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Explicit Conditions for the Oscillation of Difference Equations*

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We obtain sufficient conditions for the oscillation of all solutions of the difference equation

$$A_{n+1}-A_n+\sum_{i=1}^{m} p_i A_{n-k_i}=0, \qquad n=0, 1, 2, ...,$$

where the p_i's are real numbers and the k_i's are integers. The conditions are given explicitly in terms of the p_i's and the k_i's. © 1990 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of the difference equation

$$A_{n+1}-A_n+\sum_{i=1}^{m} p_i A_{n-k_i}=0, \qquad n=0, 1, 2, ..., \tag{1.1}$$

where

$$p_i \in \mathbb{R}-\{0\} \qquad \text{and} \qquad k_i \in \mathbf{Z}=\{... -1, 0, 1, ...\} \qquad \text{for} \quad i=1, 2, ..., m. \tag{1.2}$$

The conditions will be given explicitly in terms of the p_i's and the k_i's.

A solution $\{A_n\}$ of Eq. (1.1) is called *oscillatory* if the terms A_n of the sequence $\{A_n\}$ are neither eventually positive nor eventualy negative. Otherwise, the solution is called *nonoscillatory*.

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One may think of Eq. (1.1) as being a discrete analogue of the differential equation with deviating arguments

$$\dot{x}(t)+\sum_{i=1}^{m} p_i x(t-\tau_i)=0, \tag{1.3}$$

where

$$p_i, \tau_i \in \mathbf{R} \qquad \text{for} \quad i=1, 2, ..., m.$$

By analogy to Eq. (1.3), Eq. (1.1) is said to be of the *delay*, *advanced*, or *mixed* type provided that the k_i's are all nonnegative, all nonpositive, or some nonnegative and some nonpositive, respectively. More precisely, let

$$k=\max\{0, k_1, ..., k_m\} \qquad \text{and} \qquad l=\max\{1, -k_1, ..., -k_m\}.$$

Then Eq. (1.1) is a difference equation of order $(k+l)$. If $k \geqslant 0$ and $l=1$, we will say that Eq. (1.1) is a delay difference equation. When $k=0$ and $l \geqslant 2$, Eq. (1.1) will be called an advanced difference equation. When $k \geqslant 1$ and $l \geqslant 2$, then Eq. (1.1) is of the mixed type.

By a *solution* of Eq. (1.1) we mean a sequence $\{A_n\}$ which is defined for $n \geqslant -k$ and which satisfies Eq. (1.1) for $n \geqslant 0$. Let $a_{-k}, ..., a_0, ..., a_{l-1}$ be $(k+l)$ given real numbers. Then Eq. (1.1) has a unique solution $\{A_n\}$ which satisfies the initial conditions

$$A_i = a_i \qquad \text{for} \quad i=-k, ..., l-1.$$

It is interesting to note that no parallel to this existence theorem can be stated for the mixed type differential equation (1.3). On the other hand, the oscillation results which we will establish for Eq. (1.1) have a parallel for Eq. (1.3) and vice versa.

For the general theory of solutions of difference equations the reader is referred to [2] or [10]. From a detailed but elementary analysis of the $(k+l)$-dimensional space of solutions of Eq. (1.1) one is led to the following fundamental result for the oscillation of all solutions of Eq. (1.1). For a proof see [11].

THEOREM 1.1. *Assume that* (1.2) *is satisfied. Then the following statements are equivalent*:

(a) *Every solution of Eq.* (1.1) *oscillates.*

(b) *The characteristic equation of Eq.* (1.1)

$$\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0 \tag{1.4}$$

has no positive roots.

For delay difference equations with positive coefficients, an elementary proof of Theorem 1.1 was given in [8] which does not require any prior knowledge of the structure of solutions of Eq. (1.1). One may use the method of Z-transforms to establish a similar result for systems of difference equations and for difference equations of higher order.

The following lemma will be useful in Section 4. For the proof of part (1) see [4]. Part (b) has a similar proof.

LEMMA 1.1. *Assume that $p \in (0, \infty)$ and $k \in \{1, 2, ...\}$. Then the following statements hold*:

(a) *The delay difference inequality*

$$x_{n+1} - x_n + p x_{n-k} \leqslant 0, \qquad n = 0, 1, 2, ...$$

has an eventually positive solution if and only if the delay difference equation

$$y_{n+1} - y_n + p y_{n-k} = 0, \qquad n = 0, 1, 2, ...$$

has an eventually positive solution.

(b) *The advanced difference inequality*

$$x_{n+1} - x_n - p x_{n+k} \geqslant 0, \qquad n = 0, 1, 2, ...$$

has an eventually positive solution if and only if the advanced difference equation

$$y_{n+1} - y_n - p y_{n+k} = 0, \qquad n = 0, 1, 2, ...$$

has an eventually positive solution.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATIONS

In this section we will obtain necessary and sufficient conditions for the oscillation of all solutions of the difference equation

$$A_{n+1} - A_n + p A_{n-k} + q A_{n-l} = 0, \qquad n = 0, 1, 2, ..., \tag{2.1}$$

where

$$p, q \in \mathbf{R}, \quad k \in \mathbf{Z} \qquad \text{and} \qquad l \in \{-1, 0\}.$$

The conditions will be given explicitly in terms of p, q, k and l.

The case $q = 0$ was examined in [3, 6] where the following result was found.

THEOREM 2.1. *Consider the difference equation*

$$A_{n+1} - A_n + pA_{n-k} = 0, \qquad n = 0, 1, 2, ..., \tag{2.2}$$

where

$$p \in \mathbf{R} \qquad and \qquad k \in \mathbf{Z}. \tag{2.3}$$

Then every solution of Eq. (2.2) *oscillates if and only if one of the following conditions holds*:

(i) $k = -1$ *and* $p \leqslant -1$;

(ii) $k = 0$ *and* $p \geqslant 1$;

(iii) $k \in \{..., -3, -2\} \cup \{1, 2, ...\}$ *and* $p((k+1)^{k+1}/k^k) > 1$.

One should recall, see, for example, [9], that every solution of the differential equation

$$\dot{x}(t) + px(t - \tau) = 0, \tag{2.4}$$

where $p, \tau \in \mathbf{R}$ oscillates if and only if

$$p\tau e > 1. \tag{2.5}$$

Now observe that condition (iii) of Theorem 2.1 can be written in the form

$$p(k+1)\frac{(k+1)^k}{k^k} > 1 \tag{2.6}$$

and that

$$\frac{(k+1)^k}{k^k} = \left(1 + \frac{1}{k}\right)^k \uparrow e \qquad \text{as} \quad k \to \infty.$$

Therefore one can think of (2.6) as being the discrete analogue of (2.5) with the "delay" of (2.2) being $(k+1)$.

Necessary and sufficient conditions for the oscillation of all solutions of Eq. (2.1) where

$$p, q \in \mathbf{R} - \{0\}, \qquad k \in \mathbf{Z}, \qquad l \in \{-1, 0\}, \qquad \text{and} \qquad k \neq l \tag{2.7}$$

are given, whenever such conditions exist, in Table I. In this table K is defined to be

$$K = \frac{(k+1)^{k+1}}{k^k}.$$

TABLE I

	p	q	k	l	Necessary and sufficient conditions for oscillations
1	+	+	+	-1	$p(1+q)^k K>1$
2	+	+	0	-1	$p\geqslant 1$
3	+	+	$\leqslant -2$	-1	There exist nonoscillatory solutions
4	+	+	+	0	$q\geqslant -1$ or $q\in(0,1)$ and $pK>(1-q)^{k+1}$
5	+	+	$\leqslant -2$	0	$q\geqslant 1$
6	+	−	+	-1	$q>-1$ and $p(1+q)^k K>1$
7	+	−	0	-1	$p=-1$ or $q=1$ or $(p+1)(q-1)>0$
8	+	−	$\leqslant -2$	$-1, 0$	There exist nonoscillatory solutions
9	+	−	+	0	$pK>(1+q)^{k+1}$
10	−	+	+, 0	-1	There exist nonoscillatory solutions
11	−	+	$\leqslant -2$	-1	$p(1+q)^k K<1$
12	−	+	+	0	There exist nonoscillatory solutions
13	−	+ −	$\leqslant -2$	0	$pK<(1-q)^{k+1}$
14	−	−	+, 0	-1	$q\leqslant -1$
15	−	−	$\leqslant -2$	-1	$q\leqslant -1$ or $q>-1$ and $p(1+q)^k K<1$
16	−	−	+	0	There exist nonoscillatory solutions

As we can see from the table, in Cases 3, 8, 10, 12, and 16 there exist nonoscillatory solutions. This is because in each of these cases the characteristic equation

$$F(\lambda)\equiv \lambda-1+p\lambda^{-k}+q\lambda^{-l}=0 \tag{2.8}$$

has a positive root. This can be easily seen by computing $F(0+)$, $F(1)$, and $F(\infty)$ and by using the intermediate value theorem.

In each of the remaining cases the given condition is necessary and sufficient for the oscillation of all solutions. The proof can be obtained by computing the extreme value of $F(\lambda)$, as given by (2.8) and by applying Theorem 1.1. For example, we will give the details in Case 1 where

$$p>0, \qquad q>0, \qquad k\in\{1,2,...\}, \qquad \text{and} \qquad l=-1. \tag{2.9}$$

In this case (2.8) becomes

$$F(\lambda)=(1+q)\lambda-1+p\lambda^{-k}=0.$$

We have

$$F'(\lambda)=(1+q)-pk\lambda^{-(k+1)}$$

and

$$F''(\lambda)=pk(k+1)\lambda^{-(k+2)}>0 \qquad \text{for} \quad \lambda>0.$$

The only critical point of $F(\lambda)$ in $(0, \infty)$ is

$$\lambda_0 = \left(\frac{pk}{1+q}\right)^{1/(k+1)}$$

and $F(\lambda)$ has a minimum at $\lambda = \lambda_0$. Also $F(0+) = \infty$ and $F(\infty) = \infty$. Therefore $F(\lambda)$ has a global minimum in $(0, \infty)$ at the point λ_0.

In view of Theorem 1.1, if (2.9) holds, Eq. (2.1) oscillates if and only if $F(\lambda_0) > 0$. But

$$\begin{aligned} F(\lambda_0) &= (1+q)\lambda_0 - 1 + p\lambda_0^{-k} \\ &= \lambda_0\left[(1+q) - \frac{1}{\lambda_0} + p\lambda_0^{-(k+1)}\right] \\ &= \lambda_0\left[(1+q) - \frac{1}{\lambda_0} + \frac{1+q}{k}\right] \\ &= \lambda_0\left[(1+q)\frac{k+1}{k} - \frac{1}{\lambda_0}\right]. \end{aligned}$$

Hence $F(\lambda_0) > 0$ if and only if $\lambda_0 > (k/(k+1))(1/(1+q))$ if and only if

$$\frac{pk}{1+q} = \lambda_0^{k+1} > \left(\frac{k}{k+1}\right)^{k+1} \frac{1}{(1+q)^{k+1}} \qquad \text{if and only if } p(1+q)^k K > 1$$

which completes the proof in Case 1. The proofs in the other cases are similar and will be omitted.

3. Sufficient Conditions for Oscillations

In this section we wil obtain sufficient conditions for the oscillation of all solutions of the difference equation

$$A_{n+1} - A_n + \sum_{i=1}^{m} p_i A_{n-k_i} = 0, \qquad n = 0, 1, 2, ..., \tag{3.1}$$

where either

$$p_i \in (0, \infty) \quad \text{and} \quad k_i \in \{0, 1, 2, ...\} \quad \text{for} \quad i = 1, 2, ..., m \tag{3.2}$$

or

$$p_i \in (-\infty, 0) \quad \text{and} \quad k_i \in \{..., -3, -2, -1\} \quad \text{for} \quad i = 1, 2, ..., m. \tag{3.3}$$

Throughout this paper we will use the convention that $0^\circ = 1$.

THEOREM 3.1. *Assume that either* (3.2) *or* (3.3) *holds and suppose that*

$$\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1. \tag{3.4}$$

Then every solution of Eq. (3.1) *oscillates.*

Proof. It suffices to prove that the characteristic equation

$$F(\lambda) = \lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0 \tag{3.5}$$

has no positive roots. First, assume that (3.2) holds. Then Eq. (3.5) has no roots in $[1, \infty)$. Observe that for $i = 1, 2, ..., m$

$$\min_{0<\lambda<1} \left(\frac{\lambda^{-k_i}}{1-\lambda}\right) = \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} \qquad \text{if} \quad k_i \in \{1, 2, ...\}$$

while

$$\inf_{0<\lambda<1} \left(\frac{1}{1-\lambda}\right) = 1.$$

Hence for $0 < \lambda < 1$ we have

$$\begin{aligned} F(\lambda) &= (1-\lambda)\left(-1 + \sum_{i=1}^{m} p_i \frac{\lambda^{-k_i}}{1-\lambda}\right) \\ &\geqslant (1-\lambda)\left[-1 + \sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}}\right] \\ &> 0 \end{aligned}$$

which completes the proof when (3.2) holds.

Next, assume that (3.3) holds. Then Eq. (3.5) has no roots in $(0, 1]$. Observe that for $i = 1, 2, ..., m$

$$\min_{\lambda>1} \left(\frac{\lambda^{-k_i}}{\lambda-1}\right) = -\frac{(k_i+1)^{k_i+1}}{k_i^{k_i}}.$$

Hence for $\lambda > 1$ we have

$$\begin{aligned} F(\lambda) &= (\lambda-1)\left(1 + \sum_{i=1}^{m} p_i \frac{\lambda^{-k_i}}{\lambda-1}\right) \\ &\leqslant (\lambda-1)\left[1 - \sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}}\right] \\ &< 0 \end{aligned}$$

which completes the proof of the theorem.

The case of Theorem 3.1 where (3.2) holds was established by different techniques in [1].

For delay differential equations of the form

$$\dot{x}(t) + \sum_{i=1}^{m} p_i x(t-\tau_i) = 0, \tag{3.6}$$

where

$$p_i, \tau_i \in (0, \infty) \qquad \text{for} \quad i - 1, 2, ..., m$$

it has been proved by Hunt and Yorke [5] that every solution oscillates provided that "the sum of the torques $p_i \tau_i$ is greater than $1/e$," that is,

$$\sum_{i=1}^{m} p_i \tau_i > \frac{1}{e}. \tag{3.7}$$

If we rewrite condition (3.4) in the form

$$\sum_{i=1}^{m} p_i (k_i + 1) \frac{(k_i + 1)^{k_i}}{k_i^{k_i}} > 1$$

and if we observe that

$$\frac{(k_i + 1)^{k_i}}{k_i^{k_i}} \uparrow e \qquad \text{as} \quad k_i \to \infty,$$

then we can interpret (3.4) as the discrete analogue of (3.7).

Another condition, independent of (3.7), which implies that every solution of Eq. (3.6) oscillates is, see [9],

$$\left(\prod_{i=1}^{m} p_i \right)^{1/m} \sum_{i=1}^{m} \tau_i > \frac{1}{e}. \tag{3.8}$$

This has motivated the following result.

THEOREM 3.2. *Assume that either* (3.2) *or* (3.3) *holds and suppose that*

$$m \left(\prod_{i=1}^{m} |p_i| \right)^{1/m} \left| \frac{(k+1)^{k+1}}{k^k} \right| > 1, \tag{3.9}$$

where

$$k = \frac{1}{m} \sum_{i=1}^{m} k_i.$$

Then every solution of Eq. (1) *oscillates.*

Proof. We will prove the theorem when (3.2) holds. The proof when (3.3) holds is similar and will be omitted. In view of Theorem 1.1 it suffices to prove that the characteristic equation (3.5) has no positive roots. Clearly, Eq. (3.5) has no roots in $[1, \infty)$. On the other hand, for $0<\lambda<1$, by employing the arithmetic mean–geometric mean inequality we find

$$\begin{aligned} F(\lambda) &= (1-\lambda)\left(-1+\sum_{i=1}^{m} p_i \frac{\lambda^{-k_i}}{1-\lambda}\right) \\ &\geqslant (1-\lambda)\left[-1+m\left(\prod_{i=1}^{m} p_i\right)^{1/m} \frac{\lambda^{-k}}{1-\lambda}\right] \\ &\geqslant (1-\lambda)\left[-1+m\left(\prod_{i=1}^{m} p_i\right)^{1/m} \frac{(k+1)^{k+1}}{k^k}\right] \\ &> 0 \end{aligned}$$

and the proof is complete.

4. Oscillation in Equations with Positive and Negative Coefficients

In [7] we established the following sufficient condition for the oscillation of all solutions of the difference equation with positive and negative coefficients

$$A_{n+1}-A_n+pA_{n-k}-qA_{n-l}=0, \quad n=0, 1, 2, ..., \tag{4.1}$$

where

$$p, q \in \mathbf{R}^+ \qquad \text{and} \qquad k, l \in \mathbf{N}.$$

Theorem 4.1A. *Assume that*

$$p>q\geqslant 0, \qquad k\geqslant l\geqslant 0, \qquad q(k-l)\leqslant 1 \tag{4.2}$$

and that

$$\left.\begin{aligned} &p-q>\frac{k^k}{(k+1)^{k+1}} && \text{if } k\geqslant 1 \\ &p-q\geqslant 1 && \text{if } k=0. \end{aligned}\right\} \tag{4.3}$$

Then every solution of Eq. (4.1) *oscillates.*

The next result is the dual of Theorem 4.1A.

THEOREM 4.1B. *Assume that*

$$0 \leqslant p < q, \qquad 1 \leqslant k \leqslant l, \qquad p(l-k) \leqslant 1 \tag{4.4}$$

and that

$$\left.\begin{aligned} q-p &> \frac{(l-1)^{l-1}}{l^l} && \textit{if} \quad l \geqslant 2 \\ q-p &\geqslant 1 && \textit{if} \quad l=1. \end{aligned}\right\} \tag{4.5}$$

Then every solution of the difference equation

$$A_{n+1} - A_n + pA_{n+k} - qA_{n+l} = 0 \tag{4.6}$$

oscillates.

Proof. The case $k=l$ reduces to Theorem 2.1. So suppose $k<l$. Assume, for the sake of contradiction, that Eq. (4.6) has an eventually positive solution $\{A_n\}$. Set

$$C_n = A_n - p\left(\sum_{j=n+k}^{n+l-1} A_j\right). \tag{4.7}$$

Then

$$C_{n+1} - C_n = (A_{n+1} - A_n) - p(A_{n+l} - A_{n+k}) = (q-p)A_{n+l} > 0. \tag{4.8}$$

Thus $\{C_n\}$ is eventually strictly increasing and either

$$\lim_{n\to\infty} C_n = \infty \tag{4.9}$$

or

$$\lim_{n\to\infty} C_n = l \in \mathbf{R}. \tag{4.10}$$

Assume that (4.10) holds. Then from (4.8) and (4.7) we see that

$$\lim_{n\to\infty} A_n = 0 = \lim_{n\to\infty} C_n.$$

Hence, there exists an index n_1 such that

$$C_{n_1} < 0 \qquad \text{and} \qquad A_{n_1} \geqslant A_n > 0 \qquad \text{for} \quad n \geqslant n_1.$$

Then (4.7) yields

$$0 > C_{n_1} = A_{n_1} - p\left(\sum_{j=n_1+k}^{n_1+l-1} A_j\right) \geqslant A_{n_1}[1 - p(l-k)] \geqslant 0$$

which is a contradiction. Therefore (4.9) holds. From (4.8) and (4.7) we find

$$C_{n+1} - C_n - (q-p) C_{n+l} \geqslant 0.$$

Also

$$C_n > 0.$$

In view of Lemma 1.1(b) this implies that the difference equation

$$Y_{n+1} - Y_n - (q-p) Y_{n+l} = 0$$

has an eventually positive solution. This contradicts (4.5) and completes the proof.

By combining the results in Theorems 3.1, 4.1A, 4.1B, we obtain Table II which gives sufficient conditions for the oscillation of all solutions of the difference equation

$$A_{n+1} - A_n + pA_{n-k} + qA_{n-l} = 0, \qquad n = 0, 1, 2, ..., \tag{4.11}$$

where

$$p, q \in \mathbf{R} - \{0\}, \qquad k, l \in \mathbf{Z} - \{0, 1\}, \qquad \text{and} \qquad k > l.$$

In Table II, K and L are defined to be

$$K = \frac{(k+1)^{k+1}}{k^k} \qquad \text{and} \qquad L = \frac{(l+1)^{l+1}}{l^l}.$$

TABLE II

	p	q	k	l	Sufficient conditions for oscillation
1	+	+	+	+	$pK + qL > 1$
2	+	+	+	−	$pK > 1$
3	+	+	−	−	There exist nonoscillatory solutions
4	+	−	+	+	$1 + q(k-l) \geqslant 0$ and $(p+q)K > 1$
5	+	−	+	−	There exist nonoscillatory solutions
6	+	−	−	−	$1 - p(k-l) \geqslant 0$ and $(p+q)L > 1$
7	−	+	+	+	There exist nonoscillatory solutions
8	−	+	+	−	There exist nonoscillatory solutions
9	−	+	−	−	There exist nonoscillatory solutions
10	−	−	+	+	There exist nonoscillatory solutions
11	−	−	+	−	$qL > 1$
12	−	−	−	−	$pK + qL > 1$

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