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# Algebraic geometry methods associated to the one-dimensional Hubbard model

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## Abstract

In this paper we study the covering vertex model of the one-dimensional Hubbard Hamiltonian constructed by Shastry in the realm of algebraic geometry. We show that the Lax operator sits in a genus one curve which is not isomorphic but only isogenous to the curve suitable for the AdS/CFT context. We provide an uniformization of the Lax operator in terms of ratios of theta functions allowing us to establish relativistic like properties such as crossing and unitarity. We show that the respective R-matrix weights lie on an Abelian surface being birational to the product of two elliptic curves with distinct J-invariants. One of the curves is isomorphic to that of the Lax operator but the other is solely fourfold isogenous. These results clarify the reason the R-matrix can not be written using only difference of spectral parameters of the Lax operator.

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## 1. Introduction

The Hubbard model originates from the tight-binding formulation for solids where the electrons can hop between lattice sites but also interact through the Coulomb repulsion. In its simplest form, electron hopping takes place between nearest neighbor sites with the same kinetic energy while the Coulomb interaction occurs only for electrons at the same site with a constant

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strength  $U$ . The Hubbard Hamiltonian on a ring of size  $N$  with interaction symmetric under electron-hole transformation is given by,

$$H = - \sum_{j=1}^N \sum_{\sigma=\uparrow,\downarrow} (c_{j\sigma}^\dagger c_{j+1\sigma} + c_{j+1\sigma}^\dagger c_{j\sigma}) + U \sum_{j=1}^N (c_{j\uparrow}^\dagger c_{j\uparrow} - \frac{1}{2})(c_{j\downarrow}^\dagger c_{j\downarrow} - \frac{1}{2}), \tag{1}$$

where  $c_{j\sigma}^\dagger$  and  $c_{j\sigma}$  stand for creation and annihilation operators for an electron at site  $j$  with spin  $\sigma$ .

In a groundbreaking work Lieb and Wu showed that Hamiltonian (1) is exactly diagonalized by means of an extension of the coordinate Bethe ansatz method besides the model absence of Mott transition [1]. Over the years this solution has been used to compute many other physical properties and for a recent extensive review we refer to the monograph [2]. Exact integrability from the viewpoint of the quantum inverse scattering approach was only established many years later by Shastry in three influential papers [3–5]. An important result was the discovery of a classical two-dimensional vertex model on the square  $N \times N$  lattice whose row-to-row transfer matrix commutes with the spin version of the Hubbard Hamiltonian. This spin model was obtained by applying a generalized version of the Jordan–Wigner transformation on the bulk term of Eq. (1) which can be rewritten as [3],

$$H = \sum_{j=1}^N \sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \tau_j^+ \tau_{j+1}^- + \tau_j^- \tau_{j+1}^+ + \frac{U}{4} \sigma_j^z \tau_j^z, \tag{2}$$

where  $\sigma_j^\pm, \sigma_j^z$  and  $\tau_j^\pm, \tau_j^z$  are two commuting sets of Pauli matrices acting on the site  $j$ . Recall that strict periodic boundary conditions for electron Hamiltonian (1) lead to sector dependent twisted boundary conditions for the spin operator (2) and the precise form of this relationship can for instance be found in [6]. However, this difference on boundaries can be easily captured by introducing fermionic statistics into the integrable structures without affecting the main features of Shastry’s construction [7].

The appealing form of the spin Hamiltonian (2) led Shastry to propose that the underlying classical vertex model should be given by coupling appropriately two six-vertex models obeying the so-called free-fermion condition. Let us denote by  $L_{0j}(\omega)$  the Lax operator encoding the Boltzmann weights structure of such coupled six-vertex models. As usual the indices 0 and  $j$  refer to operators acting on the auxiliary and quantum spaces associated respectively with the degrees of freedom sited on the horizontal and vertical edges of the square lattice. In terms of Pauli matrices such Lax operator can be expressed by,

$$L_{0j}(\omega) = \exp \left[ \frac{\hbar}{2} (\sigma_0^z \tau_0^z + I_0) \right] I_j \left[ \mathcal{L}_{0j}^{(\sigma)}(a, b, c) \mathcal{L}_{0j}^{(\tau)}(a, b, c) \right] \exp \left[ \frac{\hbar}{2} (\sigma_0^z \tau_0^z + I_0) \right] I_j, \tag{3}$$

where  $I$  is the four-dimensional identity matrix and the symbol  $\omega$  denotes the set of parameters  $a, b, c$  and  $\hbar$ .

The Lax operators  $\mathcal{L}_{0j}^{(\sigma)}(a, b, c)$  and  $\mathcal{L}_{0j}^{(\tau)}(a, b, c)$  represent the weights of two copies of six-vertex models whose expressions are,

$$\mathcal{L}_{0j}^\sigma(a, b, c) = \frac{(a+b)}{2} I_0 I_j + \frac{(a-b)}{2} \sigma_0^z \sigma_j^z + c(\sigma_0^+ \sigma_j^- + \sigma_0^- \sigma_j^+), \tag{4}$$

and

$$\mathcal{L}_{0j}^\tau(a, b, c) = \frac{(a+b)}{2} I_0 I_j + \frac{(a-b)}{2} \tau_0^z \tau_j^z + c(\tau_0^+ \tau_j^- + \tau_0^- \tau_j^+), \tag{5}$$

such that the so-called free-fermion condition is fulfilled,

$$a^2 + b^2 = c^2. \tag{6}$$

In order to assure integrability the six-vertex free-fermion weights  $a, b, c$  and the dimensionless interaction  $h$  must be constrained by,

$$\sinh(2h) = \frac{Uab}{2c^2}. \tag{7}$$

In addition to that, Shastry considered the local condition that is sufficient for the commutativity of two transfer matrices built out of Lax operators with distinct weights parameters. In fact, the explicit form of the R-matrix  $R(\omega_1, \omega_2)$  operator satisfying the Yang–Baxter relation,

$$R_{12}(\omega_1, \omega_2)L_{13}(\omega_1)L_{23}(\omega_2) = L_{23}(\omega_2)L_{13}(\omega_1)R_{12}(\omega_1, \omega_2), \tag{8}$$

has been determined in references [4,5].

In recent years new insights into the Hubbard model emerged from the investigation by Beisert of integrable structures associated to the fundamental representation of centrally extended  $su(2|2)$  superalgebra [8]. This representation depends on the central elements values which have been parametrized in terms of two variables  $x_+$  and  $x_-$  constrained by the genus one curve [8],

$$E_1 \equiv x_+ + \frac{1}{x_+} - x_- - \frac{1}{x_-} - \imath U = 0. \tag{9}$$

Afterwards it has been pointed out that the intertwining operator based on such representation of the  $su(2|2)$  superalgebra can be related to the original Shastry R-matrix [9]. This equivalence was further elaborated in [10] for a factorizable S-matrix derived in the context of the  $su(2|2)$  Zamolodchikov–Faddeev algebra [11]. Such relationship occurs up to gauge transformation and when the R-matrices parameters are identified as [10],

$$x_+ = \frac{\imath a \exp(2h)}{b}, \quad x_- = \frac{-\imath b \exp(2h)}{a}. \tag{10}$$

At this point we recall that this mapping goes back at least to the parameterization used in [5,6] for the eigenvalues of the transfer matrix based on the Lax operator (3)–(7). We also note that the expression for  $E_1$  is exactly Eq. (31) of Ref. [6] taken into account identification (10).

Although the above connection suggests that the Lax operator (3)–(7) could be sited on an elliptic curve it does not mean that such underlying spectral curve is necessarily isomorphic to  $E_1$ . In this paper we shall show that the right hand side of Eq. (10) involves quadratic powers on the polynomial ring variables in which the Lax operator (3)–(7) is properly defined. This fact excludes isomorphism but leaves the possibility that the Hubbard model spectral curve  $E_2$  be isogenous to  $E_1$ . Recall that a  $n$ -fold isogeny among elliptic curves  $E_2$  and  $E_1$  is a surjective morphism that maps the distinguished point of  $E_2$  (place at “infinity”) to the distinguished point of  $E_1$  [12]. The integer  $n$  is the degree of the morphism and thus a generic point of  $E_1$  is mapped to  $n$  distinct points of  $E_2$ . In fact, it turns out that the spectral curve underlying the Shastry Lax operator is given by the following affine quartic elliptic curve,

$$E_2 \equiv (x^2 + y^2)^2 - Uxy - 1 = 0, \tag{11}$$

where the suitable ring variables  $x$  and  $y$  are related to the weights used by Shastry as,

$$x = a \exp(h), \quad y = b \exp(h). \tag{12}$$

In next section we discuss the derivation of the curve  $E_2$  from the original construction by Shastry of the covering Hubbard model. We also show that the curves  $E_2$  and  $E_1$  are not isomorphic but only have a fourth degree isogeny. In section 3 we argue that the uniformization of  $E_2$  can be performed along the lines of the symmetrical eight vertex model with weights satisfying the free-fermion condition [13]. The matrix elements of the Lax operator are then represented in terms of factorized ratios of theta functions. This allows us to present local inversion properties for the Lax operator such as crossing and unitarity relations. In section 4 we discuss the geometrical properties associated with the R-matrix of the Hubbard model. We show that the R-matrix weights lies on an Abelian surface built out of the product of two non-isomorphic elliptic curves. Our concluding remarks are in section 5 and in two appendices we present technical details of some computations omitted in the main text.

### 2. Lax operator spectral curve

The problem of finding integrable systems leads us to solve a set of polynomial relations on the product of three projective spaces originated from the Yang–Baxter equation. This means that all the matrix elements of a given Lax operator are expected to be determined by homogeneous polynomials in suitable ring variables up to an overall normalization. Inspecting the entries of the Lax operators (3)–(7) one concludes that the respective polynomial ring is  $\mathbb{C}[x, y, c]$  where the variables  $x$  and  $y$  have already been defined in Eq. (12). Upon this identification the explicit matrix form of the Lax operator is,

$$L_{12}(x, y, c) = \begin{pmatrix} x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & xy & 0 & 0 & xc & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & 0 & 0 & xc & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^2 & 0 & 0 & yc & 0 & 0 & yc & 0 & 0 & \theta(x, y) & 0 & 0 & 0 \\ 0 & xc & 0 & 0 & \frac{xy c^2}{\theta(x, y)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{x^2 c^2}{\theta(x, y)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & yc & 0 & 0 & \frac{y^2 c^2}{\theta(x, y)} & 0 & 0 & c^2 & 0 & 0 & yc & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{xy c^2}{\theta(x, y)} & 0 & 0 & 0 & 0 & 0 & 0 & xc & 0 \\ 0 & 0 & xc & 0 & 0 & 0 & 0 & 0 & \frac{xy c^2}{\theta(x, y)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & yc & 0 & 0 & c^2 & 0 & 0 & \frac{y^2 c^2}{\theta(x, y)} & 0 & 0 & yc & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{x^2 c^2}{\theta(x, y)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{xy c^2}{\theta(x, y)} & 0 & 0 & xc & 0 \\ 0 & 0 & 0 & \theta(x, y) & 0 & 0 & yc & 0 & 0 & yc & 0 & 0 & y^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & xc & 0 & 0 & 0 & 0 & 0 & xy & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & xc & 0 & 0 & xy \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 \end{pmatrix}, \tag{13}$$

where  $\theta(x, y) = x^2 + y^2$ .

The next step is the determination of the spectral curve which should constrain the variables  $x, y$  and  $c$ . This task can be done by eliminating the unwanted variables  $a, b$  and  $\exp(2h)$  with the help Eqs. (6), (12),

$$a = \frac{x}{\exp(h)}, \quad b = \frac{y}{\exp(h)}, \quad \exp(2h) = \frac{x^2 + y^2}{c^2}. \tag{14}$$

By substituting the above results in Eq. (7) we find that the desired spectral curve is,

$$\bar{E}_2 \equiv (x^2 + y^2)^2 - Uxyc^2 - c^4 = 0, \tag{15}$$

which is just the projective closure of the affine curve (11).

Let us now show that the curve  $\bar{E}_2$  is connected with the projective closure of  $E_1$  by means of a fourfold isogeny. We first note that from Eq. (9) the expression for  $\bar{E}_1$  is given by,

$$\bar{E}_1 = (x_+ - x_-)(x_+x_- - z^2) - \iota Ux_+x_-, \tag{16}$$

where the variable  $z$  refers to the extra projective coordinate.

By using Eqs. (6), (10), (12) we can establish the following morphism between the elliptic curves  $\bar{E}_2$  and  $\bar{E}_1$ ,

$$\begin{aligned} \bar{E}_2 \subset \mathbb{C}\mathbb{P}^2[x, y, c] &\xrightarrow{\psi} \bar{E}_1 \subset \mathbb{C}\mathbb{P}^2[x_+, x_-, z] \\ (x : y : c) &\longmapsto (\psi_1 : \psi_2 : \psi_3), \end{aligned} \tag{17}$$

where the polynomials map expressions are,

$$\psi_1 = \iota x^2(x^2 + y^2), \quad \psi_2 = -\iota y^2(x^2 + y^2), \quad \psi_3 = xyc^2. \tag{18}$$

Note that the above map is defined everywhere even at the singular points  $(1, \pm \iota, 0) \in \bar{E}_2$ . In fact, at these particular points one can find an alternative representation of  $\psi$  with the help of the polynomial (15), namely

$$(\psi_1 : \psi_2 : \psi_3) \sim (x^2 : -y^2 : -\frac{\iota xyc^2}{x^2 + y^2}) \sim (x^2 : -y^2 : -\frac{\iota xy(x^2 + y^2)}{c^2 + Uxy}), \tag{19}$$

and as result we obtain  $\psi(0 : \pm \iota : 0) = (1 : 1 : 0) \in \bar{E}_1$ .

The degree of the morphism (17) can be determined as the cardinality of the fiber  $\psi^{-1}(P)$  for a generic point  $P \in \bar{E}_1$ . Considering that the variables  $x, y, c$  are constrained by the curve  $\bar{E}_2$  one finds that such degree is indeed four.

An alternative way to see that the two elliptic curves are not isomorphic is through the comparison of their J-invariants. It is well known that such invariant classifies genus one curves up to isomorphism [12]. This invariant can be computed by birationally transforming a genus one curve into its Weierstrass form, namely

$$C = y_0^2 - x_0^3 - \mathbb{A}x_0 - \mathbb{B}, \tag{20}$$

with  $\mathbb{A}$  and  $\mathbb{B}$  in the complex field.

Note that if we replace  $x_0$  by  $\lambda^2 x_0$  and  $y_0$  by  $\lambda^3 y_0$  we still retain the main Weierstrass form of the curve. The only amount of ambiguity is that the coefficients  $\mathbb{A}$  and  $\mathbb{B}$  are replaced by  $\lambda^{-4}\mathbb{A}$  and  $\lambda^{-6}\mathbb{B}$  respectively. We see that under such scale of coordinates there is just one invariant which is clearly the quantity  $\mathbb{A}^3/\mathbb{B}^2$ . The J-invariant is defined as a linear fractional image of this ratio,

$$J(C) = 1728 \frac{4\mathbb{A}^3}{4\mathbb{A}^3 + 27\mathbb{B}^2}, \tag{21}$$

where the numerical prefactor is chosen for sake of compatibility with situations in which the field characteristic is non-zero [12].

The curves  $E_1$  and  $E_2$  are easily normalized to the Weierstrass form and the final results for their J-invariants are,

$$J(E_1) = \frac{(U^4 + 16U^2 + 16)^3}{U^2(U^2 + 16)} \quad \text{and} \quad J(E_2) = -\frac{(U^2 + 16U + 16)^3(U^2 - 16U + 16)^3}{U^2(U^2 + 16)^4}, \quad (22)$$

which are clearly different for generic values of  $U$  and consequently the curves  $E_1$  and  $E_2$  are not isomorphic. We also note that the denominators of the J-invariants vanish at the non-trivial values of the coupling  $U = \pm 4\iota$  in which the curves  $E_1$  and  $E_2$  become rational.

Moreover, given two elliptic curves  $C_1$  and  $C_2$  and an integer  $n$ , there is a direct way to decide if they are  $n$ -isogenous. We just have to verify that the so-called modular polynomial  $\Phi_n [J(C_1), J(C_2)]$  is zero. In our specific situation the expression of the four-level modular polynomial is [17],

$$\begin{aligned} \Phi_4[x, y] = & x^6 + y^6 - (x^5y^4 + x^4y^5) + 2976(x^5y^3 + x^3y^5) - 2533680(x^5y^2 + x^2y^5) \\ & + 561444609(x^5y + xy^5) - 8507430000(x^5 + y^5) + 7440(x^4y^4) \\ & + 80967606480(x^4y^3 + x^3y^4) + 1425220456750080(x^4y^2 + x^2y^4) \\ & + 1194227244109980000(x^4y + xy^4) + 24125474716854750000(x^4 + y^4) \\ & + 2729942049541120(x^3y^3) - 914362550706103200000(x^3y^2 + x^2y^3) \\ & + 12519806366846423598750000(x^3y + xy^3) \\ & - 22805180351548032195000000000(x^3 + y^3) \\ & + 26402314839969410496000000(x^2y^2) \\ & + 188656639464998455284287109375(x^2y + xy^2) \\ & + 158010236947953767724187500000000(x^2 + y^2) \\ & - 94266583063223403127324218750000(xy) \\ & - 36493632779675765840437500000000000(x + y) \\ & + 28094937472219537210964062500000000000. \end{aligned} \quad (23)$$

We have checked that the non-trivial identity  $\Phi_4 [J(E_1), J(E_2)] = 0$  is indeed satisfied for arbitrary values of the coupling  $U$ . This confirms the fourfold isogeny among the elliptic curves  $E_1$  and  $E_2$ .

### 3. Uniformization and local relations

We start showing that the uniformization of the curve  $\bar{E}_2$  can be implemented along the lines of the eight-vertex model satisfying the free-fermion condition [13]. To this end we first write this elliptic curve as the intersection of two quadric surfaces in the three-dimensional space. Denoting by  $w$  such extra coordinate,  $\bar{E}_2$  can be represented by the following pairs of equations,

$$x^2 + y^2 - cw = 0, \quad c^2 - w^2 + Uxy = 0, \quad (24)$$

and after performing the rotation  $c = w_1 - \iota w_2$  and  $w = w_1 + \iota w_2$  we obtain,

$$x^2 + y^2 - w_1^2 - w_2^2 = 0, \quad w_1w_2 = \frac{U}{4\iota}xy. \quad (25)$$

Inspecting Eq. (25) we recognize the well known spectral curve of the symmetric eight vertex model with weights  $x, y, w_1$  and  $w_2$  satisfying the free-fermion restriction. At this point we can follow Baxter monograph [13] and the uniformization of the weights relevant for the Hubbard model are,

$$\frac{x(\lambda)}{c(\lambda)} = \frac{\operatorname{sn}[\mathbf{K}(k) - \lambda, k]}{1 - \iota k \operatorname{sn}[\lambda, k] \operatorname{sn}[\mathbf{K}(k) - \lambda, k]}, \quad \frac{y(\lambda)}{c(\lambda)} = \frac{\operatorname{sn}[\lambda, k]}{1 - \iota k \operatorname{sn}[\lambda, k] \operatorname{sn}[\mathbf{K}(k) - \lambda, k]}, \quad (26)$$

where  $\lambda$  is the spectral parameter,  $\mathbf{K}(k)$  denotes the complete elliptic integral of the first kind of modulus  $k$  and  $\operatorname{sn}[\lambda, k]$  represents the Jacobi elliptic function. The dependence of the modulus on the coupling is,

$$k = \frac{U}{4\iota}. \quad (27)$$

Note that this uniformization when  $U \rightarrow 0$  recovers in a direct way the expected trigonometric parameterization of the weights.<sup>1</sup> This representation however involves sums in the denominator and it is not optimal for the study of analytical properties. The uniformization can alternatively be given in terms of ratios of entire functions of the spectral parameter. The first task is to locate the positions and multiplicities of the zeros and poles of the given elliptic function in the region defined by the respective pair of primitive periods. Then we can write the elliptic function as ratios of products of theta functions located at such zeros and poles within a constant factor. The multiplicative constant can be determined by the exact knowledge of the function at some suitable values of the spectral parameter. Considering this procedure we find the following factorized representation,

$$\frac{x(\lambda)}{c(\lambda)} = \iota(4k\sqrt{q})^{-1/4} \frac{H[\mathbf{K}(k) - \lambda, k] \Theta[\lambda, k]}{H[\lambda + \iota\mathbf{K}(k')/2, k] H[\mathbf{K}(k) + \iota\mathbf{K}(k')/2 - \lambda, k]} \quad (28)$$

$$\frac{y(\lambda)}{c(\lambda)} = \iota(4k\sqrt{q})^{-1/4} \frac{\Theta[\mathbf{K}(k) - \lambda, k] H[\lambda, k]}{H[\lambda + \iota\mathbf{K}(k')/2, k] H[\mathbf{K}(k) + \iota\mathbf{K}(k')/2 - \lambda, k]} \quad (29)$$

where the complementary modulus  $k'$  satisfies the usual relation  $k'^2 + k^2 = 1$  and the nome  $q = \exp[-\pi \mathbf{K}(k')/\mathbf{K}(k)]$ . For sake of completeness the explicit expressions of the theta functions are,

$$H[\lambda, k] = 2q^{1/4} \sin \left[ \frac{\pi\lambda}{2\mathbf{K}(k)} \right] \prod_{j=1}^{\infty} \left( 1 - 2q^{2j} \cos \left[ \frac{\pi\lambda}{\mathbf{K}(k)} \right] + q^{4j} \right) (1 - q^{2j}), \quad (30)$$

$$\Theta[\lambda, k] = \prod_{j=1}^{\infty} \left( 1 - 2q^{(2j-1)} \cos \left[ \frac{\pi\lambda}{\mathbf{K}(k)} \right] + q^{(4j-2)} \right) (1 - q^{2j}). \quad (31)$$

In order to express the Lax operator (13) solely in terms of ratios of entire functions we still need the representation of the polynomial combination  $\theta(x, y)$ . After some simplifications it can be given as,

$$\begin{aligned} \frac{\theta(\lambda)}{c^2(\lambda)} &= \iota \frac{\Theta[\mathbf{K}(k) + \iota\mathbf{K}(k')/2 - \lambda, k] \Theta[\lambda + \iota\mathbf{K}(k')/2, k]}{H[\mathbf{K}(k) + \iota\mathbf{K}(k')/2 - \lambda, k] H[\lambda + \iota\mathbf{K}(k')/2, k]} \\ &= \iota \frac{\Theta[\mathbf{K}(k) + \iota\mathbf{K}(k')/2 - \lambda, k] H[\lambda - \iota\mathbf{K}(k')/2, k]}{H[\mathbf{K}(k) + \iota\mathbf{K}(k')/2 - \lambda, k] \Theta[\lambda - \iota\mathbf{K}(k')/2, k]} \end{aligned} \quad (32)$$

<sup>1</sup> The regular point is at  $\lambda = 0$  in which the Lax operator (13) becomes the four-dimensional permutator.

We have now gathered the basic ingredients to discuss local properties satisfied by the Lax operator. One of them is related with the invariance of the respective partition function by  $\pi/2$  rotation of the lattice. Inspecting the structure of the operator (13) we conclude that this symmetry is directly related with the variables exchange  $x \leftrightarrow y$  which preserves the form of the spectral curve (15). Considering the above uniformization we see that this exchange is accomplished by shifting the spectral parameter by the elliptic integral  $\mathbf{K}(k)$  value. Denoting by  $L_{12}(\lambda)$  the ratio  $L_{12}(x, y, c)/c^2$  we found the following crossing relation,

$$L_{12}(\lambda) = M_1 L_{12}(\mathbf{K}(k) - \lambda)^{t_2} M_1^{-1}, \quad (33)$$

where  $t_2$  denotes transposition on the second space and the charge conjugation matrix  $M$  is,

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (34)$$

The crossing property has the immediate consequence of providing us a global symmetry constrain for the free energy of the classical vertex model at finite volume. Let  $Z_N(\lambda)$  be the partition function of the vertex model with weights  $L_{12}(\lambda)$  on the square lattice of size  $N$ . Then it follows from Eq. (33) that,

$$Z_N(\lambda) = Z_N(\mathbf{K}(k) - \lambda) \quad (35)$$

The next local property is the so-called unitarity relation which for relativistic scattering theory connects Lax operators with spectral parameters  $\lambda$  and  $-\lambda$ . Here we have attempted similar relation by studying the local properties of the Lax operator around the regular point  $\lambda = 0$ . The result of this analysis is the following expression,

$$L_{12}(\lambda)L_{12}(-\lambda) = \left[ \frac{x(\lambda)}{c(\lambda)} \right]^2 \left[ \frac{x(-\lambda)}{c(-\lambda)} \right]^2 I_1 \otimes I_2 \quad (36)$$

Note that the above relation is almost what we usually have for relativistic systems. The only difference is that the Lax operator evaluated at  $-\lambda$  is not permuted on its spaces. Here the Lax operator is not parity reversal invariant for generic values of  $U$  and as a consequence of that  $L_{12}(\lambda) \neq L_{21}(\lambda)$ . From previous experience with other solvable models it is conceivable that combination of crossing and unitarity could lead us to functional relations for the transfer matrix eigenvalues in the limit of infinite system [14,15]. This method provides an alternative way to derive relevant physical properties such as the free-energy and the dispersion relation of the low-lying excitations [16]. We hope that our theta function uniformization of the weights will shed some light on the appropriate analyticity assumptions that still has to be made for the applicability of such approach.

#### 4. R-matrix geometric properties

We start this section by presenting the explicit expression of Shastry's R-matrix in terms of the suitable ring variables describing the Lax operator. This matrix can be written as,



$$R(\lambda_1, \lambda_2) = \begin{pmatrix} \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{b} & 0 & 0 & \mathbf{c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{b} & 0 & 0 & 0 & 0 & 0 & \mathbf{c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{h} - \mathbf{a} & 0 & 0 & \mathbf{d} & 0 & 0 & \mathbf{d} & 0 & 0 & \mathbf{h} & 0 & 0 \\ 0 & \mathbf{c} & 0 & 0 & \bar{\mathbf{b}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{g} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{d} & 0 & 0 & \mathbf{q} - \mathbf{g} & 0 & 0 & \mathbf{q} & 0 & 0 & \mathbf{d} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{b} & 0 & 0 & 0 & 0 & 0 & \mathbf{c} & 0 \\ 0 & 0 & \mathbf{c} & 0 & 0 & 0 & 0 & 0 & \bar{\mathbf{b}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{d} & 0 & 0 & \mathbf{q} & 0 & 0 & \mathbf{q} - \mathbf{g} & 0 & 0 & \mathbf{d} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{g} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mathbf{b}} & 0 & 0 & \mathbf{c} \\ 0 & 0 & 0 & \mathbf{h} & 0 & 0 & \mathbf{d} & 0 & 0 & \mathbf{d} & 0 & 0 & \mathbf{h} - \mathbf{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{c} & 0 & 0 & 0 & 0 & 0 & \mathbf{b} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{c} & 0 & 0 & \mathbf{b} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{a} \end{pmatrix}. \tag{37}$$

The expressions for the R-matrix elements are obtained after performing some simplifications on the original weights determined previously by Shastry [4,5]. Considering the entry  $\mathbf{c}$  as an overall normalization we found,

$$\begin{aligned} \frac{\mathbf{a}}{\mathbf{c}} &= \frac{y_1 y_2}{\theta(x_1, y_1)} + \frac{x_1 x_2}{\theta(x_2, y_2)}, & \frac{\mathbf{b}}{\mathbf{c}} &= -\frac{x_1 y_2}{\theta(x_1, y_1)} + \frac{y_1 x_2}{\theta(x_2, y_2)}, & \frac{\bar{\mathbf{b}}}{\mathbf{c}} &= \frac{y_1 x_2}{\theta(x_1, y_1)} - \frac{x_1 y_2}{\theta(x_2, y_2)}, \\ \frac{\mathbf{d}}{\mathbf{c}} &= \frac{x_1 y_1 - x_2 y_2}{x_1^2 x_2^2 - y_1^2 y_2^2}, & \frac{\mathbf{h}}{\mathbf{c}} &= \frac{x_1 x_2 \theta(x_1, y_1) - y_1 y_2 \theta(x_2, y_2)}{x_1^2 x_2^2 - y_1^2 y_2^2}, \\ \frac{\mathbf{q}}{\mathbf{c}} &= \frac{x_1 x_2 \theta(x_2, y_2) - y_1 y_2 \theta(x_1, y_1)}{x_1^2 x_2^2 - y_1^2 y_2^2}, & \frac{\mathbf{g}}{\mathbf{c}} &= \frac{x_1 x_2}{\theta(x_1, y_1)} + \frac{y_1 y_2}{\theta(x_2, y_2)}, \end{aligned} \tag{38}$$

where the bold variables  $\mathbf{x}_j$  and  $\mathbf{y}_j$  are given as ratios of the spectral curve coordinates,

$$\mathbf{x}_j = \frac{x(\lambda_j)}{c(\lambda_j)}, \quad \mathbf{y}_j = \frac{y(\lambda_j)}{c(\lambda_j)}, \quad \text{for } j = 1, 2. \tag{39}$$

In order to understand the geometric properties associated to the R-matrix we first need to find the implicit representation of the image of the rational map,

$$\begin{aligned} \bar{E}_2 \times \bar{E}_2 \subset \mathbb{C}P^2 \times \mathbb{C}P^2 & \xrightarrow{\phi} & V \subset \mathbb{C}P^7 \\ (x(\lambda_1) : y(\lambda_1) : c(\lambda_1)) \times (x(\lambda_2) : y(\lambda_2) : c(\lambda_2)) & \longmapsto & (\mathbf{a} : \mathbf{b} : \bar{\mathbf{b}} : \mathbf{c} : \mathbf{d} : \mathbf{g} : \mathbf{h} : \mathbf{q}), \end{aligned} \tag{40}$$

where  $V$  is the algebraic variety associated to the R-matrix.

The solution of the above problem will lead us to polynomials on the R-matrix entries  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\bar{\mathbf{b}}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$ ,  $\mathbf{g}$ ,  $\mathbf{h}$  and  $\mathbf{q}$  which are the defining equations of  $V$ . This task is performed by eliminating the variables  $\mathbf{x}_j$  and  $\mathbf{y}_j$  from Eqs. (38) considering also that they are constrained by the spectral curve (11). The technical details concerning this computation are summarized in Appendix A and in what follows we only present the main results. It turns out that the variety  $V$  is formally described as the intersection of five quadrics,

$$V = \{(\mathbf{a} : \mathbf{b} : \bar{\mathbf{b}} : \mathbf{c} : \mathbf{d} : \mathbf{g} : \mathbf{h} : \mathbf{q}) \in \mathbb{C}\mathbb{P}^7 \mid Q_1 = Q_2 = Q_3 = Q_4 = Q_5 = 0\}. \tag{41}$$

The expressions of the degree two homogeneous polynomials  $Q_j$  are,

$$\begin{aligned} Q_1 &= -\mathbf{c}^2 + \mathbf{ag} + \mathbf{b}\bar{\mathbf{b}}, & Q_2 &= -\mathbf{d}^2 + \mathbf{ag} - \mathbf{gh} - \mathbf{aq} + \mathbf{hq} + \mathbf{b}\bar{\mathbf{b}}, & Q_3 &= -\mathbf{c}^2 - \mathbf{d}^2 + \mathbf{hq}, \\ Q_4 &= -\mathbf{a}^2 - \mathbf{b}^2 - \mathbf{g}^2 + \mathbf{ah} + \mathbf{gq} - \bar{\mathbf{b}}^2, & Q_5 &= U\mathbf{cd} - \mathbf{h}^2 + \mathbf{q}^2 \end{aligned} \tag{42}$$

where we recall that the above first three quadrics have been pointed before as identities among the R-matrix weights in [7]. However, to the best of our knowledge the last two are new in the literature specially  $Q_5$  since it contains the Hubbard coupling  $U$ .

We have used the computer algebra system Singular [18] to obtain some basic information on the geometric properties of the variety  $V$ . This algebraic set turns out to be an irreducible complete intersection and therefore we are dealing with a complex two-dimensional variety. This distinguishes the Hubbard and the eight-vertex models even though both have Lax operator based on elliptic curves. In fact, for the eight-vertex model the variety  $V$  is one-dimensional and the R-matrix lies on the same curve of the Lax operator [13] and the map (40) reflects the standard group law of elliptic curves. By way of contrast, the Hubbard model sits on the lower bound of the fiber dimension theorem<sup>2</sup> in which  $\phi^{-1}$  is a zero dimensional variety.

Further progress is made by noticing that the quadrics  $Q_3$  and  $Q_5$  define a nonsingular elliptic curve in  $\mathbb{C}\mathbb{P}^3[\mathbf{c}, \mathbf{d}, \mathbf{h}, \mathbf{q}]$  which is isomorphic to  $\bar{E}_2$  formulated as in Eq. (24). This means that  $V$  is a surface contained in the cone with base  $\mathbb{C}\mathbb{P}^3[\mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}, \mathbf{g}]$  over  $\bar{E}_2$  making it possible to established the following surjective map,

$$\begin{aligned} V \subset \mathbb{C}\mathbb{P}^7 & \xrightarrow{\pi} \bar{E}_2 \subset \mathbb{C}\mathbb{P}^3 \\ (\mathbf{a} : \mathbf{b} : \bar{\mathbf{b}} : \mathbf{c} : \mathbf{d} : \mathbf{g} : \mathbf{h} : \mathbf{q}) & \longmapsto (\mathbf{c} : \mathbf{d} : \mathbf{h} : \mathbf{q}). \end{aligned} \tag{43}$$

The next natural step is to investigate the properties of the fiber of  $\pi$  since this feature lies at the heart of the geometry of algebraic surfaces [20]. This study is somehow cumbersome and the main technical points of the computations have been summarized in Appendix B. The central result of this analysis is that the general fiber  $\pi^{-1}$  turns out to be a smooth curve of genus one meaning that  $V$  is an elliptic surface. From the classification theory of algebraic surfaces [20] we know that an elliptic surface fibred over a genus one curve can be either an Abelian surface, a bielliptic surface or a proper elliptic surface with Kodaira dimension one. In order to decide on the actual class of  $V$  a central ingredient is the description of the generic fiber in terms of its Weierstrass model. Now the respective pair of coefficients  $\mathbb{A}$  and  $\mathbb{B}$  are interpreted as local functions on the curve  $\bar{E}_2$  and from this data we shall be able to infer on the class of the surface. We have found that the equation for such Weierstrass fibration has a remarkable simple structure, namely

$$y_0^2 - x_0^3 + \frac{c_0^4 d_0^4 (U^4 + 256U^2 + 4096)}{48} x_0 + \frac{c_0^6 d_0^6 (32 + U^2)(U^4 - 512U^2 - 8192)}{864} = 0, \tag{44}$$

where  $c_0$  and  $d_0$  are coordinates of the affine point  $[c_0, d_0, c_0^2 + d_0^2, 1] \subset \bar{E}_2$ . For the explicit birational map dependence of  $x_0$  and  $y_0$  with the surface variables see Appendix B.

<sup>2</sup> This theorem states that if  $\phi : X \rightarrow Y$  is a surjective morphism among irreducible varieties then  $\dim(\phi^{-1}) \geq \dim(X) - \dim(Y)$ , see for example [19].

We now can just change coordinates replacing  $x_0$  by  $x_0 c_0^2 d_0^2$  and  $y_0$  by  $y_0 c_0^3 d_0^3$  and dividing through  $c_0^6 d_0^6$  we end up with coefficients not depending on  $\bar{E}_2$ . This means that locally the Weierstrass fibration can always be definable with constants  $\mathbb{A}$  and  $\mathbb{B}$  and therefore we conclude that  $V$  is an Abelian surface. More precisely, this surface is birational to the product of two elliptic curves, namely

$$V \cong \bar{E}_2 \times \bar{E}_3, \tag{45}$$

where  $\bar{E}_3$  is defined by the homogeneous polynomial,

$$\bar{E}_3 \equiv z_0 y_0^2 - x_0^3 + \frac{(U^4 + 256U^2 + 4096)}{48} x_0 z_0^2 + \frac{(32 + U^2)(U^4 - 512U^2 - 8192)}{864} z_0^3 = 0. \tag{46}$$

At this point we observe that the elliptic curves  $\bar{E}_2$  and  $\bar{E}_3$  are not isomorphic but only have a degree four isogeny. In fact, the J-invariant of  $\bar{E}_3$  is,

$$J(E_3) = \frac{(U^4 + 256U^2 + 4096)^3}{U^8(U^2 + 16)}, \tag{47}$$

such that it satisfies the modular  $\Phi_4 [J(E_2), J(E_3)] = 0$  identity.

The above analysis explain why the R-matrix associated to the Hubbard can not be written solely in terms of the difference of two spectral parameters. Besides having weights lying on a non-trivial surface only part of its geometry retains isomorphism with the one of the Lax operator.

### 5. Conclusions

The basic ingredients in the theory of solvable two-dimensional vertex model of statistical mechanics are the Lax operator and the R-matrix which are constrained by the Yang–Baxter equation (8). The Lax operator is expected to leave on some algebraic variety  $X$  while the R-matrix may generically be sitting on a yet another manifold  $Y$ . They may coincide in some special situations such as when both the Lax operator and the R-matrix are equidimensional and invariant by parity-time reversal symmetry. In fact, taking the transposition on the three spaces of Eq. (8) we obtain

$$L_{23}(\omega_2)^{t_2 t_3} L_{13}(\omega_1)^{t_1 t_3} R_{12}(\omega_1, \omega_2)^{t_1 t_2} = R_{12}(\omega_1, \omega_2)^{t_1 t_2} L_{13}(\omega_1)^{t_1 t_3} L_{23}(\omega_2)^{t_2 t_3}, \tag{48}$$

and after assuming PT symmetry for both operators we have,

$$L_{32}(\omega_2) L_{31}(\omega_1) R_{21}(\omega_1, \omega_2) = R_{21}(\omega_1, \omega_2) L_{31}(\omega_1) L_{32}(\omega_2). \tag{49}$$

Now by applying the permutation on the first and third spaces on both sides of Eq. (49) we finally find,

$$L_{12}(\omega_2) L_{13}(\omega_1) R_{12}(\omega_1, \omega_2) = R_{23}(\omega_1, \omega_2) L_{13}(\omega_1) L_{12}(\omega_2), \tag{50}$$

and direct comparison with the original relation (8) tells us that we have just exchanged the second Lax operator with the R-matrix. This means that both  $X$  and  $Y$  should be defined by the same polynomial relations.

In general situations the Yang–Baxter offers us a rational map since the R-matrix elements can be linearly eliminated from a subset of independent functional relations. Formally this map can be represented as,

$$\begin{aligned}
 X \times X \subset \mathbb{C}\mathbb{P}^{n+1} \times \mathbb{C}\mathbb{P}^{n+1} & \xrightarrow{\phi} & Y \subset \mathbb{C}\mathbb{P}^m \\
 (x_0 : \dots : x_{n+1}) \times (y_0 : \dots : y_{n+1}) & \longmapsto & (\phi_0(x_0, \dots, y_{n+1}) : \dots : \phi_m(x_0, \dots, y_{n+1})),
 \end{aligned}
 \tag{51}$$

where  $n = \dim(X)$ ,  $m$  counts the number of linearly independent R-matrix weights and  $\phi_j(x_0, \dots, y_{n+1})$  are map polynomials.

The study of the geometric properties of  $Y$  requires the implicit representation of the image of the map  $\phi$ . This is basically an elimination problem and in principle can be solved by methods based on Gröbner basis computations. In practice however it is known that this is not a simple task depending much on the number and complexity of the polynomials  $\phi_j(x_0, \dots, y_{n+1})$  as well as on the defining equations of  $X$ .

In this paper we have addressed these problems for the classical vertex model associated to the Hubbard Hamiltonian devised by Shastry [3–5]. We find that the variety  $X$  is a genus one curve and provided its uniformization in terms of factorized ratios of theta functions. This paved the way to discuss local relations for the Lax operator much like in the case of relativistic systems. On the other hand the geometric properties of  $Y$  is that of an Abelian surface birational to the product of two non-isomorphic elliptic curves. This may explain why the Bethe ansatz equations of the Hubbard model is somehow unconventional as compared with other Lattice models based on elliptic curves such as the eight-vertex and hard-hexagon models [13,21]. In the algebraic Bethe ansatz much of the input comes from the R-matrix elements which here sits in a different algebraic variety of the respective Lax operator. It seems interesting to look for alternative solutions for the transfer matrix spectrum more based on the properties of the Lax operator such as to establish finite system exact inversion identities. In this context an earlier attempt by Shastry himself [5] and the recent formulation of fusion for integrable models with R-matrix without the difference form [22] could be relevant guidelines. We hope that the uniformization given here will be useful for setting up this approach and the needed analyticity assumptions.

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### Appendix A. Elimination procedure

We start by defining the ideal  $I \subset \mathbb{C}[\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2, \mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}, \mathbf{c}, \mathbf{d}, \mathbf{g}, \mathbf{h}, \mathbf{q}]$  associated the map (40) by clearing the denominators of Eqs. (38). This can be done by choosing appropriately the weight  $\mathbf{c}$  and as result we obtain,

$$\begin{aligned}
 I = & \langle E_2(\mathbf{x}_1, \mathbf{y}_1), E_2(\mathbf{x}_2, \mathbf{y}_2), \mathbf{a} - p_1(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2), \mathbf{b} - p_2(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2), \\
 & \bar{\mathbf{b}} - p_3(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2), \mathbf{c} - p_4(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2), \mathbf{d} - p_5(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2), \\
 & \mathbf{g} - p_6(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2), \mathbf{h} - p_7(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2), \mathbf{q} - p_8(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) \rangle,
 \end{aligned}
 \tag{A.1}$$

where the symbol  $E_2(x_j, y_j)$  denotes the curve (11) on the variables  $x_j$  and  $y_j$  and the expressions for the polynomials  $p_j(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2)$  are,

$$\begin{aligned}
 p_1(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) &= [\mathbf{y}_1\mathbf{y}_2\theta(\mathbf{x}_2, \mathbf{y}_2) + \mathbf{x}_1\mathbf{x}_2\theta(\mathbf{x}_1, \mathbf{y}_1)] [\mathbf{x}_1^2\mathbf{x}_2^2 - \mathbf{y}_1^2\mathbf{y}_2^2], \\
 p_2(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) &= [\mathbf{y}_1\mathbf{x}_2\theta(\mathbf{x}_1, \mathbf{y}_1) - \mathbf{x}_1\mathbf{y}_2\theta(\mathbf{x}_2, \mathbf{y}_2)] [\mathbf{x}_1^2\mathbf{x}_2^2 - \mathbf{y}_1^2\mathbf{y}_2^2], \\
 p_3(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) &= [\mathbf{y}_1\mathbf{x}_2\theta(\mathbf{x}_2, \mathbf{y}_2) - \mathbf{x}_1\mathbf{y}_2\theta(\mathbf{x}_1, \mathbf{y}_1)] [\mathbf{x}_1^2\mathbf{x}_2^2 - \mathbf{y}_1^2\mathbf{y}_2^2], \\
 p_4(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) &= \theta(\mathbf{x}_1, \mathbf{y}_1)\theta(\mathbf{x}_2, \mathbf{y}_2) [\mathbf{x}_1^2\mathbf{x}_2^2 - \mathbf{y}_1^2\mathbf{y}_2^2], \\
 p_5(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) &= [\mathbf{x}_1\mathbf{y}_1 - \mathbf{x}_2\mathbf{y}_2] \theta(\mathbf{x}_1, \mathbf{y}_1)\theta(\mathbf{x}_2, \mathbf{y}_2), \\
 p_6(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) &= [\mathbf{x}_1\mathbf{x}_2\theta(\mathbf{x}_2, \mathbf{y}_2) + \mathbf{y}_1\mathbf{y}_2\theta(\mathbf{x}_1, \mathbf{y}_1)] [\mathbf{x}_1^2\mathbf{x}_2^2 - \mathbf{y}_1^2\mathbf{y}_2^2], \\
 p_7(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) &= [\mathbf{x}_1\mathbf{x}_2\theta(\mathbf{x}_1, \mathbf{y}_1) - \mathbf{y}_1\mathbf{y}_2\theta(\mathbf{x}_2, \mathbf{y}_2)] \theta(\mathbf{x}_1, \mathbf{y}_1)\theta(\mathbf{x}_2, \mathbf{y}_2), \\
 p_8(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) &= [\mathbf{x}_1\mathbf{x}_2\theta(\mathbf{x}_2, \mathbf{y}_2) - \mathbf{y}_1\mathbf{y}_2\theta(\mathbf{x}_1, \mathbf{y}_1)] \theta(\mathbf{x}_1, \mathbf{y}_1)\theta(\mathbf{x}_2, \mathbf{y}_2).
 \end{aligned}
 \tag{A.2}$$

The elimination of the variables  $\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2$  of the above polynomials is equivalent to find the ideal  $I_1 \subset \mathbb{C}[\mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}, \mathbf{c}, \mathbf{d}, \mathbf{g}, \mathbf{h}, \mathbf{q}]$  defined by,

$$I_1 = I \cap \mathbb{C}[\mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}, \mathbf{c}, \mathbf{d}, \mathbf{g}, \mathbf{h}, \mathbf{q}]. \tag{A.3}$$

One way of finding  $I_1$  is first to compute an alternative basis of  $I$  called Gröbner basis. The elimination theorem asserts that if  $G$  is the Gröbner basis of  $I$  then  $G \cap \mathbb{C}[\mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}, \mathbf{c}, \mathbf{d}, \mathbf{g}, \mathbf{h}, \mathbf{q}]$  is a Gröbner basis of  $I_1$ . For more details about this theorem and its properties we refer to the Book [23]. Fortunately all that can be computed using intrinsics developed in some computer algebra systems such as Singular [18]. Direct computations are however involved and we find more convenient to eliminate each pair of variables  $\mathbf{x}_j, \mathbf{y}_j$  at a time. It turns out that the elimination of the variables  $\mathbf{x}_1$  and  $\mathbf{y}_1$  leads to an intermediate ideal  $I_2 \subset \mathbb{C}[\mathbf{x}_2, \mathbf{y}_2, \mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}, \mathbf{c}, \mathbf{d}, \mathbf{g}, \mathbf{h}, \mathbf{q}]$  whose generating set of polynomials are given by,

$$\begin{aligned}
 I_2^{(1)} &= \mathbf{a}\mathbf{g} - \mathbf{c}^2 + \mathbf{b}\bar{\mathbf{b}}, \\
 I_2^{(2)} &= (\mathbf{b}^2 + \bar{\mathbf{b}}^2 + \mathbf{a}^2 - \mathbf{a}\mathbf{h})(\mathbf{h} - \mathbf{a})^3 + \mathbf{a}(\mathbf{d}^2 - \mathbf{b}\bar{\mathbf{b}})^2 - 2\mathbf{b}\bar{\mathbf{b}}(\mathbf{h} - \mathbf{a})(\mathbf{d}^2 - \mathbf{b}\bar{\mathbf{b}}), \\
 I_2^{(3)} &\equiv E_2(\mathbf{x}_2, \mathbf{y}_2) = (\mathbf{x}_2^2 + \mathbf{y}_2^2)^2 - U\mathbf{x}_2\mathbf{y}_2 - 1, \\
 I_2^{(4)} &= \mathbf{b}^2 + \mathbf{a}^2 - \mathbf{a}\mathbf{h} + \omega_1(\mathbf{x}_2, \mathbf{y}_2)\mathbf{c}\mathbf{d}, \\
 I_2^{(5)} &= \mathbf{b}\mathbf{c} + \omega_1(\mathbf{x}_2, \mathbf{y}_2)\bar{\mathbf{b}}\mathbf{d} - \omega_2(\mathbf{x}_2, \mathbf{y}_2)\mathbf{a}\mathbf{d}, \\
 I_2^{(6)} &= \omega_2(\mathbf{x}_2, \mathbf{y}_2)\mathbf{b}\mathbf{d} + \omega_1(\mathbf{x}_2, \mathbf{y}_2)\omega_2(\mathbf{x}_2, \mathbf{y}_2)\bar{\mathbf{b}}\mathbf{c} - [1 + \omega_1(\mathbf{x}_2, \mathbf{y}_2)^2](\mathbf{h} - \mathbf{a})\mathbf{c}, \\
 I_2^{(7)} &= \omega_2(\mathbf{x}_2, \mathbf{y}_2)\mathbf{a}\mathbf{d} - \omega_1(\mathbf{x}_2, \mathbf{y}_2)\omega_2(\mathbf{x}_2, \mathbf{y}_2)(\mathbf{q} - \mathbf{g})\mathbf{c} - [1 + \omega_1(\mathbf{x}_2, \mathbf{y}_2)^2]\mathbf{b}\mathbf{c},
 \end{aligned}
 \tag{A.4}$$

where we recognize that the first component  $I_2^{(1)}$  is exactly the quadratic  $Q_1$ . The functions depending on the variables  $\mathbf{x}_2$  and  $\mathbf{y}_2$  are,

$$\omega_1(\mathbf{x}_2, \mathbf{y}_2) = \frac{U\mathbf{x}_2^2\mathbf{y}_2^2}{U\mathbf{x}_2\mathbf{y}_2 + 1}, \quad \omega_2(\mathbf{x}_2, \mathbf{y}_2) = \mathbf{x}_2^2 + \frac{\mathbf{y}_2^2}{U\mathbf{x}_2\mathbf{y}_2 + 1} \tag{A.5}$$

We now proceed by eliminating the fraction field elements  $\omega_1(\mathbf{x}_2, \mathbf{y}_2)$  and  $\omega_2(\mathbf{x}_2, \mathbf{y}_2)$  out of the generators  $I_2^{(3)}, \dots, I_2^{(7)}$ . The compatibility between  $I_2^{(6)}$  and  $I_2^{(7)}$  leads us directly to the quadratic  $Q_2$  as well as to the following polynomial,

$$I_2^{(6)} = (\mathbf{c}^2 - \mathbf{b}\bar{\mathbf{b}})(\mathbf{d}^2 - \mathbf{b}\bar{\mathbf{b}}) + \mathbf{a}(\mathbf{h} - \mathbf{a}) (\mathbf{a}(\mathbf{h} - \mathbf{a}) - \mathbf{b}^2 - \bar{\mathbf{b}}^2) \tag{A.6}$$

It turns out that the above generator can be further simplified with the help of the quadrics  $Q_1$  and  $Q_2$ , namely

$$\begin{aligned}
 I_6^{(2)} &= \mathbf{a}\mathbf{g}(\mathbf{h} - \mathbf{a})(\mathbf{q} - \mathbf{g}) + \mathbf{a}(\mathbf{h} - \mathbf{a}) \left( \mathbf{a}(\mathbf{h} - \mathbf{a}) - \mathbf{b}^2 - \bar{\mathbf{b}}^2 \right) \\
 &= \mathbf{a}(\mathbf{h} - \mathbf{a}) \left[ \mathbf{g}(\mathbf{q} - \mathbf{g}) - \mathbf{a}^2 + \mathbf{h}\mathbf{a} - \mathbf{b}^2 - \bar{\mathbf{b}}^2 \right]
 \end{aligned} \tag{A.7}$$

where the last factor is just the quadric  $Q_4$  and the first two are trivial extraneous terms.

Considering these results we can now factorize the component  $I_2^{(2)}$  as follows,

$$\begin{aligned}
 I_2^{(2)} &= \mathbf{g}(\mathbf{q} - \mathbf{g})(\mathbf{h} - \mathbf{a})^3 + \mathbf{a}(\mathbf{q} - \mathbf{g})^2(\mathbf{h} - \mathbf{a})^2 - 2\mathbf{b}\bar{\mathbf{b}}(\mathbf{q} - \mathbf{g})(\mathbf{h} - \mathbf{a})^2 \\
 &= (\mathbf{q} - \mathbf{g})(\mathbf{h} - \mathbf{a})^2 \left[ \mathbf{g}(\mathbf{h} - \mathbf{a}) + \mathbf{a}(\mathbf{q} - \mathbf{g}) - 2\bar{\mathbf{b}}\mathbf{b} \right] \\
 &= (\mathbf{q} - \mathbf{g})(\mathbf{h} - \mathbf{a})^2 \left[ \mathbf{h}\mathbf{q} - \mathbf{c}^2 - \mathbf{q}^2 \right]
 \end{aligned} \tag{A.8}$$

giving rise to the quadric  $Q_3$ .

The final step is to assure the compatibilization of the fractions  $\omega_1(\mathbf{x}_2, \mathbf{y}_2)$  and  $\omega_2(\mathbf{x}_2, \mathbf{y}_2)$  with the algebraic curve  $E_2(\mathbf{x}_2, \mathbf{y}_2)$ . The elimination of the common variables  $\mathbf{x}_2, \mathbf{y}_2$  leads us to a single constraint, namely

$$\begin{aligned}
 &\left[ \omega_1(\mathbf{x}_2, \mathbf{y}_2)^2 + \omega_2(\mathbf{x}_2, \mathbf{y}_2)^2 \right]^2 - U\omega_1(\mathbf{x}_2, \mathbf{y}_2)\omega_2(\mathbf{x}_2, \mathbf{y}_2)^2 \\
 &+ 2 \left[ \omega_1(\mathbf{x}_2, \mathbf{y}_2)^2 - \omega_2(\mathbf{x}_2, \mathbf{y}_2)^2 \right] + 1 = 0.
 \end{aligned} \tag{A.9}$$

By extracting the functions  $\omega_1(\mathbf{x}_2, \mathbf{y}_2)$  and  $\omega_2(\mathbf{x}_2, \mathbf{y}_2)$  from the components  $I_2^{(4)}$  and  $I_2^{(5)}$  the constraint (A.9) becomes a polynomial in the R-matrix weights. This leads to the last quadric  $Q_5$  by considering similar simplifications as done above.

### Appendix B. Fibration analysis

In order to study the properties of a generic fiber one can take an affine point of  $\bar{E}_2$  such as  $[c_0, d_0, c_0^2 + d_0^2, 1]$  where the coordinates  $c_0$  and  $d_0$  are constrained by,

$$(c_0^2 + d_0^2)^2 + U c_0 d_0 - 1 = 0. \tag{B.1}$$

The fiber  $\pi^{-1}$  is an algebraic variety  $\subset \mathbb{C}[\mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}, \mathbf{g}]$  described by the following polynomials,

$$\begin{aligned}
 \tilde{Q}_1 &\equiv \mathbf{b}\bar{\mathbf{b}} + \mathbf{a}\mathbf{g} - c_0^2 = 0, \\
 \tilde{Q}_2 &\equiv \mathbf{b}\bar{\mathbf{b}} + \mathbf{g}(\mathbf{a} - 1) - (c_0^2 + d_0^2)\mathbf{a} + c_0^2 = 0, \\
 \tilde{Q}_4 &\equiv \mathbf{b}^2 + \bar{\mathbf{b}}^2 + \mathbf{g}^2 - (c_0^2 + d_0^2)\mathbf{g} + \mathbf{a}(\mathbf{a} - 1) = 0
 \end{aligned} \tag{B.2}$$

Using the software Singular we found that  $\pi^{-1}$  turns out to be an irreducible non-singular curve of genus one. Further information on such elliptic fibration can be obtained by eliminating the variables  $\bar{\mathbf{b}}$  and  $\mathbf{b}$  with the help of the quadrics  $\tilde{Q}_1$  and  $\tilde{Q}_2$ . After using Eq. (B.1) the polynomial  $\tilde{Q}_3$  becomes,

$$\begin{aligned}
 C &= (\mathbf{a}^2 + \mathbf{b}^2)^2 - c_0^4(2\mathbf{a} - 1)(2\mathbf{a}^2 + 2\mathbf{b}^2 - 2\mathbf{a} + 1) - U c_0 d_0 \mathbf{a} \left[ \mathbf{a}^3 + (1 + \mathbf{a})\mathbf{b}^2 \right] \\
 &- 2c_0^2 d_0^2 \left[ (2\mathbf{a} - 1)\mathbf{a}^2 + (2\mathbf{a} + 1)\mathbf{b}^2 \right]
 \end{aligned} \tag{B.3}$$

We end up with a quartic curve on the variables  $\mathbf{a}$  and  $\mathbf{b}$  which possess two double points as singularities. These are the simplest singular points we can have and the curve  $C$  can be desingularized by means of a single birational transformation bringing it into the Weierstrass form. Let us denote by  $x_0$  and  $y_0$  the corresponding affine Weierstrass coordinates then the inverse birational map is,

$$x_0 = \frac{c_0 d_0 \tilde{x}_0}{c_0^2(\mathbf{a} - 1)^2 + (d_0 \mathbf{a})^2}, \quad y_0 = \frac{c_0 d_0 U \tilde{y}_0}{c_0^2(\mathbf{a} - 1)^2 + (d_0 \mathbf{a})^2}. \tag{B.4}$$

• The variable  $\tilde{x}_0$ :

$$\begin{aligned} \tilde{x}_0 = & \frac{2\alpha_1^2}{U} \mathbf{a}^2 \left[ (d_0^2 - 5c_0^2) \mathbf{a} + \frac{3\iota}{2} (d_0^2 - 3c_0^2) \mathbf{b} \right] + 2\alpha_1 \mathbf{a}(\mathbf{a} + \iota \mathbf{b}) \left[ \mathbf{b}^2 + \frac{\alpha_1^2}{U^2} \mathbf{a}^2 \right] \\ & + 2\alpha_1 c_0 (\iota \alpha_5 \mathbf{b} + \alpha_6 \mathbf{a}) \mathbf{a} - \alpha_2 \left[ 2(2\mathbf{a} - 1) \mathbf{b}^2 + \iota \alpha_3 \mathbf{b} + \alpha_4 \mathbf{a} \right] - \iota U (3c_0^2 - d_0^2) \mathbf{b}^3 + \alpha_7 \mathbf{c}_0^3, \end{aligned} \tag{B.5}$$

where the coefficients  $\alpha_1, \dots, \alpha_7$  are determined in terms of the coordinates  $c_0$  and  $d_0$  as follows,

$$\begin{aligned} \alpha_1 = & (c_0^2 + d_0^2)U, \quad \alpha_2 = c_0^2 U, \quad \alpha_3 = 1 + 4c_0^4 - 12c_0^2 d_0^2 - 2c_0 d_0 U, \\ \alpha_4 = & \frac{32c_0 d_0}{3U} + 16c_0^4 + \frac{11c_0 d_0 U}{6} - 2, \quad \alpha_5 = 6c_0^3 - 6c_0 d_0^2 - \frac{d_0 U}{2}, \\ \alpha_6 = & \frac{8d_0}{3U} + 9c_0^3 - 3c_0 d_0^2 - \frac{d_0 U}{24}, \quad \alpha_7 = \frac{16d_0}{3} + 4c_0^3 U - 4c_0 d_0^2 U - \frac{d_0 U^2}{12}. \end{aligned} \tag{B.6}$$

• The variable  $\tilde{y}_0$ :

$$\begin{aligned} \tilde{y}_0 = & 4c_0 d_0 \alpha_1 \mathbf{a}^2 (\mathbf{b} - \iota \mathbf{a}) (\mathbf{b}^2 + \frac{\alpha_1^2}{U^2} \mathbf{a}^2) + 2 \frac{\alpha_1^2}{U^2} \mathbf{a}^3 [(\beta_1 + \alpha_1 c_0 d_0) \mathbf{b} - \iota \beta_1 \mathbf{a}] \\ & + c_0 \frac{\alpha_1}{U} \mathbf{a}^2 (2\iota \beta_2 \mathbf{a} - \frac{3}{2} \beta_3 \mathbf{b}) \\ & + 2\mathbf{a} \mathbf{b}^2 [(\beta_4 + \alpha_1 c_0 d_0) \mathbf{b} - \iota \beta_4 \mathbf{a}] + 4\iota c_0^2 \mathbf{b}^2 \left[ \left( \frac{2\alpha_1 \alpha_2}{U^2} - 1 \right) (2\mathbf{a} - 1) - \frac{c_0 d_0 U}{2} \right] \\ & + \frac{c_0}{2} (\beta_5 \mathbf{b}^3 - 4\iota \beta_6 c_0^2 \mathbf{a}^2 + \beta_7 c_0 \mathbf{a} \mathbf{b} - 4\iota c_0^2 \beta_8 \mathbf{a} + \beta_9 c_0^2 \mathbf{b} + 8\iota \beta_{10} c_0^3), \end{aligned} \tag{B.7}$$

where the coefficients  $\beta_1, \dots, \beta_{10}$  are given by,

$$\begin{aligned} \beta_1 = & 2c_0^2 - 2d_0^2 - 13c_0^3 d_0 U + 3c_0 d_0^3 U, \\ \beta_2 = & \frac{\alpha_1}{U} (8c_0 - 33c_0^2 d_0 U - d_0^3 U) + 24c_0 d_0^2 (2c_0 d_0 U - 1), \\ \beta_3 = & \frac{\alpha_1}{U} (8c_0 - 35c_0^2 d_0 U - 3d_0^3 U) + 32c_0 d_0^2 (2c_0 d_0 U - 1), \\ \beta_4 = & 2c_0^2 - 2d_0^2 - 7c_0^3 d_0 U + c_0 d_0^3 U, \\ \beta_5 = & -8c_0 + 32c_0^3 d_0^2 + 32c_0 d_0^4 + 17c_0^2 d_0 U + d_0^3 U, \\ \beta_6 = & 12c_0^3 - 36c_0 d_0^2 - d_0 U - 40c_0^4 d_0 U + 24c_0^2 d_0^3 U + c_0 d_0^2 U^2, \\ \beta_7 = & 24 - 192c_0^2 d_0^2 - 58c_0 d_0 U - 64c_0^5 d_0 U + 192c_0^3 d_0^3 U + 35c_0^2 d_0^2 U^2 - d_0^4 U^2, \\ \beta_8 = & -8c_0^3 + 8c_0 d_0^2 + 32c_0^5 d_0^2 + 32c_0^3 d_0^4 + d_0 U + 24c_0^4 d_0 U - 8c_0^2 d_0^3 U - c_0 d_0^2 U^2, \end{aligned}$$

$$\begin{aligned}\beta_9 &= -8c_0^3 - 40c_0d_0^2 + 128c_0^5d_0^2 + 128c_0^3d_0^4 - 3d_0U + 36c_0^4d_0U + 36c_0^2d_0^3U + 2c_0d_0^2U^2, \\ \beta_{10} &= \frac{\alpha_1}{U}(8c_0^2d_0^2 - 1) + c_0d_0(3c_0^2 + d_0^2)U.\end{aligned}\tag{B.8}$$

The corresponding Weierstrass equation for the variables  $x_0$  and  $y_0$  has been presented in the main text, see Eq. (44). The same analysis can be performed for the special fiber at the closed set  $\mathbf{h} = 0$ . Once again we find a non-singular genus one curve which J-invariant is the same as that of the generic fiber given by Eq. (47). This means that the surface  $V$  is normalized in terms of the product of two elliptic curves.

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