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Fredholm-valued holomorphic mappings on a Banach space ☆

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ABSTRACT

In this article we show that the pointwise existence of a regulariser for holomorphic Fredhom-valued mappings defined on pseudo-convex domains in Banach spaces with an unconditional basis implies the existence of a holomorphic regulariser.

1. INTRODUCTION

There is an extensive literature, dating back over forty years, on the parametric dependence of various types of inverses given their pointwise existence. Excellent summaries are provided in [9] and [17], and we have been strongly influenced by the results and methods in both of these articles. We are interested in holomorphic dependence on a parameter in an arbitrary Banach space while the holomorphic results in [9] and [17] concern a parameter in a finite-dimensional Banach space. The relatively recent results of Lempert [12,13] on sheaf cohomology for holomorphic sheaves defined on an infinite-dimensional Banach space provided the missing ingredient in the move from finite to infinite-dimensional spaces.

Our focus in this article is on the existence of holomorphic and meromorphic regularisers for Fredholm-valued holomorphic functions. In Section 3 we show that any Fredholm-valued holomorphic mapping defined on a pseudo-convex domain

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in a Banach space with an unconditional basis has a holomorphic regulariser. The proof of this result for domains in finite-dimensional spaces is well known and uses holomorphic liftings but this is of limited value in the infinite-dimensional case (see [3]), and we were obliged to take a different approach. In Section 4 we show, following ideas in [9] and [17], that the function $z \to f(z)^{-1}$ is meromorphic when f is a Fredholm-valued holomorphic function that is invertible at some point.

If X and Y are Banach spaces over $\mathbb{C}$ we let $\mathcal{L}(X,Y)$ (respectively $\mathcal{K}(X,Y)$, $\mathcal{F}(X,Y)$) denote the set of all continuous (respectively continuous compact, continuous finite rank) linear mappings from X to Y and we let $GL(X,Y)$ denote the subset of $\mathcal{L}(X,Y)$ consisting of all bijective mappings. We use also the standard notation $X' := \mathcal{L}(X,\mathbb{C})$, $\mathcal{L}(X) := \mathcal{L}(X,X)$, $\mathcal{K}(X) := \mathcal{K}(X,X)$, $\mathcal{F}(X) := \mathcal{F}(X,X)$ and $GL(X) := GL(X,Y)$. The space $\mathcal{K}(X)$ is a closed ideal in $\mathcal{L}(X)$ and the quotient algebra $\mathcal{C}(X) = \mathcal{L}(X)/\mathcal{K}(X)$ is called the *Calkin algebra* on X. If Ω is an open subset of a Banach space X and Y is a Banach space we let $\mathcal{H}(\Omega, Y)$ denote the space of holomorphic mappings from Ω into Y (see [1,14,15] for background material on infinite-dimensional holomorphy).

2 FREDHOLM OPERATORS

In this section we recall basic properties of Fredholm operators and prove a number of results required later. We refer to [4] and [16] for background information on Fredholm operators.

A mapping $T \in \mathcal{L}(X,Y)$ is *Fredholm* if $\ker(T)$ and $\mathrm{coker}(T) := Y/\mathrm{im}(T)$ are both finite-dimensional. The space of all Fredholm operators from X into Y is denoted by $\Phi(X,Y)$. If $T \in \Phi(X,Y)$ then $\mathrm{index}(T)$, the index of T, is defined as

$$\mathrm{index}(T) := \dim\big(\ker(T)\big) - \dim\big(\mathrm{coker}(T)\big),$$

where dim denotes dimension. We let $\Phi^n(X,Y)$ denote the set of all Fredholm mappings from X to Y with index $n \in \mathbb{Z}$. We let $\Phi(X) := \Phi(X,X)$ and $\Phi^0(X) := \Phi^0(X,X)$. The set $\Phi(X,Y)$ is an open subset of $\mathcal{L}(X,Y)$ and $GL(X,Y) \subset \Phi^0(X,Y)$.

In proving results for Fredholm operators it is often useful to be able to reduce them to the cases $\Phi(X)$ and $\Phi^0(X,Y)$. In these cases we can call on the following results: a mapping $T \in \mathcal{L}(X)$ is Fredholm if and only if its image in the Calkin algebra is invertible and, moreover, a Fredholm operator $T \in \Phi(X,Y)$ has index 0 if and only if $T = S + K = R + F$ where $S, R \in GL(X,Y)$, $K \in \mathcal{K}(X,Y)$ and $F \in \mathcal{F}(X,Y)$.

The set $\Phi(X,Y)$ may be empty. The set $\Phi^0(X)$ is never empty since the identity mapping on X belongs to it, but it is possible that $\Phi^0(X) = \Phi(X)$ and that $\Phi(X) = \bigcup_{n=1}^{\infty} \Phi^{2n}(X,Y)$ (see [5]). Recently, Argyros and Haydon answered a long standing open question by showing that there exists an infinite-dimensional Banach space X such that $\mathcal{L}(X) = \Phi^0(X)$.

Definition 1. Let X and Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. A mapping $S \in \mathcal{L}(Y,X)$ is called a regulariser for T if $ST - \mathbf{1}_X \in \mathcal{K}(X)$ and $TS - \mathbf{1}_Y \in \mathcal{K}(Y)$.

An operator $T \in \mathcal{L}(X, Y)$ has a regulariser if and only if $T \in \Phi(X, Y)$ (see, for example, [4], p. 190).

If $\Phi(X, Y)$ is non-empty we have a close relationship between X and Y, and in the following lemma we show how to take advantage of this relationship.

Lemma 1. *Let $T \in \Phi(X, Y)$ and suppose* $\dim(\ker(T)) = m$ *and* $\dim(\operatorname{coker}(T)) = n$. *Then*

$$X \oplus \mathbb{C}^n \simeq Y \oplus \mathbb{C}^m,$$

and the mapping $R(x, y) := (Tx, 0)$ belongs to $\Phi^0(X \oplus \mathbb{C}^n, Y \oplus \mathbb{C}^m)$. Moreover, if $L_Y : Y \to Y \oplus \mathbb{C}^m$ and $P_X : X \oplus \mathbb{C}^n \to X$ are the canonical mappings and S is a regulariser for R, then $P_X \circ S \circ L_Y$ is a regulariser for T.

Proof. There exists a closed subspace Z in X such that $X \simeq Z \oplus \ker(T) \simeq Z \oplus \mathbb{C}^m$, $T|_Z : Z \to T(Z)$ is a linear isomorphism and $Y \simeq T(Z) \oplus \operatorname{coker}(T) \simeq T(Z) \oplus \mathbb{C}^n$. This implies

$$X \oplus \mathbb{C}^n \simeq Z \oplus \mathbb{C}^m \oplus \mathbb{C}^n \simeq T(Z) \oplus \mathbb{C}^n \oplus \mathbb{C}^m \simeq Y \oplus \mathbb{C}^m. \tag{1}$$

Note that for most pairs of Banach spaces X and Y this implies $X \simeq Y$.

Since $\ker(R) = \ker(T) \oplus \mathbb{C}^n$, we have that $\dim(\ker(R)) = m + n$. Similarly $\operatorname{coker}(R) = \operatorname{coker}(T) \oplus \mathbb{C}^m$, and $\dim(\operatorname{coker}(R)) = n + m$. Hence $\operatorname{index}(R) = 0$ and $R \in \Phi^0(X \oplus \mathbb{C}^n, Y \oplus \mathbb{C}^m)$.

Since S is a regulariser for R, there exist $K \in \mathcal{K}(Y \oplus \mathbb{C}^m)$ and $C \in \mathcal{K}(X \oplus \mathbb{C}^n)$ such that $R \circ S = \mathbf{1}_{Y \oplus \mathbb{C}^m} + K$ and $S \circ R = \mathbf{1}_{X \oplus \mathbb{C}^n} + C$. Let $S(y, w) = (S_1(y, w), S_2(y, w))$ for all $y \in Y$ and $w \in \mathbb{C}^m$. Then $R \circ S(y, w) = (T \circ S_1(y, w), 0)$, hence $P_X \circ R \circ S(y, w) = T \circ S_1(y, w)$. If, on the other hand, we let $K(y, w) = (K_1(y, w), K_2(y, w))$ then $R \circ S(y, w) = (y + K_1(y, w), w + K_2(y, w))$, thus $P_X \circ R \circ S(y, w) = y + K_1(y, w)$. Then

$$\begin{aligned} T \circ (P_X \circ S \circ L_Y)(y) &= T \circ S_1(y, 0) \\ &= P_X \circ R \circ S(y, 0) \\ &= y + K_1(y, 0) \\ &= (\mathbf{1}_Y + K_1 \circ L_Y)(y) \end{aligned}$$

and hence $T \circ (P_X \circ S \circ L_Y) - \mathbf{1}_Y$ is compact. Let $x \in X$ and $z \in \mathbb{C}^n$, then

$$R(x, z) = L_Y \circ T(x) = [L_Y \circ T \circ P_X](x, z)$$

and $R = L_Y \circ T \circ P_X$. Hence

$$\begin{aligned} (P_X \circ S \circ L_Y) \circ T(x) &= P_X \circ S \circ (L_Y \circ T \circ P_X)(x, 0) \\ &= P_X \circ \big(S \circ R(x, 0)\big) \\ &= x + P_X \circ C(x, 0) \\ &= (\mathbf{1}_X + P_X \circ C \circ L_Y)(x), \end{aligned}$$

so $(P_X \circ S \circ L_Y) \circ T - \mathbf{1}_X$ is compact. The proof is complete. $\square$

3. HOLOMORPHIC REGULARISERS

Definition 2. Let $f \in \mathcal{H}(\Omega, \Phi(X, Y))$ where X and Y are Banach spaces and Ω is an open subset of a Banach space. A mapping $g \in \mathcal{H}(\Omega, \Phi(Y, X))$ is called a holomorphic regulariser for f if, for each $z \in \Omega$, $g(z)$ is a regulariser for $f(z)$.

For finite-dimensional domains the next proposition is due to Krein and Trofimov [11] and Gramsch [6] (see also Theorem 4.5 in [17]). It extends Proposition 9 in [3] to an arbitrary Banach space.

Theorem 1. *Let Ω be a pseudo-convex open subset of a Banach space with an unconditional basis and let X and Y be Banach spaces. If $f \in \mathcal{H}(\Omega, \Phi(X, Y))$ then there exists a holomorphic regulariser for f.*

Proof. We may suppose without loss of generality that Ω is connected. By [16], p. 29, the mapping $z \in \Omega \to \mathrm{index}(f(z))$ is continuous and as it is integer-valued it is locally constant. We first suppose that $\mathrm{index}(f) = 0$ and that $X \simeq Y$. Fix $w \in \Omega$. Since $\mathrm{index}(f(w)) = 0$, we can choose $K_w \in \mathcal{K}(X)$ such that $f(w) - K_w$ is invertible. Since the mapping $T \in GL(X) \to T^{-1} \in GL(X)$ is holomorphic we can find an open neighbourhood of w, U_w, such that $h : z \in U_w \to (f(z) - K_w)^{-1}$ is well defined and holomorphic.

On carrying out the same procedure at all points of Ω we obtain an open covering of Ω and a collection of holomorphic functions from which we can extract a countable open covering $(U_n)_{n=1}^{\infty}$ and a sequence of holomorphic mappings $(h_n)_{n=1}^{\infty}$, where $h_n(z) = (f(z) - K_n)^{-1} \in \mathcal{H}(U_n; GL(X))$ and $K_n \in \mathcal{K}(X)$. If $z \in U_n \cap U_m$, then

$$\begin{aligned}&\big(f(z) - K_n\big)^{-1} - \big(f(z) - K_m\big)^{-1}\\ &\quad= \big(f(z) - K_m\big)^{-1}(K_m - K_n)\big(f(z) - K_n\big)^{-1},\end{aligned}$$

as $u^{-1} - v^{-1} = v^{-1}(v - u)u^{-1}$. Hence

$$h_{nm} := \big[\big(f(z) - K_n\big)^{-1} - \big(f(z) - K_m\big)^{-1}\big]\big|_{U_n \cap U_m} \in \mathcal{H}\big(U_n \cap U_m, \mathcal{K}(X)\big).$$

Moreover,

$$h_{nm} + h_{mn} = 0 \quad \text{on } U_n \cap U_m$$

and

$$h_{lm} + h_{mn} + h_{nl} = 0 \quad \text{on } U_l \cap U_m \cap U_n.$$

Hence $(h_{nm})_{n,m}$ is a cocycle for the sheaf of $\mathcal{K}(X)$-valued holomorphic functions on Ω. By [12] there exists a sequence $(f_n)_{n=1}^{\infty}$, $f_n \in \mathcal{H}(U_n, \mathcal{K}(X))$, such that

$h_{nm}(z) = f_n(z) - f_m(z)$ on $U_n \cap U_m$. This implies

$$\big(f(z) - K_n\big)^{-1} - f_n(z) = \big(f(z) - K_m\big)^{-1} - f_m(z)$$

for all $z \in U_n \cap U_m$, and the function $g(z) := (f(z) - K_n)^{-1} - f_n(z)$ is a well-defined holomorphic function on Ω. Moreover,

$$\begin{aligned} g(z)f(z) &= \big(f(z) - K_n\big)^{-1} f(z) - f_n(z)f(z) \\ &= \mathbf{1}_X + \big[\big(f(z) - K_n\big)^{-1} K_n - f_n(z)f(z)\big] \end{aligned}$$

and $(f(z) - K_n)^{-1}K_n - f_n(z)f(z) \in \mathcal{K}(X)$ for all $z \in U_n$. Similarly,

$$\begin{aligned} f(z)g(z) &= f(z)\big(f(z) - K_n\big)^{-1} - f(z)f_n(z) \\ &= \mathbf{1}_X + \big[K_n\big(f(z) - K_n\big)^{-1} - f(z)f_n(z)\big] \end{aligned}$$

and $K_n(f(z) - K_n)^{-1} - f(z)f_n(z) \in \mathcal{K}(X)$ for all $z \in U_n$. This completes the proof when $\mathrm{index}(f) = 0$ and $X \simeq Y$.

Now let X and Y be arbitrary Banach spaces, and let $f \in \mathcal{H}(\Omega, \Phi(X, Y))$. Fix a point z_0 in Ω, then $\dim(\ker(f(z_0))) = m$ and $\dim(\mathrm{coker}(f(z_0))) = n$ for some $m, n \in \mathbb{N}$. By Lemma 1 $X \oplus \mathbb{C}^n \simeq Y \oplus \mathbb{C}^m$. We define $f^* \in \mathcal{H}(\Omega, \mathcal{L}(X \oplus \mathbb{C}^n))$ by letting $f^*(z)(x, y) := (f(z)(x), 0)$, so $f^*(z) \in \Phi^0(X \oplus \mathbb{C}^n)$ for all $z \in \Omega$. By the above result for index 0 there exists a holomorphic regulariser g for f^*. Let $L_Y : Y \to Y \oplus \mathbb{C}^m$ and $P_X : X \oplus \mathbb{C}^n \to X$ be the canonical mappings, then by Lemma 1, $P_X \circ g \circ L_Y$ is a holomorphic regulariser for f. The proof is complete. □

4. MEROMORPHIC INVERSES

Definition 3. Let Ω denote an open subset of the Banach space X and let Y and Z be Banach spaces.

(a) A set $A \subset \Omega$ is called an analytic set if for each $x \in \Omega$ there exists an open neighbourhood of x, V_x, a Banach space Y_x and $f_x \in \mathcal{H}(V_x, Y_x)$, $f_x \not\equiv 0$, such that $A \cap V_x = \{z \in V_x\colon\ f_x(z) = 0\}$. If each Y_x is finite-dimensional A is said to be of finite definition.

(b) A function $f : \Omega' \to Y$ is a meromorphic function on Ω if Ω' is a dense open subset of Ω and if for each $x \in \Omega$ there exists a connected open neighbourhood of x, V_x, $g_x \in \mathcal{H}(V_x, Y)$, and $h_x \in \mathcal{H}(V_x)$, $h_x \not\equiv 0$, such that

$$f(z) = \frac{g_x(z)}{h_x(z)}$$

for all $z \in V_x \cap \Omega'$. We let $\mathcal{M}(\Omega, Y)$ denote the set of all Y-valued meromorphic functions on Ω.

(c) A mapping $f \in \mathcal{M}(\Omega, \mathcal{L}(Y, Z))$ is a finite meromorphic function if for each $z_0 \in \Omega$ there exists an open neighbourhood V of z_0 such that on V f has the decomposition

$$f(z) = g(z) + \frac{1}{h(z)} k_2(z) \circ P \circ k_1(z)$$

where P is a projection onto a finite-dimensional subspace of Y, $g \in \mathcal{H}(V, \mathcal{L}(Y, Z))$, $k_1 \in \mathcal{H}(V, \mathcal{L}(Y))$, $k_2 \in \mathcal{H}(V, \mathcal{L}(Y, Z))$ and $h \in \mathcal{H}(V)$, $h \not\equiv 0$.

If A is an analytic subset of Ω then $\Omega \backslash A$ is a dense open subset of Ω. The following result is due to Krein and Trofimov [10,11]. Since the method is important and the proof in [17] is rather condensed we include the details.

Theorem 2. *Let X and Y be Banach spaces, let p be a positive integer and let*

$$\mathcal{S}_p(X, Y) := \{T \in \Phi(X, Y); \dim(\mathrm{kernel}(T)) \geqslant p\}.$$

If $\mathcal{S}_p(X, Y) \neq \emptyset$ then $\mathcal{S}_p(X, Y)$ is an analytic subset of finite definition in $\Phi(X, Y)$.

Proof. Let $T_0 \in \Phi(X, Y)$. We first suppose that $\mathrm{index}(T_0) = 0$ and $X \simeq Y$. This means we can find $T_1 \in \mathcal{F}(X)$ such that $T_0 + T_1 \in GL(X)$. Hence $T + T_1 \in GL(X)$ for all T in some neighbourhood V of T_0 in $\mathcal{L}(X)$. If $T \in V$ let $f(T) = (T + T_1)^{-1} T_1$. Then $f \in \mathcal{H}(V, \mathcal{L}(X))$, $T = (T + T_1)(\mathbf{1}_X - f(T))$ and $\ker(T) = \ker(\mathbf{1}_X - f(T))$, that is $x \in \ker(T)$ if and only if $x = f(T)x$. We now choose $(\phi_i)_{i=1}^n \subset X'$ and $(x_i)_{i=1}^n \subset X$ such that $T_1(x) = \sum_{i=1}^n \phi_i(x) x_i$ for all x in X and let $f_i(T) = (T + T_1)^{-1} x_i$ for $1 \leqslant i \leqslant n$. Then $f_i \in \mathcal{H}(V, X)$ and

$$\begin{aligned} f(T)(x) &= (T + T_1)^{-1}\left(\sum_{i=1}^n \phi_i(x) x_i\right) \\ &= \sum_{i=1}^n \phi_i(x)\big((T + T_1)^{-1} x_i\big) \\ &= \sum_{i=1}^n \phi_i(x) f_i(T) \end{aligned}$$

for all $T \in V$. Hence $x \in \ker(T)$ if and only if

$$x = [f(T)](x) = \sum_{i=1}^n \phi_i(x) f_i(T). \tag{2}$$

Let $x \in \ker(T)$ and let $\xi_j = \phi_j(x)$ for $j = 1, \ldots, n$, then

$$\xi_j = \phi_j\big(f(T)(x)\big) = \sum_{i=1}^n \phi_i(x)\phi_j\big(f_i(T)\big) = \sum_{i=1}^n \xi_i \phi_j\big(f_i(T)\big) \tag{3}$$

for $j = 1, \ldots, n$. Conversely, suppose $(\xi_j)_{j=1}^n$ satisfies (3) and $x = \sum_{j=1}^n \xi_j f_j(T)$. Then $\phi_i(x) = \sum_{j=1}^n \xi_j \phi_i(f_j(T))$ and

$$\begin{aligned} x = \sum_{j=1}^n \xi_j f_j(T) &= \sum_{j=1}^n \left(\sum_{i=1}^n \xi_i \phi_j\big(f_i(T)\big) \right) f_j(T) \\ &= \sum_{i=1}^n \xi_i \left(\sum_{j=1}^n \phi_j\big(f_i(T)\big) f_j(T) \right) \\ &= \sum_{i=1}^n \xi_i \big[f(T)\big(f_i(T)\big) \big] \\ &= f(T) \left(\sum_{i=1}^n \xi_i f_i(T) \right) \\ &= [f(T)](x). \end{aligned}$$

Hence $x \in \ker(T)$ if and only if $(\phi_i(x))_{i=1}^n$ is a solution to the finite linear system of equations (3). The system (3) has coefficient matrix $(\phi_j(f_i(T)))_{1 \leqslant i,j \leqslant n}$ and thus has at least p linearly independent solutions if and only if the determinants of all minors of order $n - p + 1$ in $(\phi_j(f_i(T)))_{1 \leqslant i,j \leqslant n}$ vanish. Since each entry in this matrix is a holomorphic function of $T \in V$, the set of all T where these vanish is an analytic subset of finite definition in V.

If $T_0 \in \Phi(X, Y)$ is arbitrary we can, by Lemma 1, choose integers r and s such that $X \oplus \mathbb{C}^r \simeq Y \oplus \mathbb{C}^s$. Let $S_0 : \mathbb{C}^r \to \mathbb{C}^s$ be the zero operator, then the operator R_0 defined by $R_0(x, y) := (T_0(x), S_0(y))$ lies in $\Phi^0(X \oplus \mathbb{C}^r, Y \oplus \mathbb{C}^s)$. By the above we can find an open neighbourhood V of R_0, a finite-dimensional Banach space Z, and $g \in \mathcal{H}(V, Z)$ such that

$$\text{(4)} \qquad \mathcal{S}_{p+r}\big(X \oplus \mathbb{C}^r, Y \oplus \mathbb{C}^s\big) \cap V = \{R \in V \colon\ g(R) = 0\}.$$

Now choose $\epsilon > 0$ such that $\{T \in \Phi(X, Y), \|T - T_0\| < \epsilon\} \times S_0 \subset V$. Let $W = \{T \in \Phi(X, Y), \|T - T_0\| < \epsilon\}$ and let $h(T) := g(T \oplus S_0)$ for all $T \in W$. Then $h(T) = 0$ if and only if $g(T \oplus S_0) = 0$, and by (4) this occurs if and only if $\dim(\ker(T \oplus S_0)) \geqslant p + r$. Since $\dim(\ker(T \oplus S_0)) = \dim(\ker(T)) + r$ this implies

$$\mathcal{S}_p(X, Y) \cap W = \{T \in W \colon\ h(T) = 0\}.$$

This completes the proof. □

The following theorem generalises results of Gramsch and Kaballo [9], and Krein and Trofimov [11].

Theorem 3. *Let X and E be Banach spaces, let Ω denote a connected open subset of E and let $f \in \mathcal{H}(\Omega, \Phi(X))$. If there exists $z_0 \in \Omega$ such that $f(z_0) \in GL(X)$ then the mapping $z \to f(z)^{-1}$ defines a $\Phi(X)$-valued finite meromorphic function on Ω.*

If, in addition, E has an unconditional basis and Ω is pseudo-convex, then on a dense open subset of Ω we have the decomposition

$$f(z)^{-1} = g(z) + h(z), \tag{5}$$

where $g \in \mathcal{H}(\Omega, \mathcal{L}(X))$ and $h \in \mathcal{M}(\Omega, \mathcal{K}(X))$.

Proof. Since $f(z_0)$ is invertible, $\text{index}(f(z_0)) = 0$ and, as f is holomorphic, and Ω is connected this implies $\text{index}(f(z)) = 0$ for all $z \in \Omega$. We then have

$$A := \{z \in \Omega\colon\ f(z) \text{ is not invertible}\} = \{z \in \Omega\colon\ \dim(\ker(f(z)) \geqslant 1\}.$$

If $A = \emptyset$ the mapping $z \in \Omega \to f(z)^{-1}$ is holomorphic and letting $g(z) = f(z)^{-1}$ and $h(z) = 0$ for all $z \in \Omega$ completes the proof.

Now suppose $A \neq \emptyset$. If $z \in A$, then by Theorem 2 there exists an open neighbourhood V of $f(z)$ in $\Phi(X)$, a finite-dimensional Banach space Y and $l \in \mathcal{H}(V, Y)$ such that $V \cap \mathcal{S}_1(X) = \{T \in V\colon\ l(T) = 0\}$. If $W = f^{-1}(V)$, then

$$A \cap W = \{z \in W\colon\ f(z) \in \mathcal{S}_1(X)\} \subset \{z \in W\colon\ l(f(z)) = 0\}.$$

Thus A is contained in an analytic subset of Ω of finite definition and hence its complement $\Omega' := \Omega \setminus A$ is a dense open subset of Ω such that $z \in \Omega' \to f(z)^{-1} \in \Phi(X)$ is holomorphic. Let $z_1 \in A$. Since $\text{index}(f(z_1)) = 0$ we can choose $F \in \mathcal{F}(X)$ such that $f(z_1) + F \in GL(X)$. By continuity there exists an open neighbourhood U of z_1 such that the mapping $z \in U \to (f(z) + F)^{-1}$ is holomorphic. Let $P \in \mathcal{L}(X)$ denote a projection onto the range X_1 of F and let $Q := \mathbf{1}_X - P$ have range X_2. Clearly $X \simeq X_1 \oplus X_2$. Let $D(z) := -F(f(z) + F)^{-1}$ for $z \in U$. Using $f(z) = (f(z) + F) - F$ we obtain

$$\begin{aligned} f(z)\big(f(z) + F\big)^{-1} &= \mathbf{1}_X - F\big(f(z) + F\big)^{-1} \\ &= \mathbf{1}_X - PF\big(f(z) + F\big)^{-1} \\ &= \mathbf{1}_X + PD(z) \\ &= \big(\mathbf{1}_X + PD(z)Q\big) \circ \big(\mathbf{1}_X + PD(z)P\big). \end{aligned} \tag{6}$$

The mappings on the left in (6) are invertible for all $z \in U' := U \cap \Omega'$. The mapping $\mathbf{1}_X + PD(z)Q$ has a holomorphic inverse on U. Indeed, $(PD(z)Q)^2 = 0$ since $QP = 0$, hence $(\mathbf{1}_X + PD(z)Q)^{-1} = \mathbf{1}_X - PD(z)Q$, and the mapping $z \in U \to \mathbf{1}_X - PD(z)Q \in \mathcal{L}(X)$ is holomorphic. Thus $z \in U \to \mathbf{1}_X + PD(z)P$ is invertible on U'. With respect to the decomposition $X_1 \oplus X_2$ of X we may present it in the following matrix form:

$$\mathbf{1}_X + PD(z)P = \begin{pmatrix} \mathbf{1}_{X_1} + PD(z)|_{X_1} & \mathbf{0}_{X_2 \mapsto X_1} \\ \mathbf{0}_{X_1 \mapsto X_2} & \mathbf{1}_{X_2} \end{pmatrix}.$$

The mapping $k\colon z \in U \to \det(\mathbf{1}_{X_1} + PD(z)|_{X_1})$ is well defined and holomorphic. If $z \in U$ and $\det(\mathbf{1}_{X_1} + PD(z)|_{X_1}) = 0$, then there exists $x_1 \neq 0$ in X_1 such that

$(\mathbf{1}_{X_1} + PD(z))(x_1) = 0$. Hence $(\mathbf{1}_X + PD(z)P)(x_1, 0_{X_2}) = 0$, implying that $z \notin U'$. Hence $z \in U' \to k(z)^{-1}$ is well defined and holomorphic. Since $\mathbf{1}_{X_1} + PD(z)|_{X_1}$ is a finite matrix for all $z \in U$, its adjoint $\mathrm{Ad}(z)$ is holomorphic on U. Hence for all $z \in U'$ we have

$$\big(\mathbf{1}_X + PD(z)P\big)^{-1} = \frac{\mathrm{Ad}(z)}{k(z)} \circ P + Q$$

and

$$\begin{aligned}&\big(\mathbf{1}_X + PD(z)P\big)^{-1} \circ \big(\mathbf{1}_X - PD(z)Q\big)\\ &\quad = \frac{\mathrm{Ad}(z)}{k(z)} \circ P \circ \big(\mathbf{1}_X - D(z)Q\big) + Q.\end{aligned}$$

By (6),

$$\begin{aligned}(7)\qquad f(z)^{-1} &= \big(f(z)+F\big)^{-1} \circ \big(\mathbf{1}_X + PD(z)\big)^{-1}\\ &= \big(f(z)+F\big)^{-1} \circ \big(\mathbf{1}_X + PD(z)P\big)^{-1} \circ \big(\mathbf{1}_X - PD(z)Q\big)\\ &= \frac{1}{k(z)}\big(f(z)+F\big)^{-1} \circ \mathrm{Ad}(z) \circ P \circ \big(\mathbf{1}_X - D(z)Q\big)\\ &\quad + \big(f(z)+F\big)^{-1} \circ Q\end{aligned}$$

on U'. Since P is a finite-dimensional projection and all of the mappings on the right in (7) are holomorphic on U, the mapping $z \to f(z)^{-1}$ is finite meromorphic on U. As the point z_1 was arbitrarily chosen in A we have shown that $z \in U \to f(z)^{-1}$ is a finite meromorphic function on Ω.

If Ω is a pseudo-convex domain and E has an unconditional basis, Theorem 1 implies that f has a holomorphic regulariser $g \in \mathcal{H}(\Omega, \Phi(X))$. If we let $h(z) = f(z)^{-1} - g(z)$ for all $z \in \Omega\backslash A$ then $h \in \mathcal{M}(\Omega, \mathcal{L}(X))$ and

$$f(z)^{-1} = g(z) + h(z).$$

Since $f(z)h(z) = \mathbf{1}_X - f(z)g(z) \in \mathcal{K}(X)$ we also have $h(z) = f(z)^{-1}(\mathbf{1}_X - f(z)g(z)) \in \mathcal{K}(X)$ for all $z \in \Omega\backslash A$ and $h \in \mathcal{M}(\Omega, \mathcal{K}(X))$. This completes the proof. □

Let X and Y be Banach spaces. A mapping $A \in \mathcal{L}(X, Y)$ is called *right invertible* if there is $B \in \mathcal{L}(Y, X)$ such that $A \circ B = \mathbf{1}_Y$. We call B a *right inverse* for A.

In [2] we considered the problem of the existence of a holomorphic right inverse for a pointwise-right-invertible holomorphic function. The following theorem shows that even right-invertibility at just one point can yield surprising results. It also generalises both Theorem 3 and a finite-dimensional result in [11].

Theorem 4. *Let Ω be a pseudo-convex connected open subset of a Banach space with unconditional basis, X and Y be Banach spaces and let $f \in \mathcal{H}(\Omega, \Phi(X, Y))$.*

If there exists $z_0 \in \Omega$ such that $f(z_0)$ is right invertible, then f has a finite meromorphic right inverse $g \in \mathcal{H}(\Omega, \Phi(Y, X))$, and on a dense open subset of Ω we have the decomposition

$$g(z) = l(z) + m(z), \tag{8}$$

where $l \in \mathcal{H}(\Omega, \mathcal{L}(Y, X))$ and $m \in \mathcal{M}(\Omega, \mathcal{K}(Y, X))$.

Proof. By Theorem 1 f has a holomorphic regulariser, $s \in \mathcal{H}(\Omega, \Phi(Y, X))$. Thus there exists $C \in \mathcal{K}(X)$ such that $s(z_0)f(z_0) = \mathbf{1}_X + C$. Let R be a right inverse for $f(z_0)$, then

$$\big(s(z_0)f(z_0)\big)R = s(z_0) = R + CR,$$

hence $R = s(z_0) + K$ where $K := -CR$ is a compact operator. Let $h(z) := s(z) + K$, then $h \in \mathcal{H}(\Omega, \Phi(Y, X))$ and

$$f(z)h(z) = f(z)s(z) + f(z)K = \mathbf{1}_Y + T(z),$$

where $T(z) \in \mathcal{K}(Y)$ for all z. Let $F(z) := f(z)h(z)$ for all z, then $F \in \mathcal{H}(\Omega, \Phi(Y))$ and since $F(z_0) = \mathbf{1}_Y$, F is invertible at z_0. Thus, by Theorem 3, F has a finitely meromorphic inverse, w, on Ω. Hence the mapping $z \to h(z)w(z)$ is a finitely meromorphic right inverse for f on Ω. The decomposition (8) follows from the existence, by Theorem 3, of a decomposition for w. The proof is complete. □

Remark.

- In the statement of Theorem 4 it suffices to have a point $z_0 \in \Omega$ such that $f(z_0)$ is surjective. Indeed, since $f(z_0)$ is Fredholm its kernel is complemented, and a surjective mapping with a complemented kernel is right invertible (see, for example, Lemma 1 in [2]).
- We stated Theorem 4 for right inverses, but it holds also for left inverses (in which case it suffices that f is injective at some point).

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