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Inverse Spectral Theory Using Nodal Points as Data—A Uniqueness Result*

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INTRODUCTION

In this paper we are concerned with inverse spectral theory for a Sturm–Liouville problem using a new kind of spectral data. In this initial uniqueness result we will consider the specific Sturm Liouville problem

$$y'' + (\lambda - q)\, y = 0 \qquad 0 < x < 1 \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where $q \in L^2(0, 1)$. The data that is given is the position of nodes (i.e., zeros) of the mode shapes (eigenfunctions). We seek to recover the potential q.

We have considered problem (1)–(2) as a prototype for second order eigenvalue problems of the form

$$(pu_x)_x - qu + \lambda \rho u = 0, \qquad 0 \leqslant x \leqslant L, \quad p > 0, \quad \rho > 0$$
$$u(0) = u(L) = 0.$$

These eigenvalue problems arise naturally in the study of vibrating systems. In these cases each eigenvalue, λ_n, is the square of a natural frequency. The most natural experiment then for finding the nodal positions is to excite the vibrating system at a natural frequency and take measurements of the positions where the system does not vibrate. These positions are the zeros (or nodes) of the eigenfunctions. What we present here is a uniqueness theorem for the case $p \equiv \rho \equiv 1$ where a subset of nodal positions are given as data.

To give some perspective on the theoretical results to be presented here, we recall three sets of spectral data that have already been successfully used

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to obtain the potential q [see [1] for a more extensive survey.] For each set of data two sequences have been given. One sequence is the set of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ for (1)–(2). The other sequence is either a set of norming constants or a second set of eigenvalues. In the case where the second sequence is a set of norming constants, two possibilities are as follows. If $y(x, q, \lambda_n)$ is the eigenfunction corresponding to the eigenvalue $\lambda = \lambda_n(q)$ then one set of norming constants could be $\rho_n(q) = \|y(\cdot, q, \lambda_n)\|^2/|y'(0, q, \lambda_n)|^2$, $n = 1, 2, \ldots$ (see [2]). Another set of norming constants, (see [3, 4]) could be $k_n(q) = \log |(y'(1, q, \lambda_n))/(y'(0, q, \lambda_n))|$, $n = 1, 2, \ldots$. In the case where the second sequence in the data set is another set of eigenvalues $\mu_n(q)$, $n = 1, 2, \ldots$, this set could be chosen as the eigenvalues for (1) together with boundary conditions different from those in (2), say

$$y(0) = 0, \qquad y'(1) = 0 \tag{3}$$

We will state now only the uniqueness results for these data sets. These results are as follows.

THEOREM 1. *Suppose* $q_1, q_2 \in L^2(0, 1)$. *Suppose* $\lambda_n(q_1) = \lambda_n(q_2)$, $\rho_n(q_1) = \rho_n(q_2)$, $n = 1, 2, \ldots$ *for* (1)–(2). *Then* $q_1 \equiv q_2$, *a.e.*

THEOREM 2. *Suppose* $q_1, q_2 \in L^2(0, 1)$. *Suppose* $\lambda_n(q_1) = \lambda_n(q_2)$, $k_n(q_1) = k_n(q_2)$, $n = 1, 2, \ldots$ *for* (1)–(2). *Then* $q_1 \equiv q_2$, *a.e.*

THEOREM 3. *Suppose* $q_1, q_2 \in L^2(0, 1)$. *Suppose* $\lambda_n(q_1) = \lambda_n(q_2)$ *for* (1)–(2) *and* $\mu_n(q_1) = \mu_n(q_2)$ *for* (1)–(3). *Then* $q_1 \equiv q_2$, *a.e.*

Roughly speaking the first two uniqueness results say that the potential q can be determined uniquely by measuring the natural frequencies of our physical system plus one measurement of each mode shape. The third uniqueness theorem says that if we set up two experiments where the left boundary condition is the same in each but the right boundary condition is dfferent, and we measure the natural frequencies of the system for each set of boundary conditions then q can be determined uniquely.

Our motivation in considering nodal points as data was our desire to determine an alternative to norming constant data, when this data is difficult to measure. We were also seeking an alternative to measuing a second set of eigenvalues. Of particular interest is the case where only one set of boundary conditions can easily be achieved experimentally. Initially we anticipated that we would have as data the eigenvalues, $\{\lambda_n(q)\}_{n=1}^{\infty}$, together with one or more positions of zeros for each eigenfunction, $n \geqslant 2$. What can be shown, however, is that just the position of one node, albeit judiciously chosen, for each eigenfunction, $n \geqslant 2$, is more than enough data to determine q uniquely. It seems then that the nodal positions in some sense contain "more" information about the potential q than either a set of eigenvalues or a set of norming constants.

The specifics of our uniqueness result is contained in Sections 1 and 2. In Section 1 we give preliminary technical lemmas. Section 2 contains the uniqueness result.

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In this section we will again consider the eigenvalue problem

$$y'' + (\lambda - q)\, y = 0, \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where $q \in L^2(0, 1)$. Known results concerning asymptotic forms for eigenvalues and eigenfunctions will be given. We will also establish the asymptotic forms for nodal positions and two denseness results.

We begin by recalling that (1)–(2) has a sequence of eigenvalues $\lambda_1 < \lambda_2 < \cdots$ with $\lim_{n \to \infty} \lambda_n = \infty$ and each eigenfunction $y(x, q, \lambda_n)$ having exactly $n-1$ zeros, $n = 1, 2, ...,$ in $0 < x < 1$.

In order to establish more results we will need the two fundamental solutions $y_1(x, q, \lambda)$, $y_2(x, q, \lambda)$ of (1) where these functions satisfy $y_1(0, q, \lambda) = 1$, $y_1'(0, q, \lambda) = 0$, and $y_2(0, q, \lambda) = 0$, $y_2'(0, q, \lambda) = 1$, respectively. Then it is known (see [3]) that

$$y_1(x, q, \lambda) = \cos \sqrt{\lambda}\, x + O\left[\frac{\exp |\operatorname{Im} \sqrt{\lambda}|\, \sqrt{x}}{\sqrt{|\lambda|}}\right],$$

$$y_2(x, q, \lambda) = \frac{\sin \sqrt{\lambda}\, x}{\sqrt{\lambda}} + O\left[\frac{\exp |\operatorname{Im} \sqrt{\lambda}|\, \sqrt{x}}{|\lambda|}\right],$$

and that the eigenvalues, λ_n, $n = 1, 2, ...,$ satisfy

$$\lambda_n = n^2\pi^2 + c_0 + \alpha_n,$$

where $\sum_{n=1}^{\infty} (\alpha_n)^2 < \infty$, that is $\{\alpha_n\} = \boldsymbol{\alpha} \in l^2$, and $c_0 = \int_0^1 q(x)\, dx$. We further observe that there is only one linearly independent eigenfunction corresponding to each eigenvalue and that any constant multiple of $y_2(x, q, \lambda_n)$ is an eigenfunction of (1)–(2) corresponding to $\lambda = \lambda_n$. Finally, we label the zeros of $y_2(x, q, \lambda_n)$ as $0 < x_n^1 < x_n^2 < \cdots < x_n^{n-1} < 1$, when $n \geqslant 2$.

It is important to establish more detailed information about the positions $x_n^j(q)$, $j = 1, ..., n-1$, $n \geqslant 2$, of the nodes. What can be shown is that a "good" approximation to the value of the points $x_n^j(q)$, for arbitrary

$q \in L^2$, can be given by the value of $x_n^j(0)$. To this end we observe that when $q \equiv 0$, the eigenvalues of (1), (2) are $\lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$, and the zeros $x_n^j(0) = j/n$, $j = 1, \ldots, n-1$, $n \geqslant 2$. We seek now to establish a bound on the difference $x_n^j(q) - x_n^j(0)$.

In order to do this we observe that the position $x_n^j(q)$, the jth zero of the nth eigenfunction $y_2(x, q, \lambda_n)$, is a continuously (Frechet) differentiable nonlinear mapping from $L^2 \to R$. In fact, we define

$$d_q x_n^j[w] = \lim_{\varepsilon \to 0} \frac{x_n^j(q + \varepsilon w) - x_n^j(q)}{\varepsilon}$$

for $q \in L^2$ and establish the following lemma.

LEMMA 1. *Let $q \in L^2(0, 1)$ and define $x_n^j(q)$ $j = 1, \ldots, n-1$, $n = 2, 3, \ldots$, as above. Then*

$$\begin{aligned} d_q x_n^j[w] = {} & \frac{1}{[y_2'(x_n^j, q, \lambda_n)]^2} \left\{ \int_0^{x_n^j} [y_2[t, q, \lambda_n)]^2 \, w(t) \, dt \right\} \\ & - \frac{\dot{y}_2(x_n^j, q, \lambda_n)}{\dot{y}_2(1, q, \lambda_n)} \frac{1}{y_2'(x_n^j, q, \lambda_n) \, y_2'(1, q, \lambda_n)} \\ & \times \left\{ \int_0^1 [y_2(t, q, \lambda_n)]^2 \, w(t) \, dt \right\}, \end{aligned}$$

where

$$\dot{y}_2(x_n^j, q, \lambda_n) = \frac{d}{d\lambda} y_2(x, q, \lambda) \,|_{\lambda = \lambda_n, x = x_n^j}$$

and

$$\dot{y}_2(1, q, \lambda_n) = \frac{d}{d\lambda} y_2(1, q, \lambda) \,|_{\lambda = \lambda_n}.$$

Proof. In order to show the formula in the lemma we observe that for all q and fixed, $j = 1, \ldots, n-1$, and $n = 2, 3, \ldots$, we have $y(x_n^j, q, \lambda_n) \equiv 0$, and $y(1, q, \lambda_n) \equiv 0$. Taking the derivative of these expressions as defined above we have

$$y_2'(x_n^j, q, \lambda_n) \, d_q x_n^j[w] + d_q \, y_2(x_n^j, q, \lambda_n)[w] + \dot{y}_2(x_n^j, q, \lambda_n) \, d_q \lambda_n[w] = 0,$$

and $d_q \, y(1, q, \lambda_n)[w] + \dot{y}(1, q, \lambda_n) \, d_q \lambda_n[w] = 0$, where $d_q \, y(x, q, \lambda)[w] = \lim_{\varepsilon \to 0} (1/\varepsilon)[y(x, q + \varepsilon w, \lambda) - y(x, q, \lambda)]$. The formula in the lemma will follow immediately once we have determined $d_q \, y(x_n^j, q, \lambda_n)[w]$ and

$d_q y(1, q, \lambda_n)[w]$. This is done easily by recalling that (or verifying directly) that

$$\frac{y_2(x, q+\varepsilon w, \lambda) - y_2(x, q, \lambda)}{\varepsilon} = \int_0^x [y_2(x, q, \lambda)\, y_1(t, q, \lambda) - y_1(x, q, \lambda)\, y_2(t, q, \lambda)] \cdot [w(t)]\, y_2(t, q+\varepsilon w, \lambda)\, dt. \tag{4}$$

Taking the limit as $\varepsilon \to 0$ we have

$$d_q\, y_2(x, q, \lambda)[w] = \int_0^x [y_2(x, q, \lambda)\, y_1(t, q, \lambda) - y_1(x, q, \lambda)\, y_2(t, q, \lambda)] \cdot y_2(t, q, \lambda)\, w(t)\, dt.$$

This expression is simplified when $\lambda = \lambda_n$ and $x = 1$ or $x = x_n^j$ since $y_2(1, q, \lambda_n) = 0$ and $y_2(x_n^j, q, \lambda_n) = 0$. In these cases we have $d_q\, y_2(x_n^j, q, \lambda_n)[w] = -y_1(x_n^j, q, \lambda_n) \int_0^{x_n^j} w(t)[y_2(t, q, \lambda_n)]^2\, dt$, and $d_q\, y_2(1, q, \lambda_n)[w] = -y_1(1, q, \lambda_n) \int_0^1 w(t)[y_2(t, q, \lambda_n)]^2\, dt$. Finally, we employ a Sturm identity to show that $y_1(1, q, \lambda_n)\, y_2'(1, q, \lambda_n) = y_1(x_n^j, q, \lambda_n)\, y_2'(x_n^j, q, \lambda_n) = 1$.

Having established the form for the derivative of $x_n^j[w]$, we observe that we can return to Eq. (4) and easily show that this derivative is also a Frechet derivative. We can also use the derived formula to help us establish an asymptotic form for $x_n^j(q)$. What we actually need, however, is really just a bound on the difference, $x_n^j(q) - x_n^j(0) = x_n^j(q) - j/n$. This is accomplished in the following lemma.

LEMMA 2. *Let $q \in L^2(0, 1)$. Consider the eigenvalue problem*

$$y'' + (\lambda - q)\, y = 0, \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

and let $y_2(x, q, \lambda_n)$ be the eigenfunction corresponding to the eigenvalue $\lambda = \lambda_n$, $n = 1, 2, \ldots$. Let x_n^j be the position of the jth *zero of $y_2(x, q, \lambda_n)$. Then*

$$x_n^j = \frac{j}{n} + 0\left(\frac{1}{n^2}\right).$$

Proof. Since x_n^j is a smooth function of q we can write

$$x_n^j(q) - x_n^j(0) = \int_0^1 \frac{d}{dt}\, x_n^j(tq)\, dt,$$

where $(d/dt)\, x_n^j(tq) = \lim_{\varepsilon \to 0} (x_n^j(tq + \varepsilon q) - x_n^j(tq))/\varepsilon = d_{tq} x_n^j[q]$.

This linear operator has been calculated explicitly in the previous lemmas. We will establish a bound on $d_{tq} x_n^j[q]$ for all t. This is done using the asymptotic forms given in the beginning of this section along with the following bounds for λ real,

$$y_2'(x, tq, \lambda_n) = \cos \sqrt{\lambda_n}\, x + O\left(\frac{1}{\sqrt{\lambda_n}}\right)$$

$$[y_2(x, tq, \lambda_n)]^2 = \left[\frac{1 - \cos 2\sqrt{\lambda_n}\, x}{2\lambda_n}\right] + O\left(\frac{1}{(\lambda_n)^{+3/2}}\right)$$

and the very crude bound

$$\frac{\dot{y}_2(x_n^j, tq, \lambda_n)}{\dot{y}_2(1, tq, \lambda_n)} = x_n^j + O(1).$$

It follows easily that

$$d_{tq} x_n^j[q] = \frac{1}{2n^2\pi^2}\left[\int_0^{x_n^j} q(s)\, ds - x_n^j \int_0^1 q(s)\, ds + O(1)\right],$$

proving the lemma.

We seek now to show that for any $q \in L^2$ an appropriately chosen subset of the positions of the nodes, $x_n^j(q)$, $j = 1, ..., n-1$, $n = 2, ...$, is dense in $[0, 1]$. The subset will contain one node for each $n \geqslant 2$. Our method will be to display a dense subset when $q \equiv 0$ to use the above theorem to obtain the result for any $q \in L^2$. From the proof it will be clear that there are many possible choices for the dense subset. The lemmas here simply display one of the dense subsets that "work." The result for $q \equiv 0$ is contained in the following lemma.

LEMMA 3. *The set of numbers* $(m+1)/(2^{k+1} - m)$, $k = 0, 1, 2, 3, ...$, $m = 0, 1, ..., 2^k - 1$ *is dense in the interval* $[0, 1]$.

Remark. The set of numbers $(2^{k+1} - m)^2 \pi^2$, $k = 0, 1, 2, ..., m = 1, 2, ..., 2^k - 1$ is the set of all eigenvalues except $\lambda = \lambda_1$, for (1)–(2) when $q \equiv 0$. The fraction $(m+1)/(2^{k+1} - m)$ is a zero for the eigenfunction corresponding to the eigenvalue $(2^{k+1} - m)^2 \pi^2$. Hence the set of rational numbers given in the above lemma represents a selection of one node (or zero) from each eigenfunction, except the eigenfunction corresponding to $\lambda = \lambda_1 = \pi^2$.

Remark. We note from the point of view of notation that for $q \equiv 0$, $x_{2^{k+1} - m}^{m+1} = (m+1)/(2^{k+1} - m)$.

Proof of Lemma 3. The proof consists merely in showing that the maximum difference between consecutive numbers in each sequence

$$0, \frac{1}{2^{k+1}}, \frac{2}{2^{k+1}-1}, \frac{3}{2^{k+1}-2}, ..., \frac{2k}{2^k+1}, 1,$$

$k = 0, 1, 2, ...,$ approaches zero as $k \to \infty$. To show this, note the following bound on the differences given below:

$$1 - \frac{2^k}{2^k+1} = \frac{1}{2^k+1},$$

$$\frac{1}{2^{k+1}} - 0 = \frac{1}{2^{k+1}},$$

and

$$\frac{m+2}{2^{k+1}-(m+1)} - \frac{m+1}{2^{k+1}-m} = \frac{2^{k+1}+1}{[2^{k+1}-(m+1)][2^{k+1}-m]} < \frac{1}{2^k+1},$$

for $m = 0, 1, ..., 2^k - 2$. Hence, the upper bound, $2/(2^k+1)$, on the differences between consecutive numbers in the kth sequence, $\{0, 1/2^{k+1}, ..., 2^k/(2^k+1), 1\}$ goes to zero as $k \to \infty$ and the lemma is proved.

We have now established a set of zeros of the eigenfunctions $\{\sin n\pi x\}_{n=2}^{\infty}$ which are dense on $(0, 1)$. These are the eigenfunctions for the potential $q \equiv 0$ and it should be said again that only one zero was chosen from each eigenfunction. We can now use this denseness result together with the bound determined in Lemma 3 to establish an analogous denseness result for nodes for the problem (1)–(2) where q is arbitrary, $q \in L^2(0, 1)$. This result is as follows:

LEMMA 4. *Let $q \in L^2(0, 1)$. For each integer $n \geqslant 2$ find $k = 0, 1, 2, 3, ...,$ and $m = 0, 1, ..., 2^{k-1}$ such that $n = 2^{k+1} - m$. For each fixed n, use the representation of n in terms of k and m to define $j(n) = m + 1$. Then with this choice $\{x_n^{j(n)}(q)\}_{n=2}^{\infty}$ is dense in $(0, 1)$.*

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We can now establish our uniqueness result. In words the uniqueness result will say that if we know the position of one node of each eigenfunction (except the first eigenfunction has no zero in $(0, 1)$) and we know the average of q on the interval, then there is at most one $q \in L^2$ which can yield that set of nodes. We state the result precisely.

UNIQUENESS THEOREM. *Let $q_1, q_2 \in L^2$, and consider the eigenvalue problem*

$$y'' + (\lambda - q_i)\, y = 0,$$
$$y(0) = y(1) = 0,$$

$i = 1, 2$. For each $n \geqslant 2$, choose $j(n)$ as in Lemma 4. Suppose that the positions of the specifically chosen zeros satisfy $x_n^{j(n)}(q_1) = x_n^{j(n)}(q_2)$, $n = 2, 3, \ldots$, and that $\int_0^1 q_1\, dx = \int_0^1 q_2\, dx$; then $q_1 \equiv q_2$ a.e.

Proof. We first imploy the fact that $\{x_n^{j(n)}\}_{n \geqslant 2}$ is dense in the following way. Let $x \in [0, 1]$, fixed but arbitrarily chosen. Then by the definition of $j(n)$, there exists a subsequence n_k, $k = 1, 2, \ldots$, such that $\lim_{k \to \infty} x_{n_k}^{j(n_k)} = x$. For ease of notation, let $x_k = x_{n_k}^{j(n_k)}$.

We next define

$$J_k = n_k^2 \pi^2 \int_0^{x_k} [\lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) - q_1 + q_2]\, y_2(t, q_1, \lambda_{n_k}(q_1)] \times y_2(t, q_2, \lambda_{n_k}(q_2))\, dt.$$

By using a standard Sturm identity we can show that $J_k = 0$, $k = 1, 2, 3, \ldots$. We seek to establish the limiting expression $\lim_{k \to \infty} J_k$. To this end we can see from the asymptotic forms for the eigenvalues that since $\int_0^1 (q_1 - q_2)\, dt = 0$ then $\lambda_{n_k}(q_1) - \lambda_{n_k}(q_2) = B_k$ where $\lim B_k = 0$ (in fact, $\sum_{k=1}^{\infty} B_k^2 < \infty$). Further, there exists a constant $M > 0$ such that for $0 \leqslant x \leqslant 1$

$$\left| (n_k \pi)^2\, y_2(x, q_1, \lambda_{n_k}(q_1))\, y_2(x, q_2, \lambda_{n_k}(q_2)) - \left[\frac{1 - \cos 2 n_k \pi x}{2} \right] \right| \leqslant \frac{M}{n_k}.$$

Finally we recall that $\int_0^{x_k} (q_2 - q_1) \cos 2 n_k \pi t\, dt \to 0$ as $k \to \infty$. Combining these results we have that most of the terms in J_k approach zero as $k \to \infty$ and taking the limit we are left with

$$0 = \lim_{k \to \infty} J_k = \int_0^x (q_1(t) - q_2(t))\, dt.$$

While the proof has been done for fixed x, x was chosen arbitrarily. Hence

$$0 = \int_0^x (q_1(t) - q_2(t))\, dt, \qquad x \in [0, 1]$$

and $q_1 - q_2 = 0$ a.e. $[0, 1]$ and the theorem is proved.

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