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Journal of Algebra 283 (2005) 723–737

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Crossed product algebras defined by separable extensions

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Received 16 January 2004

Available online 26 November 2004

Communicated by Susan Montgomery

Abstract

We generalize the classical construction of crossed product algebras defined by finite Galois field extensions to finite separable field extensions. By studying properties of rings graded by groupoids, we are able to calculate the Jacobson radical of these algebras. We use this to determine when the analogous construction of crossed product orders yield Azumaya, maximal, or hereditary orders in a local situation. Thereby we generalize results by Haile, Larson, and Sweedler.

1. Introduction

Recall that if L/K is a finite Galois field extension with Galois group G, then the crossed product algebra $(L/K, f)$ is defined as the additive group $\bigoplus_{\alpha\in G} Lu_\alpha$ with multiplication defined by the K-linear extension of the rule

$$xu_\alpha yu_\beta = x\alpha(y) f_{\alpha,\beta}u_{\alpha\beta} \tag{1}$$

for all $x, y \in L$ and all $\alpha, \beta \in G$, where f is a cocycle, that is, a map from $G \times G$ to L satisfying

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.09.007

$$f_{\alpha,\beta\gamma}\alpha(f_{\beta,\gamma}) = f_{\alpha\beta,\gamma} f_{\alpha,\beta} \tag{2}$$

for all $\alpha, \beta, \gamma \in G$ and

$$f_{\alpha,\beta} = 1 \tag{3}$$

whenever $\alpha = 1$ or $\beta = 1$. It is well known that if f is invertible, that is, if $f_{\alpha,\beta}$ is nonzero for all $\alpha, \beta \in G$, then the crossed product algebra is central and simple as an algebra over K (see, e.g., [17]). If f is not invertible, then the crossed product algebra is still central, but not simple. In fact, Haile, Larson, and Sweedler [12] have shown the following result.

Theorem 1. *The ring $(L/K, f)$ is central as an algebra over K. Furthermore, if H denotes the set of $\alpha \in G$ such that $f_{\alpha,\alpha^{-1}}$ is nonzero, then H is a subgroup of G, $\bigoplus_{\alpha\in H} Lu_\alpha$ is central and simple as an algebra over L^H, and the Jacobson radical of $(L/K, f)$ is $\bigoplus_{\alpha\in G\setminus H} Lu_\alpha$.*

If K is the field of quotients of a Dedekind domain R, S is the ring of algebraic integers in L over R, and f is an invertible cocycle taking its values in S, then the crossed product order $(S/R, f)$ is defined as the additive group $\bigoplus_{\alpha\in G} Su_\alpha$ with multiplication induced by the corresponding crossed product algebra. A lot of work has been devoted to studying the question of when crossed product orders are Azumaya, maximal, or hereditary (see [1,2, 9,11,13,17,20]). If L/K is unramified and S and R are local rings, then this question can be analyzed by calculating the Jacobson radical of the crossed product order. In fact, Haile [11] has obtained an arithmetical version of Theorem 1.

Theorem 2. *Let L/K be unramified and assume that S and R are local rings. If H denotes the set of $\alpha \in G$ such that $f_{\alpha,\alpha^{-1}}$ is a unit in S, then H is a subgroup of G and $\bigoplus_{\alpha\in H} Su_\alpha$ is Azumaya as an order over S^H. Furthermore, the Jacobson radical of $(S/R, f)$ is $m(\bigoplus_{\alpha\in H} Su_\alpha) \oplus (\bigoplus_{\alpha\in G\setminus H} Su_\alpha)$, where m is the maximal ideal of R.*

Haile loc. cit. then uses Theorem 2 to prove the following result.

Theorem 3. *Let L/K be unramified and assume that S and R are local rings. Then $(S/R, f)$ is Azumaya if and only if $f_{\alpha,\beta}$ is a unit in S for all $\alpha, \beta \in G$. Furthermore, $(S/R, f)$ is maximal if and only if it is hereditary if and only if none of the $f_{\alpha,\beta}$, $\alpha, \beta \in G$, belong to the square of the maximal ideal of S.*

If L/K is a finite separable (not necessarily normal) field extension, then the classical definition of crossed product algebras makes no sense. However, if we replace the Galois group of L/K by the set of field isomorphisms between the different conjugates of L in a normal closure of L/K, then we can define an algebra structure (see below) that generalizes the classical crossed product construction. In general this gives us rings graded by groupoids, and not just groups as in the classical case.

To be more precise, let N denote a normal closure of L/K and let G denote the Galois group of N/K. If H is a subgroup of G, then let L^H denote the set of $x \in L$ that are fixed by all $\alpha \in H$. Furthermore, let $L = L_1, L_2 \ldots, L_n$ denote the different conjugate fields of L under the action of G. If $1 \leqslant i, j \leqslant n$, then let G_{ij} denote the set of field isomorphisms from L_j to L_i. If $\alpha \in G_{ij}$, then we indicate this by writing $s(\alpha) = j$ and $t(\alpha) = i$ (s and t are abbreviations for *source* and *target*). If we let $\mathbf{G}$ be the union of the G_{ij}, $1 \leqslant i, j \leqslant n$, then $\mathbf{G}$ is no longer a group, but instead a groupoid, that is, a category where all the morphisms are isomorphisms. If $\mathbf{H}$ is a subcategory of $\mathbf{G}$ closed under taking inverses, then we say that $\mathbf{H}$ is a subgroupoid of $\mathbf{G}$. We define the crossed product algebra $(L/K, f)$ as the additive group $\bigoplus_{\alpha\in\mathbf{G}} L_{t(\alpha)}u_\alpha$ with multiplication defined by the K-linear extension of the rule (1) if $s(\alpha) = t(\beta)$, and $xu_\alpha yu_\beta = 0$ otherwise, for all $\alpha, \beta \in \mathbf{G}$ and all $x \in L_{t(\alpha)}$, $y \in L_{t(\beta)}$, where f is a cocycle on the groupoid $\mathbf{G}$. This means that (see, e.g., [18] for the details) $f_{\alpha,\beta}$ is defined precisely when $s(\alpha) = t(\beta)$ and that it satisfies $f_{\alpha,\beta} \in L_{t(\alpha)}$ and (2) for all $\alpha, \beta, \gamma \in \mathbf{G}$ such that $s(\alpha) = t(\beta)$ and $s(\beta) = t(\gamma)$. We also assume that f satisfies (3) whenever α or β is an identity map on some of the conjugate fields of L. Note that if L/K is actually Galois, then $(L/K, f)$ coincides with the usual crossed product algebra construction.

In Section 4, we prove the following generalization of Theorem 1.

Theorem 4. *The ring $(L/K, f)$ is central as an algebra over L^G. Furthermore, if $\mathbf{H}$ denotes the set of $\alpha \in \mathbf{G}$ such that $f_{\alpha,\alpha^{-1}}$ is nonzero, then $\mathbf{H}$ is a subgroupoid of $\mathbf{G}$, $\bigoplus_{\alpha\in\mathbf{H}} L_{t(\alpha)}u_{t(\alpha)}$ is central and simple as an algebra over $L^{\mathbf{H}\cap G}$ and the Jacobson radical of $(L/K, f)$ is $\bigoplus_{\alpha\in\mathbf{G}\setminus\mathbf{H}} L_{t(\alpha)}u_\alpha$.*

If K is the field of quotients of a Dedekind domain R, S (S_i) is the ring of algebraic integers in L (L_i) over R ($i = 1, \ldots, n$), and f is an invertible cocycle taking its values in $\bigcup_{i=1}^n S_i$, then we define the crossed product order $(S/R, f)$ as the additive group $\bigoplus_{\alpha\in G} S_{t(\alpha)}u_\alpha$ with multiplication induced by the corresponding crossed product algebra.

In Section 5, we prove the following generalization of Theorem 2.

Theorem 5. *Let L/K be unramified and assume that S and R are local rings. If $\mathbf{H}$ denotes the set of $\alpha \in \mathbf{G}$ such that $f_{\alpha,\alpha^{-1}}$ is a unit in $S_{t(\alpha)}$, then $\mathbf{H}$ is a subgroupoid of $\mathbf{G}$ and $\bigoplus_{\alpha\in\mathbf{H}} S_{t(\alpha)}u_\alpha$ is Azumaya as an order over $S^{\mathbf{H}\cap G}$. Furthermore, the Jacobson radical of $(S/R, f)$ is $m(\bigoplus_{\alpha\in\mathbf{H}} S_{t(\alpha)}u_\alpha) \oplus (\bigoplus_{\alpha\in\mathbf{G}\setminus\mathbf{H}} S_{t(\alpha)}u_\alpha)$, where m is the maximal ideal of R.*

In the same section, we use Theorem 5 to prove the following result which generalizes Theorem 3.

Theorem 6. *Let L/K be unramified and assume that S and R are local rings. Then $(S/R, f)$ is Azumaya if and only if $f_{\alpha,\beta}$ is a unit in $S_{t(\alpha)}$ for all $\alpha, \beta \in \mathbf{G}$ such that $s(\alpha) = t(\beta)$. Furthermore, $(S/R, f)$ is maximal if and only if it is hereditary if and only if none of the $f_{\alpha,\beta}$, $\alpha, \beta \in \mathbf{G}$, $s(\alpha) = t(\beta)$, belong to the square of the maximal ideal of $S_{t(\alpha)}$.*

As indicated above, our generalization of crossed product algebras belong to the category of rings graded by groupoids. Therefore, in Section 2, we extend some results for rings and modules graded by groups, to the groupoid graded case. Our proofs resemble their group graded counterparts (from [9,14,15]). But for the convenience of the reader we have, nonetheless, included them in full detail. In Section 3, we state and prove a result about the Jacobson radical of algebras over commutative local rings. In Sections 4 and 5, we use the results of Sections 2 and 3 to prove Theorems 4–6. For more results concerning group graded rings and modules see [4–7,10,16].

2. Graded rings and modules

In this section, we first recall the "folklore" definitions (see Definitions 1 and 2) of rings and modules graded by categories. Then we specialize these categories to be groupoids and prove that then the components of strongly graded rings are invertible bimodules (see Proposition 3). This is in turn used to prove a result (see Proposition 4) concerning the separability of strongly groupoid graded rings and the trace function (see Definition 5) that we need in later sections. The section is ended by an application (see Corollary 2) of this result to the separability of groupoid rings (see Example 1). This result is not needed in the sequel, but is interesting in its own right since it provides us with a simultaneous generalization of known separability conditions (see Corollaries 3 and 4) for group rings and matrix rings.

Let $\mathbf{C}$ be a category. If α is a morphism in $\mathbf{C}$, then we will indicate this by writing $\alpha \in \mathbf{C}$. The source and target of a morphism α in $\mathbf{C}$ will be denoted $s(\alpha)$ and $t(\alpha)$, respectively. We let $\mathbf{C}_0$ denote the collection of objects of $\mathbf{C}$. An object of $\mathbf{C}$ will often be identified with its identity morphism. For the rest of this section, we assume that $\mathbf{C}$ is small.

Let all rings be associative and equipped with multiplicative identities. We assume that ring homomorphisms respect the multiplicative identities. Furthermore, all modules (left, right and bimodules) are assumed to be unital. Let A be a ring. We let the category of left A-modules be denoted by A-mod. The center of A is denoted $C(A)$. If M is a left A-module and S and T are subsets of A and M, respectively, then ST denotes the set of all finite sums of products of the form st, $s \in S$, $t \in T$.

Definition 1. A ring A is graded if there is a set of additive subgroups A_α, $\alpha \in \mathbf{C}$, of A such that $A = \bigoplus_{\alpha \in \mathbf{C}} A_\alpha$ and for all $\alpha, \beta \in \mathbf{C}$, we have

$$A_\alpha A_\beta \subseteq \begin{cases} A_{\alpha\beta} & \text{if } s(\alpha) = t(\beta), \\ \{0\} & \text{otherwise.} \end{cases} \tag{4}$$

If there always is equality in (4), instead of just inclusion, then A is called strongly graded. A morphism of graded rings $f : A \to B$ is a morphism of rings satisfying $f(A_\alpha) \subseteq B_\alpha$ for all $\alpha \in \mathbf{C}$.

Example 1. Let R be a ring. The category ring (or groupoid ring if $\mathbf{C}$ is a groupoid) $R[\mathbf{C}]$, of R over $\mathbf{C}$, is defined to be the set of all formal sums $\sum_{\alpha \in \mathbf{C}} r_\alpha \alpha$ with $r_\alpha \in R$ and $r_\alpha = 0$ for all but finitely many $\alpha \in \mathbf{C}$. Addition is defined pointwise and multiplication is defined by the R-linear extension of the rule

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{if } s(\alpha) = t(\beta), \\ 0 & \text{otherwise,} \end{cases}$$

for all $\alpha, \beta \in \mathbf{C}$. The grading is, of course, defined by $R[\mathbf{C}]_\alpha = R\alpha$, $\alpha \in \mathbf{C}$. We now consider two special cases of interest:

(a) If $\mathbf{C}$ is a group, then $R[\mathbf{C}]$ is the usual group ring of R over $\mathbf{C}$.
(b) If $\mathbf{C} = I \times I$, where I is a finite set of cardinality n, and $\mathbf{C}$ is equipped with the operation $(i, j) \cdot (k, l) = (i, l)$ if $j = k$, then $R[\mathbf{C}]$ is R-algebra isomorphic to $M_n(R)$, the ring of $n \times n$ matrices over R.

Definition 2. If A is a graded ring, then a left A-module M is graded if there is a set of additive subgroups M_α, $\alpha \in \mathbf{C}$, of M such that $M = \bigoplus_{\alpha \in \mathbf{C}} M_\alpha$ and for all $\alpha, \beta \in \mathbf{C}$, we have

$$A_\alpha M_\beta \subseteq \begin{cases} M_{\alpha\beta} & \text{if } s(\alpha) = t(\beta), \\ \{0\} & \text{otherwise.} \end{cases} \tag{5}$$

If there always is equality in (5), instead of just inclusion, then M is called strongly graded. A morphism of graded A-modules $f : M \to N$ is a morphism of A-modules satisfying $f(M_\alpha) \subseteq N_\alpha$ for all $\alpha \in \mathbf{C}$. Let A-gr denote the category of graded left A-modules. It is easy to see that A-gr is a Grothendieck category.

Let A be a graded ring and M a graded left A-module. Any nonzero $m \in M$ has a unique decomposition $m = \sum_{\alpha \in \mathbf{C}} m_\alpha$ where $m_\alpha \in M_\alpha$ and all but finitely many of the m_α are nonzero. The nonzero elements m_α in the decomposition of m are called the homogeneous components of m. Also put $M_0 = \bigoplus_{\alpha \in \mathbf{C}_0} M_\alpha$.

For the rest of this section, we assume that $\mathbf{C}$ is a groupoid.

Proposition 1. *Let A be a graded ring. Then the multiplicative identity of A belongs to A_0. Furthermore, if we let $\mathbf{D}$ denote the set of $\alpha \in \mathbf{C}$ such that $1_{s(\alpha)}$ and $1_{t(\alpha)}$ are nonzero, then $\mathbf{D}$ is a subgroupoid of $\mathbf{C}$ with $\mathbf{D}_0$ finite and $A = \bigoplus_{\alpha \in \mathbf{D}} A_\alpha$.*

Proof. First we show that $1 \in A_0$. Let $1 = \sum_{\alpha \in \mathbf{C}} 1_\alpha$ be the homogeneous decomposition of 1 in A. Then we get that $1_\beta = 11_\beta = \sum_{\alpha \in \mathbf{C}} 1_\alpha 1_\beta$ for all $\beta \in \mathbf{C}$. But since $1_\alpha 1_\beta \in A_{\alpha\beta}$ for all $\alpha, \beta \in \mathbf{C}$, we get that $1_\alpha 1_\beta = 0$ whenever $\alpha \notin \mathbf{C}_0$. Hence, if $\alpha \notin \mathbf{C}_0$, then $1_\alpha = 1_\alpha 1 = \sum_{\beta \in \mathbf{C}} 1_\alpha 1_\beta = 0$.

Since $s(\alpha^{-1}) = t(\alpha)$, $t(\alpha^{-1}) = s(\alpha)$, $s(\alpha\beta) = s(\beta)$, $t(\alpha\beta) = t(\alpha)$ for all $\alpha, \beta \in \mathbf{C}$ with $s(\alpha) = t(\beta)$, we get that $\mathbf{D}$ is a subgroupoid of $\mathbf{C}$. Also, by the fact that $1 = \sum_{\alpha \in \mathbf{D}_0} 1_\alpha$, we get that $\mathbf{D}_0$ is finite.

Finally, take $\alpha \in \mathbf{C} \setminus \mathbf{D}$. Suppose that $1_{t(\alpha)} = 0$. Then $A_\alpha = 1A_\alpha = 1_{t(\alpha)} A_\alpha = \{0\}$. The case when $1_{s(\alpha)} = 0$ is treated similarly. □

Remark 1. By Proposition 1, it is now legitimate for us to assume for the rest of the article that $\mathbf{C}_0$ is finite and that $1 = \sum_{\alpha \in \mathbf{C}_0} 1_\alpha$ where $1_\alpha \neq 0$ for all $\alpha \in \mathbf{C}_0$. In particular, A_α is nonzero for all $\alpha \in \mathbf{C}_0$.

Before we state the next proposition, we need a definition.

Definition 3. Let i denote the inclusion map from A_0 to A. The graded restriction and induction functors

$$i_*^{\mathrm{gr}}: A\text{-gr} \to A_0\text{-mod} \quad \text{and} \quad i_{\mathrm{gr}}^*: A_0\text{-mod} \to A\text{-gr}$$

are defined by $i_*^{\mathrm{gr}}(M) = M_0$, with the induced left A_0-module structure, for all graded left A-modules M, and $i_{\mathrm{gr}}^*(N) = A \otimes_{A_0} N$, with the induced left A-module structure, and a grading defined by $(A \otimes_{A_0} N)_\alpha = A_\alpha \otimes_{A_0} N$ for all $\alpha \in \mathbf{C}$ and all left A_0-modules N. It is easy to check that i_*^{gr} is a right adjoint of i_{gr}^*. We let the corresponding unit and counit be denoted by ε and δ, respectively.

Proposition 2. *If A is a graded ring, then following three conditions are equivalent*:

(i) *The ring A is strongly graded.*
(ii) *Every graded left A-module is strongly graded.*
(iii) *The natural transformations ε and δ are natural equivalences.*

Proof. Let M be a graded left A-module. Suppose first that (i) holds. If $\alpha, \beta \in \mathbf{C}$ are chosen so that $s(\alpha) = t(\beta)$, then we get that $A_\alpha M_\beta \subseteq M_{\alpha\beta} = A_{t(\alpha)} M_{\alpha\beta} = A_\alpha A_{\alpha^{-1}} M_{\alpha\beta} \subseteq A_\alpha M_\beta$. Hence (ii) holds.

Now suppose that (ii) holds. By the assumption, δ_M is surjective. Let K denote the kernel of δ_M. Then K_0 coincides with the kernel of the isomorphism from $A_0 \otimes_{A_0} M_0$ to M_0. Hence $K_0 = 0$, and therefore, again by the assumption, $K_\alpha = A_\alpha K_0 = \{0\}$ for each $\alpha \in \mathbf{C}$. Hence δ_M is injective. Also ε_N is an isomorphism for all left A_0-modules N. In fact, the inverse of ε_N is given by the multiplication map from $A_0 \otimes_{A_0} N$ to N. Thus (iii) holds.

If (iii) holds, then trivially (ii) and hence (i) holds. $\square$

To state the next result, we need another definition.

Definition 4. For a graded left A-module M and $\alpha \in \mathbf{C}$, let $M(\alpha)$, the α-suspension of M, be M as a left A-module but with the new grading

$$M(\alpha)_\beta = \begin{cases} M_{\beta\alpha} & \text{if } s(\beta) = t(\alpha), \\ \{0\} & \text{otherwise,} \end{cases}$$

for all $\beta \in \mathbf{C}$.

Proposition 3. *Let A be a strongly graded ring.*

(a) *If M is a graded left A-module and $\beta \in \mathbf{C}$, then the multiplication map from $A \otimes_{A_{t(\beta)}} M_\beta$ to $M(\beta)$ is simultaneously an isomorphism of graded left A-modules and A-$A_{s(\beta)}$-bimodules.*
(b) *If $\alpha, \beta \in \mathbf{C}$ are chosen so that $s(\alpha) = t(\beta)$, then the multiplication map from $A_\alpha \otimes_{A_{t(\beta)}} A_\beta$ to $A_{\alpha\beta}$ is an isomorphism of $A_{t(\alpha)}$-$A_{s(\beta)}$-bimodules.*
(c) *Each A_α, $\alpha \in \mathbf{C}$, is an invertible $A_{t(\alpha)}$-$A_{s(\alpha)}$-bimodule.*

Proof. (a) follows from Proposition 2, (b) follows from (a) and (c) follows from (b) with $\beta = \alpha^{-1}$. $\square$

Recall that if a ring B is a subring of a ring C (assumed to have the same identity elements), then C is called separable over B if the multiplication map from $C \otimes_B C$ to C splits as a C-bimodule map. A ring is called Azumaya if it is separable over its center.

Now we determine a necessary and sufficient condition for a strongly graded ring A to be separable over A_0. To do that we need some more notations and a definition. By Proposition 3(c) and general theory for invertible bimodules (see, e.g., [3]), there is for each $\alpha \in \mathbf{C}$ a unique isomorphism of rings f_α from $C(A_{s(\alpha)})$ to $C(A_{t(\alpha)})$ such that

$$xa = f_\alpha(a)x \tag{6}$$

for all $x \in A_\alpha$ and all $a \in C(A_{s(\alpha)})$. By abuse of notation, we let f_α be denoted by α.

Definition 5. Let A be a strongly graded ring with $\mathbf{C}$ finite. Then the trace map $\mathrm{tr} : C(A_0) \to C(A_0)$ is defined by

$$\mathrm{tr}(a) = \sum_{\alpha \in \mathbf{C}_0} \sum_{\substack{\beta \in \mathbf{C} \\ s(\beta)=\alpha}} \beta(a_\alpha)$$

for all $a \in C(A_0)$.

Example 2. Let R be a ring and $A = R[\mathbf{C}]$ the associated groupoid ring of R over a finite groupoid $\mathbf{C}$. If $x = \sum_{\alpha \in \mathbf{C}_0} r_\alpha \alpha \in C(A_0)$, then

$$\mathrm{tr}(x) = \sum_{\alpha \in \mathbf{C}_0} r_\alpha \sum_{\substack{\beta \in \mathbf{C} \\ s(\beta)=\alpha}} t(\beta). \tag{7}$$

We now consider two special cases:

(a) If $\mathbf{C}$ is a group, then (7) reduces to

$$\mathrm{tr}(x) = nx, \tag{8}$$

where n denotes the cardinality of $\mathbf{C}$.

(b) If $\mathbf{C}$ is the groupoid from Example 1(b), then (7) reduces to

$$\mathrm{tr}(x) = \sum_{i=1}^{n} r_{(i,i)}. \tag{9}$$

Thus tr is the restriction to $C(A_0)$ of the usual trace on $M_n(R)$.

Proposition 4. *If A is a strongly graded ring, then A is separable over A_0 if and only if* $\mathbf{C}$ *is finite and the image of the trace contains* 1.

Proof. Let the multiplication map from $A \otimes_{A_0} A$ to A be denoted by μ.

First suppose that there is an A-bimodule map ν from A to $A \otimes_{A_0} A$ such that

$$\mu \circ \nu = 1. \tag{10}$$

We also put

$$s = \nu(1). \tag{11}$$

For future use, we note that, by (11), we get that

$$as = sa \tag{12}$$

for all $a \in A$. Since A is graded, we get that

$$A \otimes_{A_0} A = \bigoplus_{\alpha,\beta \in \mathbf{C}} A_\alpha \otimes_{A_0} A_\beta \cong \bigoplus_{\substack{\alpha,\beta \in \mathbf{C} \\ s(\alpha)=t(\beta)}} A_{\alpha\beta} \tag{13}$$

as additive groups. Therefore, by (10), we can assume that

$$s = \sum_{\alpha \in \mathbf{C}} \sum_{k=1}^{l_\alpha} a_{\alpha,k} \otimes b_{\alpha^{-1},k} \tag{14}$$

for some $a_{\alpha,k} \in A_\alpha$, $b_{\alpha^{-1},k} \in A_{\alpha^{-1}}$ and some positive integers l_α where $\sum_{k=1}^{l_\alpha} a_{\alpha,k} \otimes b_{\alpha^{-1},k} = 0$ for all but finitely many $\alpha \in \mathbf{C}$. By (10), (11), and (14), we get that $1 = \sum_{\alpha \in \mathbf{C}} c_{\alpha,\alpha^{-1}}$, where $c_{\alpha,\alpha^{-1}} = \sum_{k=1}^{l_\alpha} a_{\alpha,k} b_{\alpha^{-1},k}$ and

$$c_{\alpha,\alpha^{-1}} = 0 \quad \text{for all but finitely many } \alpha \in \mathbf{C}. \tag{15}$$

By (12) and (13), it follows that each $c_{\alpha,\alpha^{-1}} \in C(A_0)$. Take $\alpha, \beta \in \mathbf{C}$ such that $s(\beta) = t(\alpha)$. Then, by (12) and (13) again, we get that $ac_{\alpha,\alpha^{-1}} = c_{\beta\alpha,\alpha^{-1}\beta^{-1}} a$ for all $a \in A_\beta$. Since A is strongly graded this implies, by (6), that

$$\beta(c_{\alpha,\alpha^{-1}}) = c_{\beta\alpha,\alpha^{-1}\beta^{-1}}. \tag{16}$$

Therefore, by (15), (16), and Remark 1, it follows that $\mathbf{C}$ is finite. Now define an equivalence relation $\sim$ on $\mathbf{C}_0$ in the following way. If $\alpha, \beta \in \mathbf{C}_0$, then put $\alpha \sim \beta$ if there is $\gamma \in \mathbf{C}$ with $s(\gamma) = \alpha$ and $t(\gamma) = \beta$. Choose representatives $\alpha_1, \ldots, \alpha_r$ for the different equivalence classes and put $c = \sum_{i=1}^{r} c_{\alpha_i,\alpha_i^{-1}}$. Then $\mathrm{tr}(c) = 1$.

Now suppose that $\mathbf{C}$ is finite and that there is $c \in C(A_0)$ such that $\operatorname{tr}(c) = 1$. Since A is strongly graded we can, for each $\alpha \in \mathbf{C}$, choose a positive integer m_α and elements $a_{\alpha,k} \in A_\alpha$ and $b_{\alpha^{-1},k} \in A_{\alpha^{-1}}$ for $k = 1, \ldots, m_\alpha$, such that

$$\sum_{k=1}^{m_\alpha} a_{\alpha,k} b_{\alpha^{-1},k} = 1_{t(\alpha)}. \tag{17}$$

Now put

$$d = \sum_{\beta \in \mathbf{C}_0} \sum_{\substack{\alpha \in \mathbf{C} \\ s(\alpha)=\beta}} \sum_{k=1}^{m_\sigma} a_{\alpha,k} c_\beta \otimes b_{\alpha^{-1},k}$$

and define $\nu : A \to A \otimes_{A_0} A$ by $\nu(a) = ad$, $a \in A$. Then ν is an A-bimodule map satisfying (10). In fact, if $a \in A_\gamma$, for some $\gamma \in \mathbf{C}$, then, by (17), we get that

$$\begin{aligned}
ad &= \sum_{\beta \in \mathbf{C}_0} \sum_{\substack{\alpha \in \mathbf{C} \\ s(\alpha)=\beta}} \sum_{k=1}^{m_\alpha} a a_{\alpha,k} c_\beta \otimes b_{\alpha^{-1},k} \\
&= \sum_{\beta \in \mathbf{C}_0} \sum_{\substack{\alpha \in \mathbf{C},\ s(\alpha)=\beta, \\ t(\alpha)=s(\gamma)}} \sum_{k=1}^{m_\alpha} \sum_{l=1}^{m_{\gamma\alpha}} a_{\gamma\alpha,l} b_{\alpha^{-1}\gamma^{-1},l} a a_{\alpha,k} c_\beta \otimes b_{\alpha^{-1},k} \\
&= \sum_{\beta \in \mathbf{C}_0} \sum_{\substack{\alpha \in \mathbf{C},\ s(\alpha)=\beta, \\ t(\alpha)=s(\gamma)}} \sum_{k=1}^{m_\alpha} \sum_{l=1}^{m_{\gamma\alpha}} a_{\gamma\alpha,l} c_\beta \otimes b_{\alpha^{-1}\gamma^{-1},l} a a_{\alpha,k} b_{\alpha^{-1},k} \\
&= \sum_{\beta \in \mathbf{C}_0} \sum_{\substack{\alpha \in \mathbf{C},\ s(\alpha)=\beta, \\ t(\alpha)=s(\gamma)}} \sum_{l=1}^{m_{\gamma\alpha}} a_{\gamma\alpha,l} c_\beta \otimes b_{\alpha^{-1}\gamma^{-1},l} a \\
&= da
\end{aligned}$$

and

$$\begin{aligned}
(\mu \circ \nu)(1) = \mu(d) &= \sum_{\beta \in \mathbf{C}_0} \sum_{\substack{\beta \in \mathbf{C} \\ s(\alpha)=\beta}} \sum_{k=1}^{m_\alpha} a_{\alpha,k} c_\beta b_{\alpha^{-1},k} = \sum_{\beta \in \mathbf{C}_0} \sum_{\substack{\alpha \in \mathbf{C} \\ s(\alpha)=\beta}} \alpha(c_\beta) \\
&= \operatorname{tr}(c) = 1. \quad \square
\end{aligned}$$

Remark 2. Proposition 4 (and our proof) generalizes Proposition 2.1 (and its proof) in [14] from the group graded case to the groupoid graded situation. We have also corrected the formulation of Proposition 2.1 in loc. cit. In fact, there it is claimed that a strongly group graded ring A is separable precisely when the associated trace map $C(A_0) \to C(A_0)$ is surjective. This fails for all crossed product algebras defined by nontrivial Galois field extensions.

Definition 6. If $\mathbf{C}$ is finite and $\alpha \in \mathbf{C}_0$, then let n_α denote the number of $\beta \in \mathbf{C}$ with $s(\beta) = t(\beta) = \alpha$.

Example 3.

(a) If $\mathbf{C}$ is a finite group, then n_α equals the order of $\mathbf{C}$ for all $\alpha \in \mathbf{C}_0$.
(b) If $\mathbf{C}$ is the groupoid from Example 1(b), then $n_\alpha = 1$ for all $\alpha \in \mathbf{C}_0$.

Corollary 1. *If A is a strongly graded ring such that $\mathbf{C}$ is finite and n_α is a unit in A for all $\alpha \in \mathbf{C}_0$, then A is separable over A_0.*

Proof. By Proposition 4, we need to find $c \in A$ such that $\mathrm{tr}(c) = 1$. Choose representatives $\alpha_1, \ldots, \alpha_r \in \mathbf{C}_0$ for the different equivalence classes of the equivalence relation $\sim$ introduced in the proof of Proposition 4. A straightforward calculation shows that if we put $c = \sum_{i=1}^{r} n_{\alpha_i}^{-1} 1_{\alpha_i}$, then $\mathrm{tr}(c) = 1$. □

Lemma 1. *Let $B \subseteq C \subseteq D$ be a tower of ring extensions.*

(a) *If C is a direct product of finitely many copies of B, then C is separable over B.*
(b) *If D is separable over C, and C is separable over B, then D is separable over B.*
(c) *If D is separable over B, then D is separable over C.*

Proof. (a) is straightforward and (b) and (c) follow from a more general result concerning separable functors (see [14, Lemma 1.1]). □

Corollary 2. *If R is a ring and $\mathbf{C}$ is a groupoid, then $R[\mathbf{C}]$ is separable over R if and only if $\mathbf{C}$ is finite and n_α is a unit in R for all $\alpha \in \mathbf{C}_0$.*

Proof. Put $A = R[\mathbf{C}]$. First note that, by Lemma 1(a), A_0 is always separable over R, since A_0 is a direct product of finitely many (by Proposition 1) copies of R. Therefore, by Lemma 1(b)(c), A is separable over R if and only if A is separable over A_0.

Assume that A is separable over A_0. Then, by Proposition 4, we get that $\mathbf{C}$ is finite and that the image of the trace contains 1. But since $\mathrm{tr}(C(A_0)) = \sum_{\alpha \in \mathbf{C}_0} n_\alpha C(R)\alpha$, each n_α, $\alpha \in \mathbf{C}_0$, must be a unit in R.

On the other hand, if $\mathbf{C}$ is finite and n_α is a unit in R for all $\alpha \in \mathbf{C}_0$, then, by Corollary 1, A is separable over A_0. □

By Corollary 2 and Examples 1–3, we immediately get the following two well-known results.

Corollary 3 (Mascke's theorem). *Let R be a ring and G a group. Then $R[G]$ is separable over R if and only if G is finite and the order of G is a unit in R.*

Corollary 4 (DeMeyer and Ingraham [8]). *Let R be a ring and n a positive integer. Then $M_n(R)$ is separable over R.*

3. The Jacobson radical

In this section, we calculate the Jacobson radical of algebras with a certain property (see Proposition 6). To do that we need a well-known result (see Proposition 5).

First we recall some definitions. Let A be a ring, I a two-sided ideal of A and a an element of A. Recall that I (or a) is called nilpotent if there is a positive integer n such that $I^n = \{0\}$ (or $a^n = 0$); I is called nil if it consists of nilpotent elements. The nilradical, $\operatorname{rad}(I)$, of I is the set of $a \in A$ with the property that $a^m \in I$ for some positive integer m. The Jacobson radical, $J(A)$, of A is the intersection of the maximal left (or right) ideals of A.

Proposition 5. *Let A and B be rings.*

(a) *If B is a subring of A, and A is free as a left B-module with a finite basis consisting of elements x such that $xB = Bx$, then $J(B) = B \cap J(A)$. In particular, $J(B)A \subseteq J(A)$.*

Let C be a two-sided ideal of A.

(b) *If $C \subseteq J(A)$, then $J(A/C) = J(A)/C$.*
(c) *If C is nil, then $C \subseteq J(A)$.*
(d) *If $J(A/C) = \{0\}$, then $J(A) \subseteq C$.*

Proof. For proofs of (a), (b), (c), and (d), see Proposition 2.5.33, Proposition 2.5.6(ii), Remark 2.5.4′, and Proposition 2.5.1′(ii) respectively in [19]. □

Proposition 6. *Let A be a finitely generated algebra over a commutative local ring R with maximal ideal m. If there is an R-subalgebra B of A and a two-sided ideal I of A satisfying*

(i) *$A = B \oplus I$ as left R-modules*;
(ii) *$J(B/mB) = \{0\}$*;
(iii) *every element of I is a sum of elements of* $\operatorname{rad}(mI)$,

then $J(A) = mB \oplus I$.

Proof. By Proposition 5(a) with $B = R$, we get that $mA \subseteq J(A)$. Hence, by (i) and Proposition 5(b) with $C = mA$, we can assume that R is a field. Then, by (iii) and [19, Proposition 2.6.32], I is nilpotent and hence nil. Thus, by Proposition 5(c), $I \subseteq J(A)$. On the other hand, since, by (ii), $J(A/I) \cong J(B) = \{0\}$, we can, by Proposition 5(d), conclude that $J(A) \subseteq I$. $\square$

4. Crossed product algebras

In this section, we prove Theorem 4. We use the same notation as in Section 1. We begin by showing that the center of $A := (L/K, f)$ is L^G. Take $x = \sum_{\alpha \in \mathbf{G}} x_\alpha u_\alpha \in C(A)$, where $x_\alpha \in L_{t(\alpha)}$, for all $\alpha \in \mathbf{G}$. Then $x u_\beta = u_\beta x$ for all $\beta \in \mathbf{G}_0$. This implies that

$$\sum_{\substack{\alpha \in \mathbf{G} \\ s(\alpha) = t(\beta)}} x_\alpha u_\alpha = \sum_{\substack{\alpha \in \mathbf{G} \\ t(\alpha) = s(\beta)}} \beta(x_\alpha) u_\alpha$$

for all $\beta \in \mathbf{G}_0$. From this it follows that $x_\alpha = 0$ for all $\alpha \in \mathbf{G} \setminus \mathbf{G}_0$. Hence $x = \sum_{\alpha \in \mathbf{G}_0} x_\alpha u_\alpha \in A_0$. Take $\alpha, \beta \in \mathbf{G}_0$ and $\gamma \in \mathbf{G}$ such that $s(\gamma) = \alpha$ and $t(\gamma) = \beta$. Then the relation $x u_\gamma = u_\gamma x$ shows that $x_\beta = \gamma(x_\alpha)$ and hence that

$$x = \sum_{i=1}^{n} \alpha_i(x_{\alpha_1}) u_{\mathrm{id}_{L_i}} \tag{18}$$

where $\alpha_i : L_1 \to L_i$, $i = 1, \ldots, n$, are fixed field isomorphisms and $\alpha_1 = \mathrm{id}_{L_1}$. It also shows that $x_{\alpha_1} \in L^G$. Therefore, x can be identified with an element in L^G. On the other hand, it is easy to check that an element in A of the form (18) with $x_{\alpha_1} \in L^G$ belongs to $C(A)$.

Next we show that $\mathbf{H}$ is a subgroupoid of $\mathbf{G}$. Take $\alpha \in \mathbf{H}$. We show that $\alpha^{-1} \in \mathbf{H}$. By (2) we get that

$$f_{\alpha, \alpha^{-1}\alpha} \alpha(f_{\alpha^{-1}, \alpha}) = f_{\alpha^{-1}\alpha, \alpha} f_{\alpha, \alpha^{-1}}.$$

Hence, by (3), we get that $\alpha^{-1} \in \mathbf{H}$. Now take $\alpha, \beta \in \mathbf{H}$ with $s(\alpha) = t(\beta)$. We show that $\alpha\beta \in \mathbf{H}$. By (2) we get that

$$f_{\alpha, \alpha^{-1}\alpha\beta} \alpha(f_{\alpha^{-1}, \alpha\beta}) = f_{\alpha\alpha^{-1}, \alpha\beta} f_{\alpha, \alpha^{-1}} \quad \text{and}$$

$$f_{\beta^{-1}, \beta\beta^{-1}\alpha^{-1}} \alpha(f_{\beta, \beta^{-1}\alpha^{-1}}) = f_{\beta^{-1}\beta, \beta^{-1}\alpha^{-1}} f_{\beta^{-1}, \beta}.$$

Therefore, by (3) and the fact that $\beta^{-1} \in \mathbf{H}$, we get that $f_{\alpha,\beta}$ and $f_{\beta, \beta^{-1}\alpha^{-1}}$ are nonzero. Hence, by (2) again, and the fact that $\alpha \in \mathbf{H}$, we get that $f_{\alpha, \alpha^{-1}} \alpha(f_{\beta, \beta^{-1}\alpha^{-1}}) = f_{\alpha\beta, \beta^{-1}\alpha^{-1}} f_{\alpha,\beta}$, which, by (3), implies that $\alpha\beta \in \mathbf{H}$.

Now put $B = \bigoplus_{\alpha \in \mathbf{H}} L_{t(\alpha)} u_\alpha$ and $I = \bigoplus_{\alpha \in \mathbf{G} \setminus \mathbf{H}} L_{t(\alpha)} u_\alpha$. To show that $C(B) = L^{\mathbf{H} \cap G}$ one can proceed exactly as above, so we leave out the details for this.

Next we show that B is simple. Since, the center of B is simple, we are, by Lemma 1(a)(b), done if we can show that B is separable over B_0. By Proposition 4, we need to show that there is an element of $C(B_0) = B_0$ with trace 1. But since the extensions L_i/K,

$i = 1, \ldots, n$, are separable, the usual trace maps $\mathrm{tr}_i : L_i \to K$ are surjective. Hence, by a straightforward calculation, we get that $\mathrm{tr}(B_0) = \sum_{i=1}^{n} \mathrm{tr}_i(L_i)u_{\mathrm{id}_{L_i}} = \sum_{i=1}^{n} K u_{\mathrm{id}_{L_i}}$ which obviously contains $\sum_{k=1}^{n} u_{\mathrm{id}_{L_i}} = 1$.

Finally, we show that $J(A) = I$. By Proposition 6 we are done if we can show that every element of I is a sum of nilpotent elements. Hence, by (1), it is enough to show that u_α is nilpotent for all $\alpha \in \mathbf{G} \setminus \mathbf{H}$. Case 1: $s(\alpha) \neq t(\alpha)$. Then $u_\alpha^2 = 0$. Case 2: $s(\alpha) = t(\alpha)$. Since the set of $\beta \in \mathbf{G}$ with the property that $s(\beta) = t(\beta) = s(\alpha)$ form a finite group, we can choose a positive integer n such that $\alpha^n = \alpha^{-1}$. Then, since $f_{\alpha,\alpha^n} = f_{\alpha,\alpha^{-1}} = 0$, we get that $u_\alpha^{n+1} = (\prod_{i=0}^{n-1} \alpha^i(f_{\alpha,\alpha^{n-i}}))u_{\alpha^{n+1}} = 0$. We have now completed the proof of Theorem 4.

5. Crossed product orders

In this section, we prove Theorems 5 and 6. We begin by recalling some definitions and a well-known result (see Proposition 7) concerning orders and algebras. We use the same notation as in Section 1.

Recall that an R-order Λ in a finite dimensional K-algebra A is a subring of A such that Λ is a full R-lattice in A (that is, a finitely generated R-submodule of A such that $K\Lambda = A$) containing R as a subring. An R-order Λ is called maximal if it is not contained in a strictly larger R-order of A and it is called left (right) hereditary if every left (right) ideal of Λ is projective. It can be shown that an order is left hereditary if and only if it is right hereditary (see, e.g., [17]). Hence, in our discussion of hereditary orders, we may omit the adjectives "left" and "right." Let I be an ideal of Λ. The left (or right) order of I is defined to be the set of all $a \in A$ with the property that $aI \subseteq I$ (or $Ia \subseteq I$).

Let $\widehat{\Lambda}$ denote the $\widehat{R}$-order $\widehat{R} \otimes_R \Lambda$, where $\widehat{R}$ denotes the m-adic completion of R. Furthermore, let $\overline{\Lambda}$ denote the $\overline{R}$-algebra $\overline{R} \otimes_R \Lambda$, where $\overline{R} = R/m$.

Proposition 7. *Let Λ be an R-order in a central simple K-algebra.*

(a) *The order Λ is Azumaya (maximal, hereditary) if and only if $\widehat{\Lambda}$ is Azumaya (maximal, hereditary).*
(b) *The order Λ is Azumaya if $\overline{\Lambda}$ is Azumaya as an $\overline{R}$-algebra.*

Let R be a complete discrete valuation ring.

(c) *The order Λ is maximal if and only if it is hereditary and $J(\Lambda)$ is a maximal two-sided ideal of Λ.*
(d) *The order Λ is hereditary if and only if the left (or right) order of $J(\Lambda)$ equals Λ.*

Proof. (a) The part about Azumaya orders is proved in [8] and the part about maximal and hereditary orders can be found in [17]; (b) is proved in [8]; (c) and (d) can be found in [17]. $\square$

Now we prove Theorems 5 and 6. By Proposition 7(a), we can assume that R and S_i, $i = 1, \dots, n$, are complete discrete valuation rings with maximal ideals πR and πS_i, $i = 1, \dots, n$, respectively. Put $A = (L/K, f)$ and $\Lambda = (S/R, f)$.

To prove Theorem 5 we have (at least) two possibilities. The first possibility is to observe that Theorem 4 holds for the $\overline{R}$-algebra $\overline{\Lambda}$. Theorem 5 now follows from Proposition 7(b). The second possibility is to construct a direct proof, analogous to the proof of Theorem 4, with the use of Proposition 6.

Now we prove Theorem 6. The "Azumaya part" follows from Theorem 5. Next, by Theorems 4 and 5, we get that $\Lambda/J(\Lambda)$ is simple. Therefore, by Proposition 7(c), Λ is maximal if and only if it is hereditary. All that is left to show now is the "hereditary part" of Theorem 6. Let Λ_l denote the left order of $J(\Lambda)$.

Suppose first that π^2 does not divide any of the $f_{\alpha,\beta}$, $\alpha, \beta \in \mathbf{G}$, $s(\alpha) = t(\beta)$. We claim that $\Lambda_l = \Lambda$. If we assume that the claim holds, then, by Proposition 7(d), Λ is hereditary. Now we show the claim. Since $\Lambda \subseteq \Lambda_l$ always holds, it is enough to show that $\Lambda_l \subseteq \Lambda$. Take $x = \sum_{\alpha \in \mathbf{G}} x_\alpha u_\alpha \in \Lambda_l$, where $x_\alpha \in L_{t(\alpha)}$, $\alpha \in \mathbf{G}$. Since $\pi \in J(\Lambda)$, we get that $\pi x = \sum_{\alpha \in \mathbf{H}} \pi x_\alpha u_\alpha + \sum_{\alpha \in \mathbf{G} \setminus \mathbf{H}} \pi x_\alpha u_\alpha \in J(\Lambda)$. Therefore, by Theorem 5, $x_\alpha \in S_{t(\alpha)}$ for all $\alpha \in \mathbf{H}$. Hence, we can assume that $x = \sum_{\alpha \in \mathbf{G} \setminus \mathbf{H}} x_\alpha u_\alpha$. Take $\beta \in \mathbf{G} \setminus \mathbf{H}$. Then $x u_{\beta^{-1}} \in J(\Lambda)$, which, by Theorem 5 again, implies that $x_\beta f_{\beta,\beta^{-1}} \in \pi S_{t(\beta)}$. Since π^2 does not divide $f_{\beta,\beta^{-1}}$, we get that $x_\beta \in S_{t(\beta)}$.

On the other hand, suppose that π^2 divides some $f_{\alpha,\beta}$, $\alpha, \beta \in \mathbf{G}$, $s(\alpha) = t(\beta)$. We claim that $\Lambda_l \supsetneq \Lambda$. If we assume that the claim holds, then by Proposition 7(d), Λ is not hereditary. Now we show the claim.

Define a partial order $\leqslant$ on $\mathbf{G}$ in the following way. For $\alpha, \beta \in \mathbf{G}$, put $\alpha \leqslant \beta$ if $u_\beta \in u_\alpha \Lambda$. Note that $\alpha \leqslant \beta$ holds if and only if $t(\alpha) = t(\beta)$ and $f_{\alpha,\alpha^{-1}\beta}$ is a unit in $S_{t(\alpha)}$. Choose the largest positive integer N such that there is $\alpha \in \mathbf{G}$ with π^N dividing $f_{\alpha,\alpha^{-1}}$. Since $\leqslant$ is a partial order, we can fix $\alpha \in \mathbf{G}$ with the property that π^N divides $f_{\alpha,\alpha^{-1}}$ and if π^N divides $f_{\beta,\beta^{-1}}$ for some $\beta \in \mathbf{G}$ with $\alpha \leqslant \beta$, then also $\beta \leqslant \alpha$. We will show that $\pi^{-1} u_\alpha \in \Lambda_l$. Since $\pi^{-1} u_\alpha \notin \Lambda$, the claim will follow.

Take $\beta \in \mathbf{G}$ with $s(\alpha) = t(\beta)$. We consider three cases.

Case 1: $\beta \in \mathbf{H}$. Then $\pi^{-1} u_\alpha \pi u_\beta = f_{\alpha,\beta} u_{\alpha\beta} \in J(\Lambda)$, since $\alpha\beta \notin \mathbf{H}$.

Case 2: $\beta \notin \mathbf{H}$ but $\alpha\beta \in \mathbf{H}$. By (2), we get that $f_{\alpha,\alpha^{-1}} \alpha(f_{\beta,\beta^{-1}\alpha^{-1}}) = f_{\alpha\beta,(\alpha\beta)^{-1}} f_{\alpha,\beta}$, which, since $f_{\alpha\beta,(\alpha\beta)^{-1}}$ is a unit, implies that π^N, and therefore π^2, divides $f_{\alpha,\beta}$. Hence $\pi^{-1} u_\alpha u_\beta = \pi^{-1} f_{\alpha,\beta} u_{\alpha\beta} \in J(\Lambda)$.

Case 3: $\beta \notin \mathbf{H}$ and $\alpha\beta \notin \mathbf{H}$. By the relation $\pi^{-1} u_\alpha u_\beta = \pi^{-1} f_{\alpha,\beta} u_{\alpha\beta}$ it follows that we need to show that π divides $f_{\alpha,\beta}$. If π^N does not divide $f_{\alpha\beta,(\alpha\beta)^{-1}}$, then we can proceed exactly as in case 2.

Therefore we assume now that π^N divides $f_{\alpha\beta,(\alpha\beta)^{-1}}$. Seeking a contradiction, suppose that $f_{\alpha,\beta}$ is a unit in $S_{t(\alpha)}$. By the equality $f_{\alpha,\alpha^{-1}\alpha\beta} = f_{\alpha,\beta}$ we get that $\alpha \leqslant \alpha\beta$. But by the choice of α this implies that $\alpha\beta \leqslant \alpha$, that is, that $f_{\alpha\beta,(\alpha\beta)^{-1}\alpha} = f_{\alpha\beta,\beta^{-1}}$ is a unit also. By (2), we get that $f_{\alpha,\beta} \alpha(f_{\alpha\beta,\beta^{-1}}) = f_{\beta,\beta^{-1}}$. Since the left hand side of this equation is a unit and the right hand side is divisible by π this gives us the desired contradiction. We have now completed the proof of Theorem 6.

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