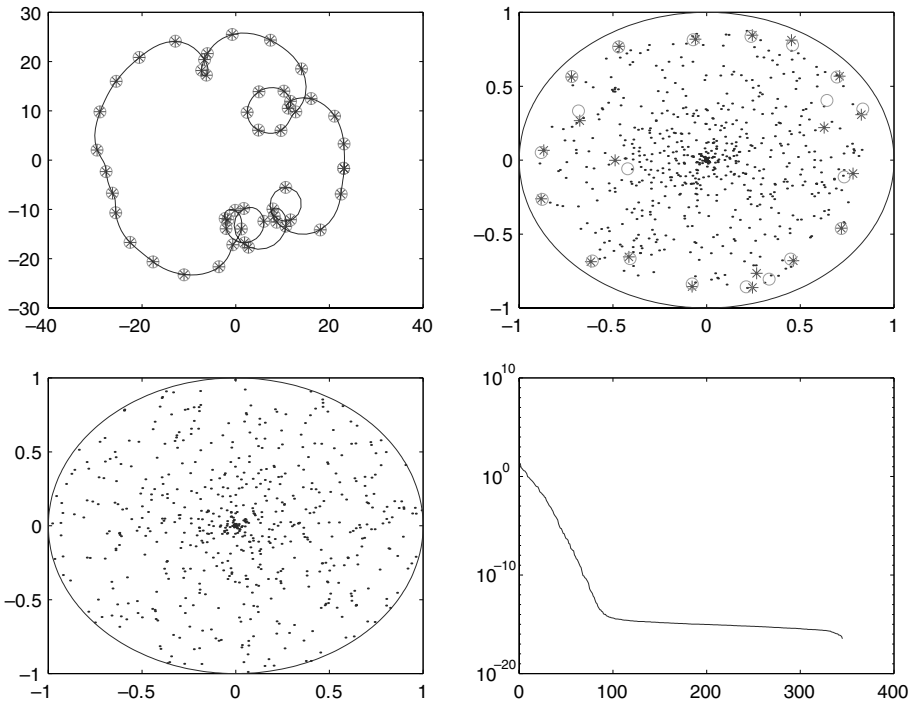

Fig. 8. $[r_\Sigma, \nu, n, N] = [0.9, 600, 20, 1000]$.

and in Fig. 6.2.2 the decreasing sequence of the singular values σ_k of Σ in "semilogy"-style. The plot of the sequence $\sigma_1, \ldots, \sigma_{40}$ hereby supports the observation derived from Fig. 6.1.1, that in the considered case a 10-dimensional reduced system allows a reasonable approximation of the original one.

In the case where r_Σ tends to 1 the system Σ becomes instable, and in particular for $r_\Sigma \approx 1$ numerical difficulties are to be expected. To illustrate the case, where r_Σ is almost 1, we set $[r_\Sigma, \nu, n, N] = [0.986, 400, 5, 1000]$, and for $k = 1, \ldots, \nu$

$$\alpha_k := r_\Sigma \exp(2\pi \mathbf{i} k/\nu),$$
$$g_k := \cos(1 + 3\pi k/\nu) + \cos(3 + 4\pi k/\nu) + \cos(2 + 6\pi k/\nu).$$

Then for

$$\Sigma := (\mathrm{diag}(\alpha_k)_{k=1}^{\nu}, [1, \ldots, 1]^{\mathrm{T}}, [g_1, \ldots, g_\nu]), \tag{17}$$

one obtains Fig. 7.

Fig. 7.1.1 shows the sets $G(\mathbb{T})$, $\widehat{G}(\mathbb{T})$, $G_{\mathrm{trunc}}(\mathbb{T})$ (solid line, stars, circles), Fig. 7.1.2 the poles of G, $\widehat{G}$, G_{trunc}, Fig. 7.2.1 the distribution of the partial fraction coefficients g_k of G, and Fig. 7.2.2 the singular values σ_k in "semilogy"-style. Obviously, reduction dimension $n = 5$ already is sufficient.

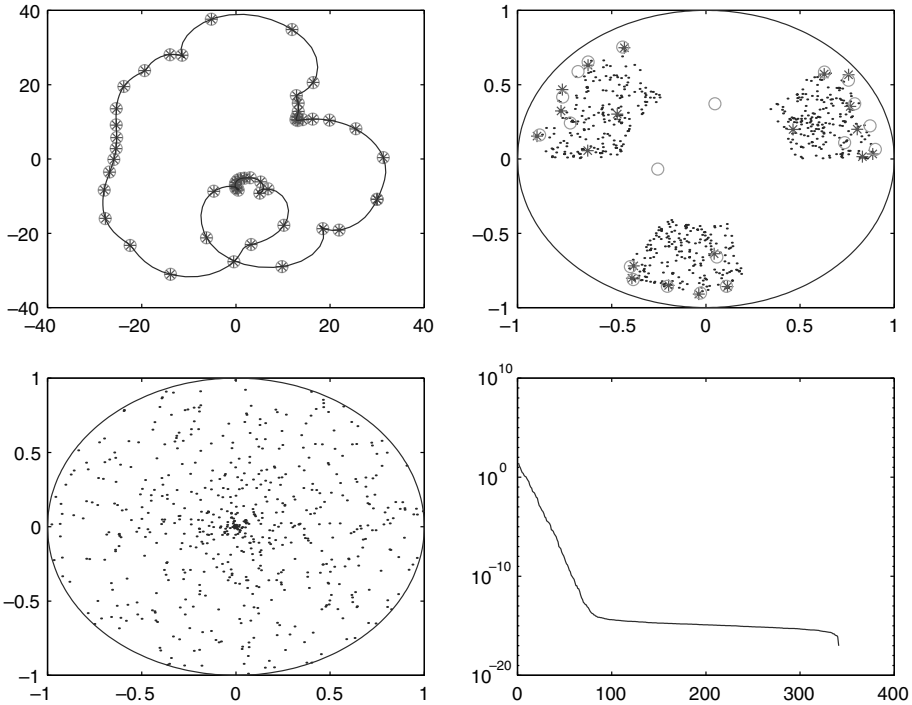

Fig. 9. $[r_\Sigma, \nu, n, N] = [0.9, 600, 20, 1000]$.

Let now the parameters α_k, $g_k \in \mathscr{D}$ of the system (17) be generated at random. For $[r_\Sigma, \nu, n, N] = [0.9, 600, 20, 1000]$, for example one obtains Fig. 8. Finally we consider the case where the eigenvalues of the system (17) are clustered in three regions. For $[r_\Sigma, \nu, n, N] = [0.93, 360, 30, 500]$, for example one obtains Fig. 9. Here, in every cluster 200 eigenvalues of $\mathscr{A}$ are located. In virtue of Fig. 9.1.2 the question for the distribution of the poles of $\widehat{G}$ (stars) depending on the distributions of the poles of G (dots) arise. Problems of this kind will be considered by the authors in a forthcoming paper.

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