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Chip firing and all-terminal network reliability bounds

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ABSTRACT

The *(all-terminal) reliability* of a graph G is the probability that all vertices are in the same connected component, given that vertices are always operational but edges fail independently each with probability p. Computing reliability is #P-complete, and hence is expected to be intractable. Consequently techniques for efficiently (and effectively) bounding reliability have been the major thrust of research in the area. We utilize a deep connection between reliability and chip firings on graphs to improve previous bounds for reliability.

1. Introduction

Let G be a finite undirected multigraph on vertex set V and edge set E. We assume that every vertex is always operational, but each edge is independently operational with probability $p \in [0, 1]$. The *(all-terminal) reliability* of G, $\mathrm{Rel}(G, p)$ (or simply $\mathrm{Rel}(p)$ if G is understood) is the probability that the spanning subgraph of the operational edges is connected. The reliability of graphs has been well studied (see [1]), and is a useful measure of the robustness of a network.

A main focus concerns methods of calculation and estimation. Calculating reliability is #P-complete, and hence the fact that the exact algorithms described are quite inefficient come as no surprise. Nevertheless, in assessing the reliability of a network, it is imperative that the assessment can be completed in a "reasonable" amount of time. The conflicting desires for fast computation and for great accuracy have led to a varied collection of methods for estimating reliability measures.

One method for estimation of reliability involves *bounding* methods for finding upper and lower bounds for reliability. Current techniques for bounding attempt to find combinatorial or algebraic structure in the reliability problem, permitting the deduction of structural information upon examination of a small fraction of the states. The goal of bounding is to produce *absolute* upper and lower bounds on the reliability measure. In this paper we dramatically improve the best bounds known for reliability via a connection to chip firing games on graphs.

2. Sperner, Kruskal–Katona, and Ball–Provan Bounds

The reliability of a graph G (with n vertices and m edges) can be expressed as a polynomial in p. A subgraph with operational edges $E' \subseteq E$ arises with probability $p^{|E'|}(1-p)^{|E-E'|}$. Consequently, the probability of obtaining a subgraph

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depends only on the number of edges it contains. Let F_i denote the number of subsets S of E of cardinality i such that $G - S$ is connected. The reliability of G is then

$$\mathrm{Rel}(p) = \sum_{i=0}^{m} F_i p^{m-i}(1-p)^i.$$

This expansion is called the *F-form* of the reliability. Typically we can calculate in polynomial time the bottom and top coefficients [1]; in particular, if c is the minimum size of an edge cut (that is, the minimum size of a subset of edges whose removal disconnects G) and $d = m - n + 1$, then $F_i = \binom{m}{i}$ for $i < c$ and F_d is the number of spanning trees of G (all of c, F_c and F_d can be calculated in polynomial time, the last as a determinant via the well-known Matrix Tree Theorem).

One set of bounding techniques rely on the interpretation of the coefficients F_i. A *(simplicial) complex* $\mathcal{C}$ on a finite set X is a nonempty collections of subsets of X, closed under containment, i.e. $Y \in \mathcal{C}$ and $Z \subseteq Y$ implies that $Y \in \mathcal{C}$. The elements of $\mathcal{C}$ are called *faces* of the complex and the maximal faces with respect to containment are called *facets* or *bases*. The *dimension* $d = d(\mathcal{C})$ of $\mathcal{C}$ is the maximum size of a facet. The sequence $\langle F_0, F_1, \ldots, F_d\rangle$ is called the *F-vector* of the complex.

For a graph G with edge set E, consider the complex on E whose faces are $S \subseteq E$ such that $G-S$ is connected; this complex is called the *cographic matroid* of G. It is clear from the definition that in fact the coefficients of the F-form of the reliability are precisely the F-vector of the cographic matroid of G. This connection leads to the application of Sperner's bounds [2] and the Kruskal–Katona bounds [3–5] for complexes, both of which calculate lower and upper bounds based on the exact values of the coefficients in the tails.

The Kruskal–Katona bounds are the best possible for complexes. Provan and Billera [6] prove a powerful result about the structure of matroids, which (together with later results) constrains their F-vectors by observing that matroids are "shellable" complexes. The importance of the Provan–Billera result in our reliability investigations is that they suggest the possibility of exploiting shellability to improve on the Kruskal–Katona bounds. Of course, this requires that we obtain structure theorems for shellable systems. An *interval* $[L, U]$ is a family of subsets $\{S : L \subseteq S \subseteq U\}$. An interval partition of a complex is a collection of disjoint intervals for which every set in the complex belongs to precisely one interval. A complex is *partitionable* if it has an interval partition $[L_i, U_i]$, $1 \le i \le J$ with U_i a base for all i. Shellable complexes are all partitionable.

Ball and Nemhauser [7] developed the application of the partition property to reliability. Consider a shellable complex with b bases; let $\{[L_i, U_i] | 1 \le i \le b\}$ be an interval partition for this complex. $[L_i,U_i]$ is a compact encoding of all sets in this interval; the probability that any one of these sets arises is then $p^{m-|U_i|}(1-p)^{|L_i|}$. In other words, $|L_i|$ edges must fail, and $m - |U_i|$ edges must operate; the state of the remaining edges is of no consequence. Every U_i is a base in the complex; hence the cardinality of each U_i is the same, the rank d of a base. Hence a Boolean formula for all-terminal reliability can be represented as a disjunction of conjuncts whose number is equal to the number of sets of maximum cardinality in the complex; a conjunct contains the Boolean variables corresponding to the negations of elements of L_i, and those corresponding to elements of U_i. This form is in orthogonal DNF.

However, the ranks of the L_i are not all identical; we therefore define $H_i = |\{L_j : 1 \le j \le b, |L_j| = i\}|$. This gives rise to an *H-vector* $\langle H_0, \ldots, H_d\rangle$. The coefficient H_i counts intervals in the partition whose lower set has rank i.

The importance of the Provan–Billera result is that it leads to the *H-form* of the reliability of a graph G on n vertices and m edges, namely

$$\mathrm{Rel}(p) = p^{n-1} \sum_{i=0}^{d} H_i (1-p)^i, \tag{1}$$

where $d = m - n + 1$. The sequence $\langle H_0, H_1, \ldots, H_d\rangle$ is called the *H-vector* of the cographic matroid. Equating the F-vector and H-vector forms of the reliability polynomial gives an expression for H_i in terms of the F-vector:

$$H_k = \sum_{r=0}^{k} F_r (-1)^{k-r} \binom{d-r}{k-r}.$$

This expression allows us to efficiently compute $H_0, \ldots, H_s$ from $F_0, \ldots, F_s$. Another obvious, but useful, fact is that $F_d = \sum_{i=0}^{d} H_i$, that is, the sum of the terms in the H-vector is the number of facets in the cographic matroid (which is the number of spanning trees in the graph).

Following pioneering research of Macaulay [8], Stanley [9–13] has studied H-vectors in an algebraic context, as "Hilbert functions of graded algebras". Stanley's theorem provides a lower bound $H_i^{\langle i-1/i\rangle}$ on H_{i-1}, given H_i, that is tight for shellable complexes in general; this in turn gives an upper bound $H_{i-1}^{\langle i/i-1\rangle}$ on H_i given H_{i-1}. The precise definition of $x^{\langle k/i\rangle}$, the (k, i)th *upper pseudopower of* x, can be found in [1, pg. 65], but for our purposes, three things are important. First of all, for $k \ge j \ge i$, $x^{\langle k/i\rangle} = (x^{\langle j/i\rangle})^{\langle k/j\rangle}$. Secondly, given x, j and i we can compute $x^{\langle j/i\rangle}$ efficiently. Finally, whenever $x \ge y$, $x^{\langle j/i\rangle} \ge y^{\langle j/i\rangle}$.

Stanley's theorem can be used to obtain efficiently computable bounds on the reliability polynomial. Given a prefix $(F_0, \ldots, F_s)$ of the F-vector, we can efficiently compute a prefix $(H_0, \ldots, H_s)$ of the H-vector. Knowing this prefix, we obtain some straightforward bounds.

$$p^{n-1} \sum_{i=0}^{s} H_i (1-p)^i \le \mathrm{Rel}(p) \le p^{n-1} \left[\sum_{i=0}^{s} H_i (1-p)^i + \sum_{i=s+1}^{d} H_s^{\langle i/s\rangle} (1-p)^i \right].$$

This simple formulation ignores a substantial piece of information, the number of spanning trees. This is introduced by recalling that $F_d = \sum_{i=0}^{d} H_i$. Ball and Provan [14,15] develop bounds that incorporate this additional information; they suggest a very useful pictorial tool for thinking about the problem. Associate with each H_i a "bucket". Now suppose we have F_d "balls." Our task is to place all of the balls into buckets, so that the number of balls in the ith bucket, n_i, satisfies $n_i \leq n_{i-1}^{\langle i/i-1\rangle}$.

How do we distribute the balls so as to maximize or minimize the reliability polynomial? These distributions, when found, give an upper and a lower bound on the reliability polynomial. Consider carefully the sum in the reliability polynomial: $\sum_{i=0}^{d} H_i(1-p)^i$. Since $0 < p < 1$, the sum is larger when the lower order coefficients are larger. In fact, for two H-vectors $(H_0, \ldots, H_d)$ and $(J_0, \ldots, J_d)$, whenever $\sum_{j=0}^{i} H_j \geq \sum_{j=0}^{i} J_j$ for all i, the reliability polynomial for the H_i dominates the reliability polynomial for the J_i.

This last simple observation suggests the technique for obtaining bounds. In the pictorial model, an upper bound is obtained by placing balls in the leftmost possible buckets (with buckets $0, \ldots, d$ from left to right); symmetrically, a lower bound is obtained by placing balls in the rightmost possible buckets. We are not totally without constraints in making these placements, as we know in advance the contents of buckets $0, \ldots, s$.

With this picture in mind, Algorithm 2.1 gives a more precise description, producing coefficients $\overline{H}_i$ for an upper bound polynomial, and $\underline{H}_i$ for a lower bound polynomial.

Algorithm 2.1. The Ball–Provan Process, given $(H_0, \ldots, H_s)$ and F_d.

1. For $i = 0, \ldots, s$, set $\underline{H}_i = H_i = \overline{H}_i$.
2. For $i = s+1, s+2, \ldots, d$, set

$$\underline{H}_i = \min\left\{ r : \sum_{j=0}^{i-1} \underline{H}_j + \sum_{j=i}^{d} r^{\langle j/i\rangle} \geq F_d \right\}.$$

$$\overline{H}_i = \min\left(\overline{H}_{i-1}^{\langle i/i-1\rangle}, F_d - \sum_{j=0}^{i-1} \overline{H}_j \right).$$

An explanation in plain text is in order. In each bound, we determine the number of balls in each bucket from 0 to d in turn; as we remarked, the contents of buckets $0, \ldots, s$ are known. For subsequent buckets, the upper bound is determined as follows. The number of balls which can go in the current bucket is bounded by Stanley's theorem, and is also bounded by the fact that there is a fixed number of balls remaining to be distributed. If there are more balls remaining than we can place in the current bucket, we place as many as we can. If all can be placed in the current bucket, we do so; in this case, all balls have been distributed and the remaining buckets are empty. The lower bound is determined by placing as few balls as possible.

The determination of the number of balls to be placed in each bucket can be carried out efficiently. For the upper bound, a single pseudopower must be calculated. For the lower bound, we must choose the smallest value r for which $0 \leq \sum_{j=i}^{d} r^{\langle j/i\rangle} \geq F_d - \sum_{j=0}^{i-1} \underline{H}_j$. When $j \geq i$, $r^{\langle j/i\rangle} \geq r$ and is a nondecreasing function of r. Thus the smallest suitable r certainly lies in the range $0 \leq r \leq F_d - \sum_{j=0}^{i-1} \underline{H}_j$. To locate the minimum value of r that meets the required inequality, a binary search in this range can be undertaken in a number of trials no larger than $\log F_d$; then because F_d is bounded by $n!$, the number of candidate values of r to be examined is bounded by $n \log n$.

We obtain a very powerful set of bounds, the *Ball–Provan* bounds:

$$p^{n-1} \sum_{i=0}^{d} \underline{H}_i (1-p)^i \leq \mathrm{Rel}(p) \leq p^{n-1} \sum_{i=0}^{d} \overline{H}_i (1-p)^i.$$

Unlike the Kruskal–Katona bounds, in the case of the Ball–Provan bounds it is not generally the case that $\underline{H}_i \leq H_i \leq \overline{H}_i$. Brown, Colbourn and Devitt [16] observe that a number of simple network transformations can be used to determine bounds $L_i \leq H_i \leq U_i$ efficiently.

Algorithm 2.2. The Ball–Provan Process with Coefficient Bounds, given $(H_0, \ldots, H_s)$, F_d, and bounds L_i and U_i so that $L_i \leq H_i \leq U_i$ for $0 \leq i \leq d$. Without loss of generality, we suppose that $L_i \geq L_{i+1}^{\langle i/i+1\rangle}$ and $U_{i+1} \leq U_i^{\langle i+1/i\rangle}$ for $0 \leq i < d$.

1. For $i = 0, \ldots, s$, set $\underline{H}_i = H_i = \overline{H}_i$.
2. For $i = s+1, s+2, \ldots, d$, set

$$\underline{H}_i = \min\left\{ r : r \geq L_i \text{ and } \sum_{j=0}^{i-1} \underline{H}_j + \sum_{j=i}^{d} \min(r^{\langle j/i\rangle}, U_j) \geq F_d \right\}.$$

$$\overline{H}_i = \min\left(U_i, \overline{H}_{i-1}^{\langle i/i-1\rangle}, F_d - \sum_{j=i+1}^{d} L_j - \sum_{j=0}^{i-1} \overline{H}_j \right).$$

When $L_i = 0$ and $U_i = F_d$ for $0 \leq i \leq d$, Algorithm 2.2 coincides with Algorithm 2.1. We shall see that by better understanding the associated order of monomials we can further improve the Ball–Provan bounds. The improvement to follow can be viewed as a means of iteratively improving the lower bounds $\{L_i\}$ in Algorithm 2.2.

3. Improving the Ball–Provan bounds

While the cographic matroid is known to be shellable, not all shellable complexes arise from cographic matroids. Indeed not all shellable complexes arise from matroid complexes. Most importantly, the complexes produced in the Ball–Provan bounding process do not arise, in general, from matroid complexes. This suggests the importance of obtaining a characterization of the complexes that is tighter than shellability.

Using techniques in commutative algebra, Stanley's characterization of H-vectors of shellable complexes employs a bijection between faces in the shellable complex and a set of F_d monomials closed under divisibility (an *order ideal of monomials*). Indeed the coefficient H_i in the H-vector counts the monomials of degree i. A further bijection that maps a monomial to a multiset that contains each variable a number of times equal to its exponent in the monomial maps the faces in a shellable complex to a collection of multisets, closed under taking submultisets (a *multicomplex*). We call this an $\mathcal{H}$*-multicomplex* of the graph (we shall often move back and forth between the multicomplex and the associated order ideal of monomials without comment). Shelling, in essence, tells us that this process yields a multicomplex, but tells us nothing about its structure. Stanley [17] conjectured that every shellable complex has an $\mathcal{H}$-multicomplex that is *pure*, in that every maximal multiset is of cardinality d (equivalently, every maximal monomial is of maximum degree).

Progress on characterizing $\mathcal{H}$-multicomplexes hinge on the development of a combinatorial mapping from the shelling to the multicomplex. Brown et al. [18–21] and Hibi [22,23] establish relationships among the terms in the H-vector that are not implied by shellability alone. Chari [24,25] developed connections with topological spaces to explain these relationships; his approach, while complementary to Stanley's, does not appear to provide a simple mechanism for determining the $\mathcal{H}$-multicomplex. Understanding more about the structure of the $\mathcal{H}$-multicomplex arises from a different research direction, which we explore next.

3.1. Chip firing and degree bounds

Let $G = (V, E)$ be a connected multigraph without loops. Let $V = \{q\} \cup \{1, \ldots, n-1\}$. A *configuration* on G is a function $\theta : V \mapsto \mathbb{Z}$ for which $\theta(v) \geq 0$ for all $v \in V \setminus \{q\}$ and $\theta(q) = -\sum_{v \neq q} \theta(v)$. In configuration θ, vertex v is *ready to fire* if $\theta(v) \geq \deg(v)$; vertex q is ready to fire if and only if no other vertex is ready. *Firing* vertex v changes the configuration from θ to θ', where $\theta'(v) = \theta(v) - \deg(v)$ and, for $w \neq v$, $\theta'(w) = \theta(w) + \ell(v, w)$ where $\ell(v, w)$ is the number of edges connecting v and w. A configuration is *stable* when $\theta(v) < \deg(v)$ for all $v \neq q$; in such a configuration only q is ready to fire.

A *firing sequence* $\Theta = (\theta_0, \theta_1, \ldots, \theta_k)$ is a sequence of configurations in which, for $1 \leq i \leq k$, θ_i is obtained from θ_{i-1} by firing one vertex that is ready to fire. It is *nontrivial* when $k > 0$. We write $\theta_0 \to \theta_k$ when some nontrivial firing sequence starting with θ_0 and ending with θ_k exists. Configuration θ is *recurrent* if $\theta \to \theta$. Stable, recurrent configurations are called *critical*.

This is a *chip firing game* on the graph G. Initial motivation for its study arises in the so-called abelian sandpile models to study self-critical systems [26,27]. Biggs [26,28] (who called this the *dollar game*) and Merino [29] studied critical configurations and showed that the enumeration of critical configurations classified by the sum $\sum_{v \neq q} \theta(v)$ is an evaluation of the two variable Tutte polynomial of the graph G. Indeed Merino [29] establishes that a particular evaluation yields the generating function of the H-vector of the cographic matroid, which is explained by Theorem 3.1.

Theorem 3.1 (*Merino [30]*). *Let $\mathcal{C}$ be the set of all critical configurations of G. For each $v \in V \setminus \{q\}$, let x_v be an indeterminate. For each $\theta \in \mathcal{C}$, define a monomial $m_\theta = \prod_{v \in V \setminus \{q\}} x_v^{\deg(v)-1-\theta(v)}$. Then the set $\{m_\theta : \theta \in \mathcal{C}\}$ is an order ideal of monomials associated with an $\mathcal{H}$-multicomplex of G.*

Closure under division is easily seen, by observing that if θ is a critical configuration with $\theta(v) < \deg(v) - 1$, the assignment to v can be increased by one to obtain another critical configuration. That the number of critical configurations of degree δ is the same as H_Δ with $\Delta = \left(\sum_{v \neq q}(\deg(v) - 1)\right) - \delta$ follows from the fact that it is the same Tutte invariant.

As an illustration, we provide an explicit order ideal of monomials for the cographic matroid of complete graph K_n on n vertices. There is no known explicit formula for $\mathrm{Rel}(K_n, p)$, though there is a recursion [31] to calculate it efficiently:

$$\mathrm{Rel}(K_n, p) = 1 - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \mathrm{Rel}(K_j, p)(1-p)^{j(n-j)}. \tag{2}$$

Now for any graph G with m edges, in the H-form (1) of the reliability the coefficient of q^m is precisely $(-1)^{n-1}H_d$. Together the recursion for $\mathrm{Rel}(K_n, p)$ we can deduce that for the complete graph K_n, $H_d = (n-1)!$. What is an explicit order ideal of monomials for the corresponding cographic matroid? We fix vertex v_1 of K_n. Assigning $0, 1, \ldots, n-2$ chips in some order to the remaining vertices produces a critical configuration of the chip firing game. There are $(n-1)!$ such assignments to

$K_n - v_1$, each a critical configuration. There are exactly H_d critical configurations, and $H_d = (n-1)!$ for K_n. Thus *all* of the critical configurations have been accounted for, and hence the order ideal of monomials can be described as the set of all divisors of the monomials of the form $x_{i_1}^1 x_{i_2}^2 \cdots x_{i_{n-2}}^{n-2}$ where $\{i_1, i_2, \ldots, i_{n-2}\}$ is a subset of size $n-2$ from $\{2, 3, \ldots, n\}$.

Let us explore some other consequences of the connections between chip firings and the order ideals of monomials.

Proposition 3.2. *Let $d_1, \ldots, d_n$ be the vertex degrees of graph G. Then a corresponding order ideal for the cographic matroid of G can be represented as monomials in the $n-1$ variables $x_1, \ldots, x_{n-1}$ of degrees at most $d_1 - 1, \ldots, d_{n-1} - 1$, respectively.*

The selection of the nth vertex is precisely the choice of q, which is arbitrary. It seems natural to choose q to have either lowest degree or highest degree; although its choice does not impact the true H-vector, it can affect the bound. We assume that d_n is maximum, and that $d_1 \geq \cdots \geq d_{n-1}$. In essence, then, a strategy like the Kruskal–Katona or Ball–Provan bounds results, if we can specify inequalities between numbers of multisets of cardinalities i and $i+1$ in a multicomplex. Kruskal and Katona determined these precisely when the multisets are sets; Macaulay [8,32,33] determined these precisely when there are no constraints on multiplicities in the multisets. When such restrictions on multiplicities are known, Clements and Lindström [34] proved a common generalization. We examine this next.

First we describe a fundamental ordering on multisets. Suppose that S and T are multisets of elements from a set $x_1, \ldots, x_\ell$. Then S precedes T in *colexicographic order* if and only if for some k satisfying $1 \leq k \leq \ell$, the number of occurrences of x_i is the same in S and in T when $k < i \leq \ell$, but S contains x_k fewer times than T does.

Consider multisets $\mathcal{M}$ of symbols $x_1, \ldots, x_{n-1}$ in which element x_i appears at most $e_i = d_i - 1$ times, and let $\mathcal{M}_r$ be the collection of multisets of cardinality r. Treating the order ideal as a multicomplex makes it a submulticomplex of $\mathcal{M}$. When $\mathcal{N}_r \subseteq \mathcal{M}_r$, its *shadow* is the collection of all multisets in $\mathcal{M}_{r-1}$ contained in at least one member of $\mathcal{N}_r$.

Lemma 3.3 (*[34]*). *Suppose that $d_1 \geq \cdots \geq d_{n-1}$. Then a subset of N_r sets in $\mathcal{M}_r$ that has the smallest shadow is obtained by taking the first N_r multisets in colexicographic order.*

The first N_r multisets in colexicographic order admit a simple description. Indeed, once the last of the multisets in colexicographic order is identified, the remainder can be determined as all those multisets preceding the last in colexicographic order. Let $\binom{e_1,\ldots,e_k}{r}$ denote the number of ways to choose distinct multisets of size r from a multiset consisting of k types of elements, the ith type containing e_i indistinguishable elements. To compute $\binom{e_1,\ldots,e_k}{r}$, observe that it is the coefficient of x^r in $\prod_{i=1}^{k} \sum_{j=0}^{e_i} x^j$. Let $\Pi_\ell = \prod_{i=1}^{\ell} \sum_{j=0}^{e_i} x^j$. Then taking $\Pi_0 = 1$, the polynomial Π_ℓ can be calculated as $\Pi_{\ell-1}(\sum_{j=0}^{e_\ell} x^j)$, naively using $(e_\ell + 1)(1 + \sum_{i=1}^{\ell-1} e_i)$ multiplications. It follows that when each e_i is bounded by n, and k is bounded by n, the polynomial Π_k can be computed in time that is polynomial in n, and hence so can $\binom{e_1,\ldots,e_k}{r}$.

Algorithm 3.4. The N_rth multiset (of size r) in colexicographic order.

1. Find the value κ of k for which $\binom{e_1,\ldots,e_k}{r} \leq N_r < \binom{e_1,\ldots,e_{k+1}}{r}$.
2. Choose α to be the value of a for which $\sum_{j=0}^{a} \binom{e_1,\ldots,e_\kappa}{r-j} \leq N_r < \sum_{j=0}^{a+1} \binom{e_1,\ldots,e_\kappa}{r-j}$.
3. Set $\widehat{N}_{r-\alpha-1} = N_r - \sum_{j=0}^{\alpha} \binom{e_1,\ldots,e_\kappa}{r-j}$. Choose the $(\widehat{N}_{r-\alpha-1})$th multiset of $\mathcal{M}_{r-\alpha-1}$ inductively, and adjoin $x_{\kappa+1}$ to it $\alpha + 1$ times.

Computing the size of the shadow of the first N_r multisets of size r is straightforward using Algorithm 3.4. From the first $\sum_{j=0}^{\alpha} \binom{e_1,\ldots,e_\kappa}{r-j}$ multisets, one obtains a shadow of $\binom{e_1,\ldots,e_\kappa,\alpha}{r-1} = \sum_{j=0}^{\alpha} \binom{e_1,\ldots,e_\kappa}{r-j-1}$ multisets. Proceed inductively for the remaining $\widehat{N}_{r-\alpha-1}$ sets, noting that the only new elements in the shadow are those that contain x_κ exactly $\alpha + 1$ times. All others appear already in the shadow. This number of multisets in the shadow is the *CL-pseudopower* $N_r^{[i-1/i]}$. In general, when $j < i-1$, define $x^{[j/i]} = (x^{[j/j+1]})^{[j+1/i]}$. When $x = y^{[i/i+1]}$ we write $y = x^{[i+1/i]}$, and extend the definition in the natural way to $x^{[j/i]}$ for all $j \geq i$.

Proposition 3.2 and Lemma 3.3 combine to yield inequalities on coefficients in the H-vector:

Lemma 3.5. *For $0 \leq i < d$, $H_i \geq H_{i+1}^{[i/i+1]}$ and equivalently $H_{i+1} \leq H_i^{[i+1/i]}$.*

Using CL-pseudopowers in place of upper pseudopowers in Algorithm 2.1 yields the *Clements–Lindström bounds*. Upper pseudopowers are, in fact, CL-pseudopowers in which the vertex degrees are not employed; hence $x^{\langle i-1/i \rangle} \leq x^{[i-1/i]}$. Thus the Clements–Lindström bounds can be no looser than the Ball–Provan ones. Moreover, as in the Ball–Provan process, the computation of κ and α in Algorithm 3.4 can be efficiently completed with binary search, so that CL-pseudopowers can be calculated efficiently.

3.2. Pure multicomplexes

Stanley [17] conjectured that the $\mathcal{H}$-multicomplex is pure. The bijection with critical configurations enabled Merino [30] to prove this. The idea is straightforward. Consider a critical configuration, and a recurrent firing sequence for it. Imagine that each vertex has a number of "chips" equal to its current value in the configuration. Then firing involves the movement of some chips off the vertex to neighbouring vertices. Observing that a recurrent firing sequence for a critical configuration

fires each vertex exactly once, and keeping track of chips that are moved, it is easy to see that every unmoved chip could be deleted from the initial stable configuration. Now simple counting ensures that every monomial that is not of maximum degree must divide a monomial in the order ideal that has larger degree. This ensures purity (see [30] for details).

A result of Hibi states a further constraint on the H-vector of a pure multicomplex.

Theorem 3.6 (*Hibi [23,30]*). *If $\langle H_0, H_1, \ldots, H_d\rangle$ is the H-vector of the cographic matroid of graph G, then $H_{d-i} \geq H_i$ for $i \leq d/2$.*

Theorem 3.6 permits a modest improvement in the upper bound; unlike the Ball–Provan bounds, we are constrained to place some balls in the rightmost buckets. However purity is much more informative in conjunction with the degree bounds, as we see next.

3.3. M-shellability

In the same manner that shellability (or at least partitionability) of the $\mathcal{F}$-complex permitted a representation as intervals, and ultimately as the $\mathcal{H}$-multicomplex, one can ask whether the $\mathcal{H}$-multicomplex admits a similar representation. Chari [25] first explored this. An *M-interval* in a pure multicomplex $\mathcal{M}$ is specified by a lower multiset L and an upper multiset U, and contains all multisets containing L and contained in U; here U must be maximal. Then $\mathcal{M}$ is M-partitionable if it admits a partition into M-intervals. It is *M-shellable* if the M-intervals of an M-partitioning can be ordered so that the union of every prefix of M-intervals itself forms a multicomplex. Following a conjecture of Chari [25], using a clever deletion/contraction argument Merino [30] proved:

Theorem 3.7. *The $\mathcal{H}$-multicomplex is M-shellable.*

Theorem 3.7 enforces further combinatorial structure. In exploiting this structure, however, we are handicapped in two ways that did not arise in the Ball–Provan (or Clements–Lindström) bounds. In the shelling of $\mathcal{F}$, one knows the number of sets of each cardinality in an interval simply by knowing the size of its lower set (and hence, the height of the interval). One also knows that the number of intervals to be produced equals F_d, a quantity that we can compute efficiently. In the M-shelling of $\mathcal{H}$, the height of an M-interval does not alone permit us to determine the number of multisets in the M-interval of each cardinality (except the smallest and largest, of course). Moreover although we know that the number of M-intervals equals H_d, we have no general efficient method for its calculation (indeed it is #P-complete [35]).

One strategy would be to bound H_d; another is to constrain the "shapes" of the M-intervals when we cannot determine them exactly. We pursue the latter approach. Knowing the cardinality of a lower multiset for an M-interval, we know its height. Without further information, the M-interval could be as thin as a chain, or arbitrarily thick. However, the thinness of an M-interval is constrained by restrictions on exponents of indeterminates in the corresponding monomials. Its thickness is restricted by the limitation on the number of indeterminates.

Proposition 3.8. *Let s be an integer. Let G be an n-vertex m-edge graph with degrees $d_1 \geq \cdots \geq d_{n-1}$ (for vertices other than q). Let $d = m - n + 1$. For $i = 1, \ldots, n-1$, let $\lambda_i = \min(d_i - 1, d - s - \sum_{j=1}^{i-1} \lambda_j)$. For $i = n-1, \ldots, 1$, let $u_i = \min(d_i - 1, \lfloor \frac{d-s-\sum_{j=i+1}^{n-1} u_j}{i} \rfloor)$. In an M-interval for G whose lower multiset has size s the number of multisets of size $s + \alpha$ is bounded below by the coefficient of x^α in $\prod_{i=1}^{n-1}(1 + x + \cdots + x^{\lambda_i - 1} + x^{\lambda_i})$ and above by the coefficient of x^α in $\prod_{i=1}^{n-1}(1 + x + \cdots + x^{u_i - 1} + x^{u_i})$.*

Proof. Let $(z_1, \ldots, z_{n-1})$ be a sequence of nonnegative integers whose sum is fixed. The coefficient of x^α in $\prod_{i=1}^{n-1}(1 + x + \cdots + x^{z_i - 1} + x^{z_i})$ is $\binom{z_1,\ldots,z_{n-1}}{\alpha}$. For β satisfying $1 \leq \beta \leq z_j$, consider 'shifting' β from x_j to x_i. Now $\binom{z_1,\ldots,z_i,\ldots,z_j,\ldots,z_{n-1}}{\alpha} - \binom{z_1,\ldots,z_i+\beta,\ldots,z_j-\beta,\ldots,z_{n-1}}{\alpha}$ is nonnegative if $z_i \geq z_j - \beta$, and nonpositive if $z_i \leq z_j - \beta$; it is zero when $z_i = z_j - \beta$.

Proposition 3.2 bounds the entries in the sequence of exponents. By shifting repeatedly without violating the degree bounds, $(\lambda_1, \ldots, \lambda_{n-1})$ yields a minimum for the coefficient of x^α. Informally, the thinnest M-interval is no thinner than one that uses the fewest indeterminates allowed to the largest exponents allowed. By the same token, if two entries differ by two or more in the sequence, and they can be made more equal without violating the degree bounds, shifting would not reduce the coefficient of x^α; hence $(u_1, \ldots, u_{n-1})$ yields a maximum for the coefficient of x^α. Informally, the thickest M-interval is no thicker than one that uses instead as many indeterminates as possible, with exponents as equal as possible subject to the degree constraints. □

For simple graphs, Proposition 3.8 can be improved upon.

Lemma 3.9. *Let G be a simple graph with vertex degrees $d_1 \geq \cdots \geq d_{n-1}$ (for vertices other than q). Define $d_i' = \min(d_i, n - 1 - i)$. Then the thickest M-interval for G is no thicker than one obtained by choosing exponents for the variables $x_1, \ldots, x_{n-1}$ as equal as possible so that the exponent of x_i is at most d'_i.*

Proof. All monomials in the order ideal have the corresponding indeterminate x_i raised to the power at most $d_i - 1$. Now consider a specific M-interval, and consider the monomial corresponding to its upper multiset. This corresponds to a specific critical configuration, which when fired through a recurrent sequence has no "spare" chips. The vertex v fired immediately

after q in this sequence must have been allocated exactly $\deg(v) - \ell(q, v)$ chips. It follows that when G is a simple graph, no maximum degree monomial can contain all $n-1$ indeterminates to a nonzero power, since the vertex v fired immediately after q has $\deg(v) - 1$ chips, and hence has power 0 in the corresponding monomial. When the graph is simple, proceed inductively to establish that for every $1 \leq i \leq n-1$ there are at least i indeterminates of the $n-1$ whose power in the maximum degree monomial is at most $i-1$. (This is a statement about each maximum degree monomial independently; a restriction on the power of an indeterminate in one need not constrain the power of the same indeterminate in another.) Therefore for simple graphs, we can replace d_i by d_i', because within any specific interval, no more than i variables can have degree at least $n-1-i$. □

Suppose that the Clements–Lindström bounds have been calculated. There may be no suitable corresponding assignment of M-intervals. Indeed for the lower bound, the M-intervals with the smallest lower multisets may exhaust more sets at some higher level than the bounding process has assigned, even under the restriction that the M-interval examined is as thin as Proposition 3.8 allows. For the upper bound, even the thickest M-intervals force a number of "balls" to be placed in the rightmost buckets, so the upper bound distribution must place balls more to the right.

Whether Proposition 3.8 or Lemma 3.9 is used, let $\lambda(s, i)$ and $u(s, i)$ be the resulting lower and upper bounds on the number of multisets of size $s+i$ in an M-interval whose lower multiset has size s. (These are easily calculated knowing s and the degree sequence of the graph.) At each stage L_i denotes the number of balls already placed in bucket i as a result of the M-intervals selected. Algorithm 3.10 describes the process for the lower bound.

Algorithm 3.10. The BCN Lower Bound.

for $i = 0, \ldots, d$, compute $\underline{H}_i$ via the Clements–Lindström bounds.
for $i = 0, \ldots, d$, set $L_i = 0$.
for $i = 0, \ldots, d$,
  set $B = \underline{H}_i - L_i$.
  for $j = i, \ldots, d$, set $L_j = L_j + B \cdot \lambda(i, j-i)$.
  set $\underline{H}_i = \underline{H}_i + B$.
  recalculate $\underline{H}_j$ for $j = i+1, \ldots, d$ via the Clements–Lindström bounds,
    using the $\{L_j\}$ as lower bounds on the coefficients.
return $p^{n-1} \sum_{i=0}^{d} \underline{H}_i (1-p)^i$.

In Algorithm 3.10, intuitively our objective is to make many short M-intervals (hence having large lower multisets) to permit balls to be placed as far to the right as possible. Every M-interval $[L, U]$ is symmetric in terms of number of balls placed at height $|L| + i$ and $|U| - i$, and hence our objective is to associate each multiset with the largest possible lower multiset in its M-interval. For each size of lower multiset in turn, starting with size 0, we determine the number of balls not already placed in buckets for this size, and determine the thinnest M-interval for which each could be a lower multiset. This accounts for a number of balls at each higher level. It may happen that at some higher level we account for more balls than remain available in the current values $\{\underline{H}_i\}$. We therefore recalculate the Clements–Lindström bound treating the numbers $\{L_i\}$ of balls already placed as coefficient lower bounds. This has the effect of reallocating balls further to the left than before (increasing the lower bound).

When the Ball–Provan bounds are applied to $\mathcal{F}$, the H-vectors determined for both the lower and the upper bound correspond to partitionable complexes; in that sense, they are as tight as possible for the class of partitionable complexes. However, when Algorithm 3.10 is applied, there is no assurance that the H-vector determined corresponds to an M-partitionable multicomplex (although by the same reasoning as for the Ball–Provan bounds, it does correspond to a partitionable complex). This occurs because it may not be possible to form an M-partitioning in which every M-interval is the thinnest permitted by Proposition 3.8. Hence Algorithm 3.10 may not be tight for M-partitionable multicomplexes.

Turning to the upper bound, a similar technique is applied in Algorithm 3.11.

Algorithm 3.11. The BCN Upper Bound.

for $i = 0, \ldots, d$, compute $\overline{H}_i$ via the Clements–Lindström bounds.
for $i = 0, \ldots, d$, set $L_i = 0$.
for $i = 0, \ldots, d$,
  set $T = \sum_{j=i}^{d} u(i, j-i)$.
  compute $A = \lfloor (F_d - \sum_{j=0}^{i-1} \overline{H}_j - \sum_{j=i}^{d} L_j)/T \rfloor$.
  set $B = \min(A, \overline{H}_i - L_i)$.
  for $j = i+1, \ldots, d$, set $L_j = L_j + B \cdot u(i, j-i)$.
  set $\overline{H}_i = L_i + B$.
  recalculate $\overline{H}_j$ for $j = i+1, \ldots, d$ via the Clements–Lindström bounds,
    using the $\{L_j\}$ as lower bounds on the coefficients.
return $p^{n-1} \sum_{i=0}^{d} \overline{H}_i (1-p)^i$.

Table 1
Bounds for $K_{4,4}$. Bounds arise from the polynomials having these coefficients; the coefficients themselves do not individually bound each true coefficient.

i	Polynomials for Lower Bound			Polynomials for Upper Bound		
	BP	CL	BCN	BCN	CL	BP
0	1	1	1	1	1	1
1	7	7	7	7	7	7
2	28	28	28	28	28	28
3	84	84	84	84	84	84
4	202	202	202	202	202	202
5	231	336	336	407	407	428
6	404	552	552	707	707	819
7	656	790	790	1073	1073	1450
8	1006	993	993	1110	1439	1077
9	1477	1103	1103	477	148	0

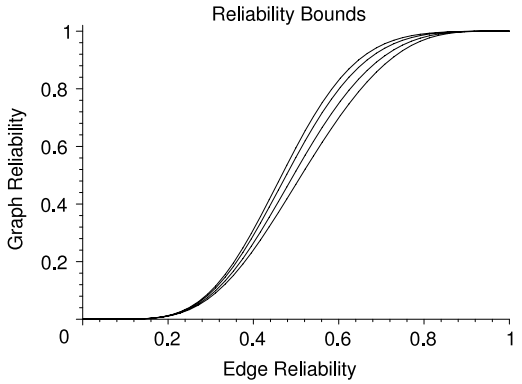

Fig. 1. BP and BCN reliability bounds for $K_{4,4}$.

Here the goal is to permit balls to remain as far to the left as possible, and hence we choose thickest M-intervals. For the rightmost buckets, there are typically few balls initially placed (often $\overline{H}_d$ is 0 in the initial Clements–Lindström upper bound.) By the computation of A, balls are redistributed to provide enough balls at levels $i+1, \ldots, d$ to meet the current demand. In this way balls are moved to the right (reducing the upper bound) when needed.

In terms of efficiency, Algorithms 3.10 and 3.11 recalculate the Clements–Lindström bounds at most d times, and hence remain polynomial time methods.

4. Numerical examples

We have implemented the Ball–Provan (BP) bounds from Section 2, bounds applying the Clements–Lindström (CL) inequalities on the degrees of the monomials as in Section 3, and bounds (BCN) that employ purity (actually M-partitioning) as described in Section 3.3. The CL bounds result in a substantial improvement on the Ball–Provan bounds, while the additional assumption of purity thereafter has a smaller effect. We examine a small graph, $K_{4,4}$, to illustrate this (see Table 1).

Enforcing the degree constraints (in the CL bound) causes both the lower and the upper bound to improve. However, in addition employing constraints from M-partitioning in this case has no effect on the lower bound; in the upper bound, improvement results from requiring $H_d = H_9$ to be increased, which in turn causes a compensating reduction in $H_{d-1} = H_8$. Fig. 1 shows the improvements graphically; four curves are shown giving all-terminal reliability bounds as a function of p, the edge operation probability. From bottom to top, the curves are the BP lower, BCN lower, BCN upper, and BP upper bounds.

Next we consider a larger example, the circulant graph on 21 vertices with all pairs of vertices at distance 1 or 2 connected (Table 2 and Fig. 2).

For two larger examples, we show only the graphs of the bounds. Fig. 3 shows improvements for $K_{6,10}$, and Fig. 4 shows the results for $K_{4,5,6}$.

To get a quantitative measure of the improvement in the upper and lower bounds, we can consider two measures. For a pair of lower and upper bound functions (f, g) for $\mathrm{Rel}(G, p)$ (that is, $f(p) \le \mathrm{Rel}(G, p) \le g(p)$ for all $p \in [0, 1]$), we define

$$\text{maximum diff}(f, g) = \max\{f(p) - g(p) : p \in [0, 1]\}$$

and

$$\text{avg diff}(f, g) = \int_0^1 (f - g)\, \mathrm{d}p.$$

Table 2
Bounds for 21-vertex circulant with distances $\pm\{1, 2\}$.

i	Lower Bound			Upper Bound		
	BP	CL	BCN	BCN	CL	BP
0	1	1	1	1	1	1
1	20	20	20	20	20	20
2	210	210	210	210	210	210
3	1540	1540	1540	1540	1540	1540
4	8834	8834	8834	8834	8834	8834
5	8359	19092	33726	42085	42085	42275
6	26027	67200	96985	172710	172710	175370
7	72880	208763	227480	625671	625671	647615
8	187028	582091	582091	2035887	2035887	2170256
9	446133	1474891	1474891	6027503	6027503	6696151
10	1000047	3428177	3428177	16398064	16398064	19237614
11	2124769	7363917	7363917	41244771	41314437	51926525
12	4308628	14704636	14704636	94973504	97002884	132651350
13	8385892	27426362	27426361	193397920	213335686	322656765
14	15738546	47966986	47966975	332382809	441345908	751058790
15	28594035	78917058	78916994	462819401	861930498	1228839920
16	50455630	122463591	122463330	506631755	835871298	0
17	86713051	179641222	179640389	425580473	0	0
18	145494459	249549525	249547323	267387984	0	0
19	238837261	328780144	328775158	121269769	0	0
20	384274912	411315011	411305129	37455338	0	0
21	606960355	489074711	489057313	7047465	0	0
22	942474619	553119254	553091756	609522	0	0

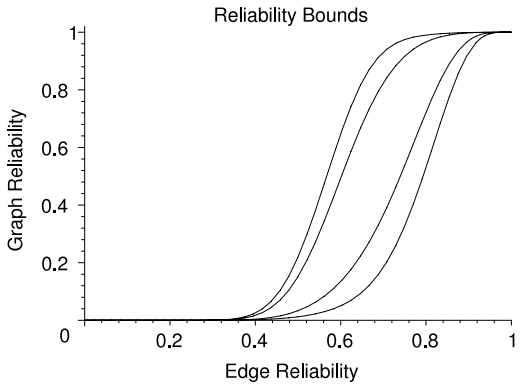

Fig. 2. BP and BCN reliability bounds for 21-vertex circulant.

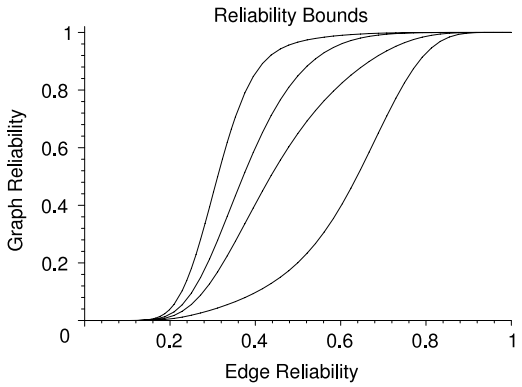

Fig. 3. BP and BCN reliability bounds for $K_{6,10}$.

(The latter is just the average value of the difference over the interval [0, 1], and is the same as the area between the two curves as the interval has length 1.) For the graph $K_{4,5,6}$ we find that for the maximum difference between the Ball–Provan lower and upper bounds is approximately 0.8246 while for our bounds it is only approximately 0.2216. The average

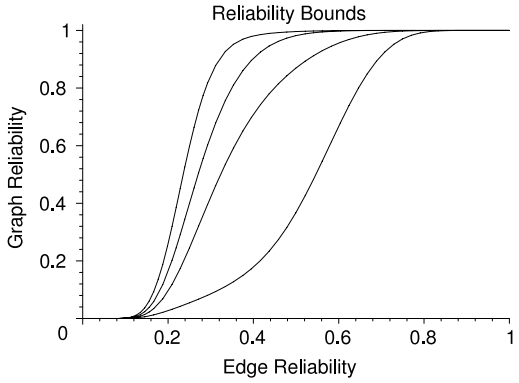

Fig. 4. BP and BCN reliability bounds for $K_{4,5,6}$.

difference of the lower and upper Ball–Provan bounds is approximately 0.2848 while for our bounds it is only 0.0636, a 78% reduction.

5. Concluding remarks

It has been expected that a more accurate characterization of matroid complexes would prove useful in improving bounds on all-terminal network reliability. In this paper we have extended the very effective Ball–Provan bounding methods to take information on degree bounds, purity, and M-shellability into account. The resulting improvement in the bounds is striking. The improvement in the lower bound is particularly useful. Not all of the combinatorial characterization of the complexes is employed, as a consequence of the need to maintain efficiency in the bounding techniques. Despite approximations using thinnest and thickest possible intervals in an M-shelling, the requirement for M-shellability is powerful. It remains of substantial interest whether a more precise characterization of matroid complexes, in particular complexes of cographic matroids, can be found to improve these bounds further yet.

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